HYPER QUASI-MV ALGEBRAS AND IDEALS

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Abstract. In this paper, we introduce hyper quasi-MV algebras as the generalizations of hyper MV-algebras and quasi-MV algebras. First we give the definition of hyper quasi-MV algebras and investigate some basic properties of hyper quasi-MV algebras. Second we introduce ideals and weak ideals in a hyper quasi-MV algebra. Especially, we study two types of (weak) ideals and discuss the relationship between them. We also present the dual notions of ideals and weak ideals in this paper. Finally, we show the properties of ideals and weak ideals under the homomorphism of hyper quasi-MV algebras.
Keywords: hyper MV-algebras, hyper quasi-MV-algebras, homomorphisms, ideals.

1. Introduction

The theory of hyper algebraic structures was firstly introduced by Marty in 1934 [15]. Unlike the classical algebraic structures, the composition of two elements in the hyper algebraic structure is a subset. Since the theory was introduced, various hyper structures have been studied such as hyper K-algebras [3], hyper BCK-algebras [13], hyper MV-algebras [8], hyper pseudo MV-algebras [1], hyper effect algebras [6] and so on. Among them, the notion of hyper MV-algebras as a generalization of MV-algebras was paid more attentions and several related results were obtained. In [9, 10], authors discussed quotient structures and category theory of hyper MV-algebras. Hyper MV-ideals of hyper MV-algebras were investigated by Torkzadeh and Ahadpanah in [17]. Rasouli and Davvaz also studied hyper MV-ideals and introduced homomorphism, dual homomorphism and strong homomorphism of hyper MV-algebras in [20]. Meanwhile,

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they showed the properties of hyper MV-ideals under these mappings. Jun et al. introduced some new types of hyper MV-deductive systems in hyper MV-algebras and discussed their relationships [11, 12]. In addition, many other properties of hyper MV-algebras can be seen in [2, 7, 16, 18, 19].

Quasi-MV algebras [14] arising from quantum computational logic can be regarded as a generalization of MV-algebras. In this paper, we want to define hyper quasi-MV algebras as a hyper structural generalization of quasi-MV algebras and a generalization of hyper MV-algebras and study ideals of hyper quasi-MV algebras. The paper is organized as follows. In Section 2, we recall some definitions and results which will be used in the following. In Section 3, we define hyper quasi-MV algebras and investigate the related properties. In Section 4, we discuss two types of ideals and weak ideals and discuss the relationship between them. We also present the dual notions of ideals and weak ideals in this section. In Section 5, we show the properties of ideals and weak ideals under the homomorphism of hyper quasi-MV algebras.

2. Preliminary

In this section, we recall some definitions and show more results which will be used in the subsequent sections.

In [14], Ledda et al. introduced the quasi-MV algebra as a first significant step towards an algebraic characterization of the quantum computational logic. A *quasi-MV algebra* is an algebra $\mathbf{A} = \langle A; \oplus, ', 0, 1\rangle$ of type $\langle 2, 1, 0, 0\rangle$ satisfying the following identities for any $x, y, z \in A$:

(Q1) $x \oplus (y \oplus z) = (x \oplus z) \oplus y$;
(Q2) $x'' = x$;
(Q3) $x \oplus 1 = 1$;
(Q4) $(x' \oplus y)' \oplus y = (y' \oplus x)' \oplus x$;
(Q5) $(x \oplus 0)' = x' \oplus 0$;
(Q6) $(x \oplus y) \oplus 0 = x \oplus y$;
(Q7) $0' = 1$.

Obviously, any MV-algebra is a quasi-MV algebra. Conversely, any quasi-MV algebra with $x \oplus 0 = x$ is an MV-algebra. On any quasi-MV algebra, we can define some operations: $x \odot y = (x' \oplus y')'$, $x \vee y = (x' \oplus y)' \oplus y$ and $x \wedge y = (x' \vee y')'$. We can also define a relation $x \leq y$ if and only if $x \vee y = y \oplus 0$. Below we list some elementary properties of them.

Proposition 2.1 ([14])**.** *Let $\boldsymbol{A}$ be a quasi-MV algebra. Then for any $x, y, z \in A$, we have*

(1) $x \vee y = y \vee x$ *and* $x \wedge y = y \wedge x$*;*
(2) $(x \vee y) \vee z = x \vee (y \vee z)$ *and* $(x \wedge y) \wedge z = x \wedge (y \wedge z)$*;*
(3) $x \wedge y \leq x, y$ *and* $x, y \leq x \vee y$*;*
(4) $x \leq x \wedge x$ *and* $x \wedge x \leq x$*;*
(5) $x \leq x \oplus 0$ *and* $x \oplus 0 \leq x$*;*

(6) $x \odot y \leq z$ *if and only if* $x \leq y' \oplus z$*;*
(7) *if* $x \leq y$*, then* $y' \leq x'$*;*
(8) $0 \leq x \leq 1$.

Let $\mathbf{A}$ be a quasi-MV algebra and I be a non-empty subset of A. We say that I is an *ideal* of $\mathbf{A}$ if, for any $x, y \in I$, the following conditions are satisfied: (I1) $0 \in I$; (I2) if $x, y \in I$, then $x \oplus y \in I$; (I3) if $x \in I$ and $y \in A$ with $y \leq x$, then $y \in I$. On the other hand, I is called a *weak ideal* of $\mathbf{A}$ if, for any $x, y \in I$, the following conditions are satisfied: (WI1) $0 \in I$; (WI2) if $x, y \in I$, then $x \oplus y \in I$; (WI3) if $x \in I$ and $y \in A$, then $x \odot y \in I$. In [14], authors proved that I is an ideal of $\mathbf{A}$ if and only if (1) I is a weak ideal of $\mathbf{A}$ and (2) $x \in I \Leftrightarrow x \oplus 0 \in I$.

Hyper MV-algebras were introduced in [8] as a generalization of MV-algebras. In a hyper MV-algebra M, the composition of x and y is a subset of M.

Definition 2.1 ([8])**.** A hyper MV-algebra is a non-empty set M endowed with a binary hyper operation $\oplus$, a unary operation * and a constant 0 satisfying the following conditions for any $x, y, z \in M$:
(hmv 1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
(hmv 2) $x \oplus y = y \oplus x$;
(hmv 3) $(x^*)^* = x$;
(hmv 4) $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$;
(hmv 5) $0^* \in x \oplus 0^*$;
(hmv 6) $0^* \in x \oplus x^*$;
(hmv 7) if $x \ll y$ and $y \ll x$, then $x = y$, where $x \ll y$ is defined as $0^* \in x^* \oplus y$.

For every non-empty subset A, B of M, we define $A \oplus B = \cup\{x \oplus y | x \in A, y \in B\}$. If A has just only one element x, then we write $x \oplus B = \{x\} \oplus B$. In addition, $A \ll B$ if and only if there exist $x \in A$ and $y \in B$ such that $x \ll y$.

Proposition 2.2 ([8])**.** *Let M be a hyper MV-algebra. Then for any $x, y, z \in M$ and for any non-empty subset A, B and C of M the following hold*
(1) $(A \oplus B) \oplus C = A \oplus (B \oplus C)$*;*
(2) $0 \ll x$*;*
(3) $x \ll x$*;*
(4) *if* $x \ll y$*, then* $y^* \ll x^*$ *and* $A \ll B$ *implies* $B^* \ll A^*$*;*
(5) $x \ll 1$*;*
(6) $A \ll A$*;*
(7) $A \subseteq B$ *implies* $A \ll B$*;*
(8) $x \ll x \oplus y$*,* $A \ll A \oplus B$*;*
(9) $0 \oplus 0 = \{0\}$*;*
(10) $x \in x \oplus 0$*;*
(11) *if* $y \in x \oplus 0$*, then* $y \ll x$*;*
(12) *if* $x \oplus 0 = y \oplus 0$*, then* $x = y$.

For hyper MV-algebras, ideals and filters are defined by many authors. Please see the following. However, all these definitions have similar yet slightly different formats. In Section 4, we present the relationship between these notions.

Definition 2.2. Let M be a hyper MV-algebra and I be a non-empty subset of M. Then

(HF) [8] I is called a *hyper MV-filter* of M, if (i) $1 \in I$; (ii) if $I \ll x^* \oplus y$ and $x \in I$, then $y \in I$.

(HI1) [20] I is called a *hyper MV-ideal* of M, if (i) $x, y \in I$ imply $x \oplus y \subseteq I$; (ii) if $x \ominus y \ll I$ and $y \in I$, then $x \in I$, where $x \ominus y = (x^* \oplus y)^*$.

(HI2) [17] I is called a *hyper MV-ideal* of M, if (i) $x, y \in I$ imply $x \oplus y \subseteq I$; (ii) if $y \ll x$ and $x \in I$, then $y \in I$.

(HDS) [11] I is called a *hyper MV-deductive system* of M, if (i) $0 \in I$; (ii) if $(x^* \oplus y)^* \ll I$ and $y \in I$, then $x \in I$.

3. Hyper quasi-MV algebras

In this section, we give the definition of hyper quasi-MV algebras and list some properties of them.

Definition 3.1. A *hyper quasi-MV algebra* is a non-empty set M endowed with a binary hyper operation $\oplus$, a unary operation * and a constant 0 satisfying the following conditions for any $x, y, z \in M$:

(hqmv 1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
(hqmv 2) $x \oplus y = y \oplus x$;
(hqmv 3) $(x^*)^* = x$;
(hqmv 4) $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$;
(hqmv 5) $0^* \in x \oplus 0^*$;
(hqmv 6) $0^* \in x \oplus x^*$;
(hqmv 7) $x \oplus y \subseteq x \oplus y \oplus 0$;
(hqmv 8) $0 \in 0 \oplus 0$;
(hqmv 9) if $x \ll y$ and $y \ll x$, then $x \oplus 0 = y \oplus 0$, where $x \ll y$ is defined as $0^* \in x^* \oplus y$.

Denote $0^* = 1$. If $0 \neq 1$, then we say that hyper quasi-MV algebra is *non-trivial* or *non-flat*[1]. In what follows, let M denote a non-trivial hyper quasi-MV algebra unless otherwise specified.

Example 3.1. Any hyper MV-algebra is a hyper quasi-MV algebra.

Indeed, let M be a hyper MV-algebra. Following from Proposition 3.7 of [8], we have $x \in x \oplus 0$, so $0 \in 0 \oplus 0$ and $x \oplus y \subseteq x \oplus y \oplus 0$. If $x \ll y$ and $y \ll x$, then $x = y$, it turns out that $x \oplus 0 = y \oplus 0$. Hence M is a hyper quasi-MV algebra.

1. A quasi-MV algebra with $0 = 1$ is called *flat*.

Example 3.2. Any quasi-MV algebra is a hyper quasi-MV algebra.

Let $\langle M, \oplus, ', 0\rangle$ be a quasi-MV algebra. We define $x \oplus^H y = \{x \oplus y\}$ and $x^* = x'$. Then it is easy to verify that $\langle M, \oplus^H, ^*, 0\rangle$ is a hyper quasi-MV algebra.

Example 3.3. Let $M = \{0, a, b, 1\}$. Consider the following tables:

$\oplus$	0	a	b	1
0	$\{0\}$	$\{0,b\}$	$\{0,b\}$	M
a	$\{0,b\}$	M	M	M
b	$\{0,b\}$	M	M	M
1	M	M	M	M

$*$	0	a	b	1
	1	a	b	0

Then $\langle M, \oplus, ^*, 0\rangle$ is a hyper quasi-MV algebra. Moreover, since $a \notin a \oplus 0$, we have that M is not a hyper MV-algebra.

Example 3.4. Let $M = [0,1] \times [0,1]$. Define the operations as follows: $\langle a, b\rangle \oplus \langle c, d\rangle = [0, \min\{1, a + c\}] \times [0,1]$; $\langle a, b\rangle^* = \langle 1 - a, 1 - b\rangle$; $0 = \langle 0, \frac{1}{2}\rangle$. Then $\langle M, \oplus, ^*, 0\rangle$ is a hyper quasi-MV algebra.

The following proofs are similar to the case of hyper MV-algebra in [8, 17].

Proposition 3.1. *Let $\langle M_1, \oplus_1, ^{*_1}, 0_1\rangle$ and $\langle M_2, \oplus_2, ^{*_2}, 0_2\rangle$ be hyper quasi-MV algebras and $M = M_1 \times M_2$. We define a hyper operation $\oplus$ on M as follows: $\langle a_1, b_1\rangle \oplus \langle a_2, b_2\rangle = \langle a_1 \oplus_1 a_2, b_1 \oplus_2 b_2\rangle$, a unary operation * on M as follows: $\langle a, b\rangle^* = \langle a^{*_1}, b^{*_2}\rangle$, and $0 = \langle 0_1, 0_2\rangle$. Then*

(1) $0^* \in \langle a_1, b_1\rangle^* \oplus \langle a_2, b_2\rangle$ *if and only if* $0_1^{*_1} \in a_1^{*_1} \oplus_1 a_2$ *and* $0_2^{*_2} \in b_1^{*_1} \oplus_2 b_2$.

(2) $\langle M, \oplus, ^*, 0\rangle$ *is a hyper quasi-MV algebra.*

Proposition 3.2. *Let M be a hyper quasi-MV algebra. Then for any $x, y, z \in M$ and for any non-empty subset A, B and C of M the following hold*

(hq1) $(A \oplus B) \oplus C = A \oplus (B \oplus C)$;

(hq2) $0 \ll x$;

(hq3) $x \ll x$;

(hq4) *if* $x \ll y$, *then* $y^* \ll x^*$ *and* $A \ll B$ *implies* $B^* \ll A^*$;

(hq5) $x \ll 1$;

(hq6) $A \ll A$;

(hq7) $A \subseteq B$ *implies* $A \ll B$;

(hq8) $x \ll x \oplus y$, *especially,* $x \ll x \oplus 0$.

According to (hp3), we note that the relation $\ll$ is reflexive. However, unlike hyper MV-algebras, it is not antisymmetric. It is easy to see that if the relation $\ll$ is antisymmetric, any hyper quasi-MV algebra is a hyper MV-algebra. Meanwhile, it is not transitive either. Hence if $\ll$ is transitive, then M is called a *transitive hyper quasi-MV algebra.*

On any hyper quasi-MV algebra M, we can define some hyper operations as follows:

$x \odot y = (x^* \oplus y^*)^*$;

$x \vee y = (x^* \oplus y)^* \oplus y$;
$x \wedge y = (x^* \vee y^*)^*$.

Below we will see some properties of these operations.

Lemma 3.1. *Let M be a hyper MV-algebra. For any $x, y \in M$ the following conditions are equivalent:*

(1) $1 \in x^* \oplus y$;
(2) $0 \in x \odot y^*$.

Proposition 3.3. *Let M be a hyper quasi-MV algebra. Then for any $x, y, z \in M$ the following hold:*

(hq9) $x \odot (y \odot z) = x \odot (y \odot z)$;
(hq10) $x \odot y = y \odot x$;
(hq11) $0 \in x \odot x^*$;
(hq12) $0 \in x \odot 0$;
(hq13) $x \odot y \ll x, y$, *especially*, $x \odot 1 \ll x$;
(hq14) $x \odot y \subseteq x \odot y \odot 1$;
(hq15) $x \vee y = y \vee x$ *and* $x \wedge y = y \wedge x$;
(hq16) $x \oplus y \subseteq (x \vee y) \oplus (x \wedge y)$;
(hq17) *if* $x \in x \oplus y$, *then* $0 \in x^* \wedge y$;
(hq18) *if* $x \in x \odot y$, *then* $1 \in x^* \vee y$;
(hq19) $x \odot y \ll x \wedge y \ll x, y$;
(hq20) $x, y \ll x \vee y \ll x \oplus y$;
(hq21) *if* $x \in x \oplus x$, *then* $x \ll x \odot x$, *if* $x \in x \odot x$, *then* $x \ll x \oplus x$;
(hq22) $x \ll y$ *implies* $x \wedge z \ll y \wedge z$ *and* $x \vee z \ll y \vee z$;
(hq23) $x \ll y$ *implies* $x \in x \wedge y$ *and* $y \in x \vee y$;
(hq24) *if* $x \ll y$, *then* $x \oplus z \ll y \oplus z$ *and* $x \odot z \ll y \odot z$;
(hq25) $z \odot x \ll y$ *if and only if* $z \ll x^* \oplus y$.

4. Ideals, weak ideals and duality

In this section, we introduce some types of (weak) ideals for hyper quasi-MV algebras and investigate related properties of them. Let M be a hyper quasi-MV algebra and I be a non-empty subset of M. Denote

$\mathbb{I}_1(M) = \{I \subseteq M |$ if $x \in I$ and $y \in M$ such that $y \odot x^* \ll I$, then $y \in I\}$;
$\mathbb{I}_2(M) = \{I \subseteq M |$ if $x \in I$ and $y \in M$ such that $y \ll x$, then $y \in I\}$;
$\mathbb{I}_3(M) = \{I \subseteq M |$ if $x \in I$ and $y \in M$ such that $(y \odot x^*) \cap I \neq \emptyset$, then $y \in I\}$.

Proposition 4.1. *Let M be a hyper quasi-MV algebra and I be a non-empty subset of M. If $I \in \mathbb{I}_1(M)$, then we have*

(1) $0 \in I$;
(2) *if* $x, y \in I$, *then* $x \oplus y \ll I$;
(3) *if* $x, y \in I$, *then* $(x \oplus y) \cap I \neq \emptyset$;
(4) $I \in \mathbb{I}_2(M)$;
(5) $I \in \mathbb{I}_3(M)$.

Proof. (1) Since I is a non-empty set, we can suppose that $x \in I$. Then by (hqmv 5) and Lemma 3.1, we have $0 \in 0 \odot x^*$. Because $0 \ll x$, we have $0 \odot x^* \ll I$, it follows that $0 \in I$.

(2) Let $x, y \in I$. Then by (hq 11), $0 \in (x \oplus y) \odot (x \oplus y)^* = (x \oplus y) \odot (x^* \odot y^*) = ((x \oplus y) \odot x^*) \odot y^*$, it turns out that there exists $a \in (x \oplus y) \odot x^*$ such that $0 \in a \odot y^*$. By (1), $0 \in I$, we have $a \odot y^* \ll I$, so $a \in I$ and $(x \oplus y) \odot x^* \ll I$. Similarly, there exists $b \in x \oplus y$ such that $b \odot x^* \ll I$. Using the hypothesis again, we have $b \in I$. Hence $x \oplus y \ll I$.

(3) Let $x, y \in I$. By (2), we have $x \oplus y \ll I$, so there exist $a \in x \oplus y$ and $b \in I$ such that $a \ll b$, it turns out that $0 \in a \odot b^*$, so $a \odot b^* \ll I$, we have $a \in I$, thus $(x \oplus y) \cap I \neq \emptyset$.

(4) Let $x \in I$ and $y \in M$ such that $y \ll x$. Then $0 \in y \odot x^*$, so by (1), $y \odot x^* \ll I$, we have $y \in I$. Hence $I \in \mathbb{I}_2(M)$.

(5) Let $x \in I$ and $y \in M$ such that $(y \odot x^*) \cap I \neq \emptyset$, we suppose that $a \in (y \odot x^*) \cap I$, then $a \in y \odot x^*$ and $a \in I$, so $y \odot x^* \ll I$, it follows that $y \in I$. Hence $I \in \mathbb{I}_3(M)$. □

Remark 4.1. Let $I \in \mathbb{I}_1(M)$. Then $x, y \in I$ may not imply $x \oplus y \subseteq I$ in general. See the following example.

Example 4.1. Let $M = \{0, a, b, 1\}$ and the operations are given by the following tables:

$\oplus$	0	a	b	1
0	$\{0\}$	$\{b\}$	$\{b\}$	$\{1\}$
a	$\{b\}$	$\{1\}$	$\{1\}$	$\{1\}$
b	$\{b\}$	$\{1\}$	$\{1\}$	$\{1\}$
1	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$

$*$	0	a	b	1
	1	a	b	0

Then $\langle M, \oplus, ^*, 0\rangle$ is a hyper quasi-MV algebra. Denote $I = \{0, a\}$. Then $I \in \mathbb{I}_1(M)$. However, $a \oplus a = \{1\}$, and $1 \notin I$.

Lemma 4.1. *Let M be a hyper quasi-MV algebra and I be a non-empty subset of M. If $I \in \mathbb{I}_3(M)$ and $0 \in I$, then $I \in \mathbb{I}_1(M)$.*

Proof. Let $x \in I$ and $y \in M$ such that $y \odot x^* \ll I$. Then there exist $a \in y \odot x^*$ and $b \in I$ such that $a \ll b$, so $0 \in a \odot b^*$. Since $0 \in I$, we have $(a \odot b^*) \cap I \neq \emptyset$, it follows that $a \in I$. Hence $(y \odot x^*) \cap I \neq \emptyset$ and we have $y \in I$. □

Remark 4.2. The condition "$0 \in I$" is necessary in Lemma 4.1. See the following example.

Example 4.2. Let $M = \{0, a, b, a^*, 1\}$ and the operations are given by the following tables:

$\oplus$	0	a	b	a	1
0	$\{0\}$	$\{0\}$	$\{b\}$	$\{1\}$	$\{1\}$
a	$\{0\}$	$\{0\}$	$\{b\}$	$\{1\}$	$\{1\}$
b	$\{b\}$	$\{b\}$	$\{1\}$	$\{1\}$	$\{1\}$
a^*	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
1	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$

$*$	
0	1
a	a^*
b	b
a^*	a
1	0

Then $\langle M, \oplus, ^*, 0\rangle$ is a hyper quasi-MV algebra. Denote $I = \{a^*, 1\}$. Then $I \in \mathbb{I}_3(M)$. However, $I \notin \mathbb{I}_1(M)$. Indeed, $0 \odot a^{**} \ll I$ but $0 \notin I$.

Proposition 4.2. *Let M be a hyper quasi-MV algebra and I be a non-empty subset of M. Then the following conditions are equivalent*

(1) $I \in \mathbb{I}_1(M)$ *and if* $x, y \in I$, *then* $x \oplus y \subseteq I$;

(2) $I \in \mathbb{I}_2(M)$ *and if* $x, y \in I$, *then* $x \oplus y \subseteq I$;

(3) $I \in \mathbb{I}_3(M)$ *and* $0 \in I$ *and if* $x, y \in I$, *then* $x \oplus y \subseteq I$.

Proof. (1) $\Rightarrow$ (2) By Proposition 4.1.

(2) $\Rightarrow$ (1) Let $x \in I$ and $y \in M$ with $y \odot x^* \ll I$. Then there exist $a \in y \odot x^*$ and $b \in I$ such that $0 \in a \odot b^*$, it follows that $0 \in (y \odot x^*) \odot b^* = y \odot (x^* \odot b^*) = y \odot (x \oplus b)^*$, so there exists $m \in x \oplus b$ such that $0 \in y \odot m^*$, we have $y \ll m$. Note that $x, b \in I$, we have $x \oplus b \subseteq I$, thus $m \in I$. Since $I \in \mathbb{I}_2(M)$, we get $y \in I$.

(1) $\Leftrightarrow$ (3) By Proposition 4.1 and Lemma 4.1. □

Now, we give the definition of ideals in hyper quasi-MV algebras.

Definition 4.1. Let M be a hyper quasi-MV algebra and I be a non-empty subset of M. Then I is called a *hyper quasi-MV type*-1 *ideal* (*type*-1 *ideal*, for short), if it satisfies any one of the equivalent conditions in Proposition 4.2.

Proposition 4.3. *Let M be a hyper quasi-MV algebra and $\{I_i | i \in S\}$ be a family of type-1 ideals of M. Then $\bigcap_{i\in S} I_i$ is also a type-1 ideal of M.*

Proof. Since $0 \in \bigcap_{i\in S} I_i$, we have that $\bigcap_{i\in S} I_i$ is a non-empty subset of M. Let $x, y \in \bigcap_{i\in S} I_i$. Then for any $i \in S$, we have $x \in I_i$ and $y \in I_i$, it follows that $x \oplus y \subseteq I_i$, so $x \oplus y \subseteq \bigcap_{i\in S} I_i$. Now, suppose that $x \in \bigcap_{i\in S} I_i$ and $y \in M$ such that $y \odot x^* \ll \bigcap_{i\in S} I_i$. Then for any $i \in S$, $x \in I_i$ and $y \odot x^* \ll I_i$, we have $y \in I_i$ for any $i \in S$, so $y \in \bigcap_{i\in S} I_i$. Hence $\bigcap_{i\in S} I_i$ is a type-1 ideal of M. □

Proposition 4.4. *Let M be a hyper quasi-MV algebra with $0 \oplus 0 = \{0\}$. Then $(0] = \{x \in M | x \ll 0\}$ is a type-1 ideal of M.*

Proof. Suppose that $x, y \in (0]$. Then $x \ll 0$ and $y \ll 0$, it follows that $x \oplus 0 = 0 \oplus 0 = \{0\}$ and $y \oplus 0 = 0 \oplus 0 = \{0\}$. We have $x \oplus y \subseteq x \oplus y \oplus 0 \subseteq x \oplus 0 \oplus y \oplus 0 = 0 \oplus 0 \oplus 0 \oplus 0 = \{0\}$, so for any $a \in x \oplus y$, then $a = 0$, we have $a \ll 0$. Hence $x \oplus y \subseteq (0]$. Now, let $x \in (0]$ and $y \in M$ such that $y \ll x$. Then $1 \in y^* \oplus x \subseteq y^* \oplus x \oplus 0 = y^* \oplus 0 \oplus 0 = y^* \oplus 0$, it turns out that $y \ll 0$, so $y \in (0]$. Hence $(0] = \{x \in M | x \ll 0\}$ is a type-1 ideal of M. □

According to Proposition 4.1, if $I \in \mathbb{I}_1(M)$, then $x \oplus y \ll I$ for any $x, y \in I$. However, we need to point out that if $I \in \mathbb{I}_2(M)$, then for any $x, y \in I$, the result $x \oplus y \ll I$ may be not true.

Example 4.3. Let $M = \{0, b, 1\}$ and the operations are given by the following tables:

$\oplus$	0	b	1
0	$\{0\}$	$\{0, b\}$	M
b	$\{0, b\}$	$\{1\}$	M
1	M	M	M

$*$	0	b	1
	1	b	0

Then $\langle M, \oplus, ^*, 0\rangle$ is a hyper MV-algebra [17] and so it is a hyper quasi-MV algebra. Denote $I = \{0, b\}$. Then $I \in \mathbb{I}_2(M)$, however, $b \oplus b = \{1\} \not\ll I$.

Proposition 4.5. *Let M be a hyper quasi-MV algebra and I be a non-empty subset of M. Then the following conditions are equivalent*

(1) $I \in \mathbb{I}_2(M)$ *and if* $x, y \in I$, *then* $x \oplus y \ll I$*;*

(2) $I \in \mathbb{I}_2(M)$ *and if* $x, y \in I$, *then* $(x \oplus y) \cap I \neq \emptyset$.

Proof. (1) $\Rightarrow$ (2) Let $x, y \in I$ and $x \oplus y \ll I$. Then there exist $a \in x \oplus y$ and $b \in I$ such that $a \ll b$. Note that $I \in \mathbb{I}_2(M)$, we have $a \in I$. Hence $(x \oplus y) \cap I \neq \emptyset$.

(2) $\Rightarrow$ (1) Suppose that $x, y \in I$ and $(x \oplus y) \cap I \neq \emptyset$. Then we can suppose that $a \in (x \oplus y) \cap I$, it turns out that $a \in x \oplus y$ and $a \in I$, so $x \oplus y \ll I$. □

Definition 4.2. Let M be a hyper quasi-MV algebra and I be a non-empty subset of M. Then I is called a *hyper quasi-MV type*-1 *weak ideal* (*type*-1 *weak ideal*, for short), if it satisfies any one of the equivalent conditions in Proposition 4.5.

Proposition 4.6. *Let M be a hyper quasi-MV algebra. Then any type*-1 *ideal of M is a type*-1 *weak ideal of M.*

Proof. Follows from Proposition 4.2 and Proposition 3.2(hq7). □

Below we give another type of ideals in hyper quasi-MV algebras.

Definition 4.3. Let M be a hyper quasi-MV algebra and I be a non-empty subset of M. Then I is called a *hyper quasi-MV type*-2 *ideal* of M (*type*-2 *ideal*, for short), if it satisfies the following conditions:

(1) $0 \in I$;

(2) if $x, y \in I$, then $x \oplus y \subseteq I$;

(3) if $x \in I$ and $y \in M$, then $(x \odot y) \cap I \neq \emptyset$.

Proposition 4.7. *Let M be a hyper quasi-MV algebra and $\{I_i | i \in S\}$ be a family of type*-2 *ideals of M. Then $\bigcap_{i \in S} I_i$ is also a type*-2 *ideal of M.*

Proof. Since $0 \in \bigcap_{i\in S} I_i$, we have that $\bigcap_{i\in S} I_i$ is a non-empty subset of M. Let $x, y \in \bigcap_{i\in S} I_i$. Then for any $i \in S$, we have $x \in I_i$ and $y \in I_i$, it follows that $x \oplus y \subseteq I_i$, so $x \oplus y \subseteq \bigcap_{i\in S} I_i$. Now, suppose that $x \in \bigcap_{i\in S} I_i$ and $y \in M$. Then for any $i \in S$, $x \in I_i$ and $(x \odot y) \cap I_i \neq \emptyset$, so $(x \odot y) \cap (\bigcap_{i\in S} I_i) \neq \emptyset$. Hence $\bigcap_{i\in S} I_i$ is a type-2 ideal of M. □

Proposition 4.8. *Let M be a hyper quasi-MV algebra. Then any type-1 ideal of M is a type-2 ideal of M.*

Proof. Let I be a type-1 ideal of M and $x \in I$. Since $0 \ll x$, we have $0 \in I$. Now, suppose that $x \in I$ and $y \in M$. Since $x \odot y \ll x$ by (hq13), there exists $a \in x \odot y$ such that $a \ll x$, it follows that $a \in I$. Hence $(x \odot y) \cap I \neq \emptyset$. □

Proposition 4.9. *Let M be a hyper quasi-MV algebra. Then I is a type-1 ideal of M if and only if (1) I is a type-2 ideal of M and (2) $y \in I \Leftrightarrow y \odot 1 \ll I$.*

Proof. Let I be a type-1 ideal of M and $x \in I$. Then I is a type-2 ideal of M by Proposition 4.7. Moreover, if $y \in I$, then $y \odot 1 \ll y$ by (hq13), so $y \odot 1 \ll I$. Conversely, if $y \odot 1 \ll I$, then $y \odot 0^* \ll I$. Since $0 \in I$ and I is a type-1 ideal, it follows that $y \in I$.

Conversely, suppose that I is a type-2 ideal of M and let $x \in I$ and $y \in M$ such that $y \ll x$. We want to prove $y \in I$. By (2), we only show that $y \odot 1 \ll I$. Because $1 \in y^* \oplus x \subseteq y^* \oplus x \oplus 0 = (y^* \oplus 0) \oplus x = (y \odot 1)^* \oplus x$, there exists $a \in y \odot 1$ such that $1 \in a^* \oplus x$, it follows that $a \ll x$. Note that $x \in I$, we get $y \odot 1 \ll I$. □

Definition 4.4. Let M be a hyper quasi-MV algebra and I be a non-empty subset of M. Then I is called a *hyper quasi-MV type-2 weak ideal* of M (*type-2 weak ideal*, for short), if it satisfies the following conditions

(1) $0 \in I$;

(2) if $x, y \in I$, then $x \oplus y \ll I$;

(3) if $x \in I$ and $y \in M$, then $(x \odot y) \cap I \neq \emptyset$.

Proposition 4.10. *Let M be a hyper quasi-MV algebra. Then $\{0\}$ is a type-2 weak ideal of M.*

Proof. Suppose that $x, y \in \{0\}$. Then $x \oplus y = 0 \oplus 0$. Since $0 \in 0 \oplus 0$, we have $x \oplus y \ll \{0\}$. Let $x \in \{0\}$ and $y \in M$. Then we have $x \odot y = 0 \odot y$. And by (hq12), we have $0 \in 0 \odot y$, so $(x \odot y) \cap \{0\} \neq \emptyset$. □

Proposition 4.11. *Let M be a hyper quasi-MV algebra. Then any type-2 ideal of M is a type-2 weak ideal of M.*

Proof. It is obvious to get the result by (hq7). □

Proposition 4.12. *Let M be a hyper quasi-MV algebra. Then any type-1 weak ideal of M is a type-2 weak ideal of M.*

Proof. The proof is similar to Proposition 4.8. □

Proposition 4.13. *Let M be a hyper quasi-MV algebra and I be a non-empty subset of M. Then I is a type-1 weak ideal of M if and only if* (1) *I is a type-2 weak ideal of M and* (2) $y \in I \Leftrightarrow y \odot 1 \ll I$.

Proof. The proof is similar to Proposition 4.9. □

Theorem 4.1. *Let M_1 and M_2 be hyper quasi-MV algebras and consider the hyper quasi-MV algebra $M_1 \times M_2$. Then*

(1) *If I_1 and I_2 are two type-1/type-2 ideals (type-1/type-2 weak ideals) of M_1, M_2, respectively, then $I_1 \times I_2$ is a type-1/type-2 ideal (type-1/type-2 weak ideal) of $M_1 \times M_2$.*

(2) *If I is a type-1/type-2 ideal (type-1/type-2 weak ideal) of $M_1 \times M_2$, then there are two unique type-1/type-2 ideals (type-1/type-2 weak ideals) I_1 and I_2 of M_1 and M_2, respectively, such that $I = I_1 \times I_2$.*

Proof. The proof is based on Proposition 3.1 and the definitions of ideals and weak ideals. □

Recall that in a hyper MV-algebra, Ghorbani introduced the notions of hyper-MV filters and weak hyper-MV filters in [8]. Furthermore, Davvaz et al. defined hyper-MV ideals in [20] and clarified the connection between hyper-MV filters and hyper-MV ideals. In the following, we generalize these results in a hyper quasi-MV algebra.

Definition 4.5. Let M be a hyper quasi-MV algebra and F be a non-empty subset of M. Then

(1) F is called a *hyper quasi-MV type-1 filter*(*type-1 filter*, for short) of M, if it satisfies the following conditions:

(hf11) if $x, y \in F$, then $x \odot y \subseteq F$;

(hf12) if $x \in F$ and $y \in M$ such that $F \ll x^* \oplus y$, then $y \in F$.

(2) F is called a *hyper quasi-MV type-1 weak filter*(*type-1 weak filter*, for short) of M, if it satisfies the following conditions:

(hwf11) if $x, y \in F$, then $x \odot y \ll F$;

(hwf12) if $x \in F$ and $y \in M$ such that $x \ll y$, then $y \in F$.

(3) F is called a *hyper quasi-MV type-2 filter*(*type-2 filter*, for short) of M, if it satisfies the following conditions:

(hf21) $1 \in F$;

(hf22) if $x, y \in F$, then $x \odot y \subseteq F$;

(hf23) if $x \in F$ and $y \in M$, then $(x \oplus y) \cap F \neq \emptyset$.

(4) F is called a *hyper quasi-MV type-2 weak filter*(*type-2 weak filter*, for short) of M, if it satisfies the following conditions:

(hwf21) $1 \in F$;

(hwf22) if $x, y \in F$, then $x \odot y \ll F$;

(hwf23) if $x \in F$ and $y \in M$, then $(x \oplus y) \cap F \neq \emptyset$.

Proposition 4.14. *Let M be a hyper quasi-MV algebra and I be a non-empty subset of M. Then*

(1) *I is a type-1 ideal of M if and only if I^* is a type-1 filter of M;*

(2) *I is a type-1 weak ideal of M if and only if I^* is a type-1 weak filter of M;*

(3) *I is a type-2 ideal of M if and only if I^* is a type-2 filter of M;*

(4) *I is a type-2 weak ideal of M if and only if I^* is a type-2 weak filter of M.*

Proof. We only prove (1) and the others can be proved similarly. Let I be a type-1 ideal of M. Then for any $x^*, y^* \in I^*$, we have $x \odot y = (x^* \oplus y^*)^*$. Since $x^*, y^* \in I$ and $x^* \oplus y^* \subseteq I$, we have $x \odot y \subseteq I^*$. Suppose that $x \in I^*$ and $y \in M$ such that $I^* \ll x^* \oplus y$. Then $(x^* \oplus y)^* \ll I$, i.e., $(x^*)^* \odot y^* \ll I$. Since $x^* \in I$ and $y^* \in M$, we have $y^* \in I$, so $y \in I^*$. The converse can be proved similarly. □

Remark 4.3. Let M be a hyper MV-algebra and I be a non-empty subset of M. Following from our results, I is a hyper MV-filter of M defined in [8] if and only if I^* is a hyper MV-deductive system of M defined in [11]. Meanwhile, the conditions of these two definitions are non-independent. The condition (i) in hyper MV-filters can be implied by condition (ii). On the other hand, hyper MV-ideals defined in [20] are same as hyper MV-ideals defined in [17] and they are hyper MV-deductive systems.

In the end of this section, we present some properties of subalgebras for hyper quasi-MV algebras.

Definition 4.6. Let M be a hyper quasi-MV algebra and S be a non-empty subset of M. If S is a hyper quasi-MV algebra with respect to the hyper operation $\oplus$ and the unary operation * of M, then S is called a *hyper quasi-MV subalgebra* of M.

Proposition 4.15. *Let M be a hyper quasi-MV algebra and S be a non-empty subset of M. Then S is a hyper quasi-MV subalgebra of M if and only if $x^* \in S$ and $x \oplus y \subseteq S$ for any $x, y \in S$.*

Proof. It is similar to the case of hyper MV-algebras in [8]. □

Corollary 4.1. *Let M be a hyper quasi-MV algebra and S be a non-empty subset of M. If S is a hyper quasi-MV subalgebra of M, then $0 \in S$ and $x^* \oplus y \subseteq S$.*

Proposition 4.16. *Let M be a hyper quasi-MV algebra and I be a proper type-2 ideal of M. Then it is not a subalgebra of M.*

Proof. If not, then I is a subalgebra of M, so by Proposition 4.15, we have $x^* \in I$, it turns out that $1 \in x \oplus x^* \subseteq I$, this is a contradiction with I proper. □

5. Homomorphisms

In this section, we introduce the homomorphism, weak homomorphism and dual weak homomorphism of hyper quasi-MV algebras and investigate some properties of ideals and weak ideals under these mappings.

Definition 5.1. Let M_1 and M_2 be hyper quasi-MV algebras.

A mapping $f : M_1 \to M_2$ is called a *homomorphism*, if for any $x, y \in M_1$, we have

(1) $f(x \oplus y) = f(x) \oplus f(y)$;
(2) $f(x^*) = f(x)^*$;
(3) $f(0) = 0$.

A mapping f is called a *weak homomorphism*, if for any $x, y \in M_1$, we have

(1) $f(x \oplus y) \subseteq f(x) \oplus f(y)$;
(2) $f(x^*) = f(x)^*$;
(3) $f(0) = 0$.

A mapping f is called a *dual weak homomorphism*, if for any $x, y \in M_1$, we have

(1) $f(x \oplus y) \supseteq f(x) \oplus f(y)$;
(2) $f(x^*) = f(x)^*$;
(3) $f(0) = 0$.

Obviously, any homomorphism is a (dual) weak homomorphism. Moreover, if f is a (dual) weak homomorphism, then $f(1) = 1$.

Proposition 5.1. *Let M_1 and M_2 be hyper quasi-MV algebras and $f : M_1 \to M_2$ be a homomorphism. Then*

(1) *$f(x \odot y) = f(x) \odot f(y)$;*
(2) *$f(x \vee y) = f(x) \vee f(y)$;*
(3) *$f(x \wedge y) = f(x) \wedge f(y)$.*

The following results can be proved similarly as hyper MV-algebras [20].

Proposition 5.2. *Let M_1 and M_2 be hyper quasi-MV algebras, $f : M_1 \to M_2$ be a weak homomorphism and $g : M_1 \to M_2$ be a dual weak homomorphism. Then the following properties hold:*

(1) *for any $x, y \in M_1$, if $x \ll y$, then $f(x) \ll f(y)$.*

(2) *if A is a subalgebra of M_1, then $g(A)$ is a subalgebra of M_2 and if B is a subalgebra of M_2, then $f^{-1}(B)$ is a subalgebra of M_1.*

(3) *if $h : M_1 \to M_2$ is a homomorphism, $x \ll y$, A and B are subalgebras of M_1 and M_2, respectively, then $h(x) \ll h(y)$, $h(A)$ is a subalgebra of M_2 and $h^{-1}(B)$ is a subalgebra of M_1.*

Proposition 5.3. *Let M_1 and M_2 be hyper quasi-MV algebras, $f : M_1 \to M_2$ be a mapping. Then*

(1) *if M_2 is a quasi-MV algebra and M_1 is a hyper quasi-MV algebra, then f is a weak homomorphism if and only if f is a homomorphism.*

(2) *if M_1 is a quasi-MV algebra and M_2 is a hyper quasi-MV algebra, then f is a dual weak homomorphism if and only if f is a homomorphism.*

(3) *if M_1 and M_2 are quasi-MV algebras, then f is a weak homomorphism (dual weak or a homomorphism) if and only if f is the usual homomorphism of quasi-MV algebras.*

Proposition 5.4. *Let M_1, M_2 and M_3 be hyper quasi-MV algebras and $f : M_1 \to M_2$, $g : M_2 \to M_3$ be weak homomorphisms (dual weak homomorphism) while $g \circ f$ be a dual weak homomorphism (weak homomorphism). Then*

(1) *if f is onto, then g is a homomorphism.*

(2) *if g is one-to-one, then f is a homomorphism.*

Below we will see the properties of ideals and weak ideals under homomorphisms.

Theorem 5.1. *Let M_1 and M_2 be hyper quasi-MV algebras and $f : M_1 \to M_2$ be a homomorphism. Then*

(1) *if I is a type-1 (weak) ideal of M_2, then $f^{-1}(I)$ is a type-1 (weak) ideal of M_1; if I is a type-2 (weak) ideal of M_2, then $f^{-1}(I)$ is a type-2 (weak) ideal of M_1.*

(2) *Denote* $\ker(f) = \{x \in M_1 | f(x) = 0\}$. *then* $\ker(f)$ *is a type-2 weak ideal of M_1.*

(3) *if f is one-to-one, then* $\ker(f) = \{0\}$.

(4) *if f is onto and I is a type-2 (weak) ideal of M_1, then $f(I)$ is a type-2 (weak) ideal of M_2; if f is onto and I is a type-1 (weak) ideal of M_1 which contains* $\ker(f)$, *then $f(I)$ is a type-1 (weak) ideal of M_2.*

Proof. (1) Suppose that $x, y \in f^{-1}(I)$. Then there exist $a, b \in I$ such that $f(x) = a$ and $f(y) = b$, we have $f(x \oplus y) = f(x) \oplus f(y) = a \oplus b \subseteq I$, thus $x \oplus y \subseteq f^{-1}(I)$. Let $x \in f^{-1}(I)$ and $y \in M_1$ such that $y \ll x$. Then $f(y) \ll f(x)$ by Proposition 5.2. Since $f(x) \in I$ and I is a type-1 ideal of M_2, we have $f(y) \in I$, so $y \in f^{-1}(I)$. Hence $f^{-1}(I)$ is a type-1 ideal of M_1. The case of type-1 weak ideal can be proved similarly.

Now, we prove the case of type-2 ideal. Obviously, $0 \in f^{-1}(I)$. Suppose that $x, y \in f^{-1}(I)$. Then $x \oplus y \subseteq f^{-1}(I)$ can be proved as above. Let $x \in f^{-1}(I)$ and $y \in M$. Then by Proposition 5.1, $f(x \odot y) \cap I = (f(x) \odot f(y)) \cap I \neq \emptyset$, so there exists $a \in f(x \odot y) \cap I$, we have $a \in f(x \odot y)$ and $a \in I$, it turns out that there exists $m \in x \odot y$ such that $a = f(m) \in I$, so $m \in f^{-1}(I)$ and $(x \odot y) \cap f^{-1}(I) \neq \emptyset$. The case of type-2 weak ideal can be proved similarly.

(2) By (1) and Proposition 4.10.

(3) Obviously, $\{0\} \subseteq \ker(f)$. For any $x \in \ker(f)$, then $f(x) = 0 = f(0)$. Since f is one-to-one, we have $x = 0$.

(4) Since $0 = f(0)$ and $0 \in I$, we have $0 \in f(I)$. Let $x, y \in f(I)$. Then there exist $a, b \in I$ such that $f(a) = x$ and $f(b) = y$. We have $x \oplus y = f(a) \oplus f(b) = f(a \oplus b)$. Because $a \oplus b \subseteq I$, it follows that $x \oplus y \subseteq f(I)$. Let $x \in f(I)$ and

$y \in M_2$. Then there exist $a \in I$ and $b \in M_1$ such that $x = f(a)$ and $y = f(b)$. We have $(x \odot y) \cap f(I) = (f(a) \odot f(b)) \cap f(I) = f((a \odot b) \cap I)$. Since $(a \odot b) \cap I \neq \emptyset$, it turns out that $(x \odot y) \cap f(I) \neq \emptyset$. The case of type-2 weak ideal can be proved similarly.

Now, let $x \in f(I)$ and $y \in M_2$ such that $y \ll x$. Then there exist $a \in I$ and $b \in M_1$ such that $f(a) = x$, $f(b) = y$ and $f(b) \ll f(a)$, it turns out that $0 \in f(b) \odot f(a)^* = f(b \odot a^*)$, so there exist $m \in b \odot a^*$ such that $0 = f(m)$, we have $m \in \ker(f) \subseteq I$, thus $b \odot a^* \ll I$. Since I is type-1 ideal of M_1, we get $b \in I$ and so $y \in f(I)$. The case of type-1 weak ideal can be proved similarly. □

Definition 5.2. Let M be a hyper quasi-MV algebra and I be a proper (weak) ideal of M. Then I is called *maximal*, if $I \subseteq J \subseteq M$ for some (weak) ideal J of M, then $J = I$ or $J = M$.

Theorem 5.2. *Let M_1 and M_2 be hyper quasi-MV algebras and $f : M_1 \to M_2$ be an epimorphism. Then*

(1) *if I is a maximal type-1 (weak) ideal of M_2, then $f^{-1}(I)$ is a maximal type-1 (weak) ideal of M_1 which contains $\ker(f)$.*

(2) *if I is a maximal type-1 (weak) ideal of M_1 which contains $\ker(f)$, then $f(I)$ is a maximal type-1 (weak) ideal of M_2.*

(3) *the mapping $I \mapsto f(I)$ is bijective corresponding between the maximal type-1 (weak) ideals of M_1 containing $\ker(f)$ and maximal type-1 (weak) ideals of M_2.*

Proof. (1) By Theorem 5.1 (1), we have that $f^{-1}(I)$ is a type-1 (weak) ideal of M_1. Now, let J be a type-1 (weak) ideal of M_1 such that $f^{-1}(I) \subsetneqq J \subseteq M_1$. Then $I \subsetneqq f(J)$. Since I is a maximal type-1 (weak) ideal of M_2, we have $f(J) = M_2$, so $J = M_1$. If not, there exists $x \in M_1 \setminus J$ and $f(x) \in M_2$, so there exists $a \in J$ such that $f(a) = f(x)$, we have $0 \in f(x) \odot f(a)^* = f(x \odot a^*)$, so there is $m \in x \odot a^*$ such that $0 = f(m)$, it turns out that $m \in \ker(f) \subseteq f^{-1}(I) \subsetneqq J$, thus $x \odot a^* \ll J$ and then $x \in J$, this is a contradiction. Hence $f^{-1}(I)$ which contains $\ker(f)$ is a maximal type-1 (weak) ideal of M_1.

(2) By Theorem 5.1 (4), we have that $f(I)$ is a type-1 (weak) ideal of M_2. Suppose that J be a type-1 (weak) ideal of M_2 such that $f(I) \subsetneqq J \subseteq M_2 = f(M_1)$. Then $J = f(J')$ and $I \subsetneqq J' \subseteq M_1$ where J' is a type-1 (weak) ideal of M_1. Since I is a maximal ideal of M_1, we have $J' = M_1$, so $J = M_2$.

(3) Based on (1) and (2). □

6. Conclusion

In this paper, we introduce hyper quasi-MV algebras and mainly study ideals and weak ideals of hyper quasi-MV algebras. In the future work, on the one hand, we will focus on the quotient algebra of hyper quasi-MV algebra. On the other hand, the non-commutative generalization of quasi-MV algebras had been introduced in [4, 5], it is natural to study the corresponding hyper structure as the generalization of hyper quasi-MV algebras and pseudo-quasi-MV algebras.

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