STUDY OF EXTENDED WEYL k-FRACTIONAL INTEGRAL VIA CHEBYSHEV INEQUALITIES

M. K. Azam
University of Management and Technology (UMT)
Lahore Pakistan
mkhurshidaazam@gmail.com

Fiza Zafar
CASPAM
B. Z. University
Multan, Pakistan
fizazafar@gmail.com

M. A. Rehman
University of Management and Technology (UMT)
Lahore Pakistan
aziz.rehman@umt.edu.pk

F. Ahmad*
Dean Faculty of Sciences
Punjab Higher Education Department
Govt. Islamia College
Civil Lines
Lahore
farooqgujar@gmail.com

Shahid Qaisar
COMSATS Institute of Information Technology Sahiwal
Pakistan
shahidqaisar90@ciitsahiwal.edu.pk

Abstract. In this paper, we introduce the extended Weyl k-fractional integral and then present its results and some inequalities. These results and inequalities hold true for: (i) k-Weyl fractional integral when $s \to 0$; (ii) Weyl fractional integral when $s \to 0$ and $k \to 1$.
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1. Introduction

Considerable work has been done in recent years on fractional integrals because of their applications in many fields of science and technology. The fractional integrals are powerful tools in applied mathematics to solve many problems

∗. Corresponding author

from different fields of science and engineering. By the help of fractional integrals many useful results are found in mathematical physics, thermodynamics, control theory, hydrology, astrophysics and bioengineering.

The objective and motivation of this work is to find the generalized results of the Weyl fractional integrals and to highlight their applications. Many authors had studied fractional integrals [1-21].

Diaz and Pariguan [7] paved the way for the extensions in fractional calculus by introducing the k-gamma and k-beta functions. Now, many new results are introduced which are based on these functions. They defined $k-$gamma function as

$$\Gamma_k(u) = \int_0^\infty e^{-\frac{x^k}{k}} x^{u-1} dx = k^{k-1}\Gamma(\frac{u}{k}), \mathrm{Re}(u) > 0, \tag{1}$$

They defined the k-beta function as

$$B_k(u,v) = \frac{1}{k}\int_0^1 z^{\frac{u}{k}-1}(1-z)^{\frac{v}{k}-1} dz, \mathrm{Re}(u) > 0, \mathrm{Re}(v) > 0, \tag{2}$$

From (1) and (2), we have

$$\Gamma(u) = \lim_{k\to 1}\Gamma_k(u),\ \Gamma_k(k) = 1,\ \Gamma_k(u+k) = u\Gamma_k(u). \tag{3}$$

and

$$B_k(c,d) = \frac{\Gamma_k(c)\Gamma_k(d)}{\Gamma_k(c+d)} = \frac{1}{k}B(\frac{c}{k},\frac{d}{k}). \tag{4}$$

Romero and Luque [15] have used the idea of Diaz and Pariguan to define the $k-$Weyl fractional integral as

$$W_k{}^{\alpha}[f(x)] = \frac{1}{k\Gamma_k(\alpha)}\int_x^\infty (t-x)^{\frac{\alpha}{k}-1} f(t)dt,\ t > 0. \tag{5}$$

Here, we introduce the extended weyl $k-$fractional integral [$(k,s)-$ weyl fractional integral] of order α. Let f be continuous on $[0,\infty)$ and $\alpha \in (0,1), k \in (0,\infty), s > -1$. Then $\forall\, x \geqslant 0$, the extended $k-$weyl fractional integral is defined by

$${}^s_kW^{\alpha}[f(x)] \ = \ \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_x^\infty (t^{s+1}-x^{s+1})^{\frac{\alpha}{k}-1} t^s f(t)dt,\ t > 0. \tag{6}$$

Note: (i) For $s \to 0$, k -Weyl fractional integral is recaptured.

(ii) For $s \to 0$ and $k \to 1$, usual Weyl fractional integral is recaptured.

For a real scalar function g , the Mellin transform is dened by

$$g^*(s_1) = M[g(x)] = \int_0^\infty x^{s_1-1} g(x)dx, \mathrm{Re}(s_1) > 0. \tag{7}$$

whenever the integral exists and s_1 is parameter.

The introduction of the extended Weyl k-fractional integral and the study of its properties and results will help many researchers in solving the differential as well as integral problems. These results will be very useful in solving the problems as existing in the paper texture enhancement for medical images based on fractional differential masks [12].

2. Main results and discussion

Theorem 2.1 (semi group property and commutative law)**.** *Let f be continuous on $[0,\infty)$ and $\alpha,\beta\in(0,1), k\in(0,\infty), s>-1$. Then* $\forall\, x\geqslant 0$

$$ {}^{s}_{k}W^{\alpha}\left[{}^{s}_{k}W^{\beta}f(x)\right] = {}^{s}_{k}W^{\alpha+\beta}f(x) = {}^{s}_{k}W^{\beta}\left[{}^{s}_{k}W^{\alpha}f(x)\right]. \tag{8}$$

Proof. Using the result (6) in the LHS of the equation (8), we have

$$\begin{aligned} {}^{s}_{k}W^{\alpha}\left[{}^{s}_{k}W^{\beta}f(x)\right] &= \frac{(s+1)^{2-\frac{\alpha}{k}-\frac{\beta}{k}}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_x^\infty (t^{s+1}-x^{s+1})^{\frac{\alpha}{k}-1}t^s \\ &\quad \cdot [\int_t^\infty (u^{s+1}-t^{s+1})^{\frac{\beta}{k}-1}u^s f(u)du]dt, \end{aligned} \tag{9}$$

Using Fubini's theorem, we get

$$\begin{aligned} {}^{s}_{k}W^{\alpha}\left[{}^{s}_{k}W^{\beta}f(x)\right] &= \frac{(s+1)^{2-\frac{\alpha}{k}-\frac{\beta}{k}}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_x^\infty u^s f(u) \\ &\quad \cdot [\int_x^u (t^{s+1}-x^{s+1})^{\frac{\alpha}{k}-1}(u^{s+1}-t^{s+1})^{\frac{\beta}{k}-1}t^s dt]du, \end{aligned} \tag{10}$$

Substituting $z=\frac{u^{s+1}-t^{s+1}}{u^{s+1}-x^{s+1}}$

$$\begin{aligned} {}^{s}_{k}W^{\alpha}\left[{}^{s}_{k}W^{\beta}f(x)\right] &= \frac{(s+1)^{1-\frac{\alpha}{k}-\frac{\beta}{k}}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_x^\infty (u^{s+1}-x^{s+1})^{\frac{\alpha+\beta}{k}-1}u^s f(u) \\ &\quad \cdot \left[\int_0^1 (1-z)^{\frac{\alpha}{k}-1}(z)^{\frac{\beta}{k}-1}dz\right]du, \end{aligned} \tag{11}$$

Using (2), (4) and (6), we get

$$ {}^{s}_{k}W^{\alpha}\left[{}^{s}_{k}W^{\beta}f(x)\right] = {}^{s}_{k}W^{\alpha+\beta}f(x), \tag{12}$$

Similarly, we can prove

$$ {}^{s}_{k}W^{\beta}\left[{}^{s}_{k}W^{\alpha}f(x)\right] = {}^{s}_{k}W^{\alpha+\beta}f(x)$$

and hence we get (8). □

Theorem 2.2. *Let* $\alpha \in (0,1)$, $\beta, k \in (0,\infty)$, $s > -1$. *Then* $\forall\, x \geqslant 0$,

$$ {}_k^sW^{\alpha}[(x^{s+1})^{-\frac{\beta}{k}}] = \frac{\Gamma_k(\beta-\alpha)(x^{s+1})^{\frac{\alpha}{k}-\frac{\beta}{k}}}{(s+1)^{\frac{\alpha}{k}}\Gamma_k(\beta)}. \tag{14} $$

Proof. Using the result (6) in the LHS of the equation (14), we have

$$ {}_k^sW^{\alpha}[(x^{s+1})^{-\frac{\beta}{k}}] = \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_x^{\infty} (t^{s+1}-x^{s+1})^{\frac{\alpha}{k}-1} t^s[(t^{s+1})^{-\frac{\beta}{k}}]dt, \tag{15} $$

Substituting $y = \frac{x^{s+1}}{t^{s+1}}$

$$ {}_k^sW^{\alpha}[(x^{s+1})^{-\frac{\beta}{k}}] = \frac{(s+1)^{-\frac{\alpha}{k}}(x^{s+1})^{\frac{\alpha}{k}-\frac{\beta}{k}}}{k\Gamma_k(\alpha)} \left[\int_0^1 (1-y)^{\frac{\alpha}{k}-1}(y)^{\frac{(\beta-\alpha)}{k}-1}dy\right]. \tag{16} $$

Using the result (2), we get (14). □

Theorem 2.3. *Let* $\alpha \in (0,1)$, $\mu, k \in (0,\infty)$, $s > -1$. *Then* $\forall\, x \geqslant 0$,

$$ {}_k^sW^{\alpha}(e^{-\mu x^{s+1}}) = \frac{e^{-\mu x^{s+1}}}{[\mu k(s+1)]^{\frac{\alpha}{k}}}. \tag{17} $$

Proof. Using the result (6) in the LHS of the equation (17), we have

$$ {}_k^sW^{\alpha}(e^{-\mu x^{s+1}}) = \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_x^{\infty} (t^{s+1}-x^{s+1})^{\frac{\alpha}{k}-1} t^s[e^{-\mu t^{s+1}}]dt, \tag{18} $$

Substituting $t^{s+1} - x^{s+1} = z^{s+1}$

$$ {}_k^sW^{\alpha}(e^{-\mu x^{s+1}}) = \frac{(s+1)^{1-\frac{\alpha}{k}}e^{-\mu x^{s+1}}}{k\Gamma_k(\alpha)} \int_0^{\infty} (z^{s+1})^{\frac{\alpha}{k}-1} z^s\, e^{-\mu z^{s+1}}dz. \tag{19} $$

Substituting $\mu z^{s+1} = y$

$$ {}_k^sW^{\alpha}(e^{-\mu x^{s+1}}) = \frac{(s+1)^{-\frac{\alpha}{k}}e^{-\mu x^{s+1}}}{\mu^{\frac{\alpha}{k}}k\Gamma_k(\alpha)} \int_0^{\infty} (y)^{\frac{\alpha}{k}-1}e^{-y}dy. \tag{20} $$

Using (1), we get (17). □

Theorem 2.4. *Let* h, f *be continuous on* $[0,\infty)$ *and let* $\alpha \in (0,1), k \in (0,\infty), s > -1$. *Then* $\forall\, x \geqslant 0$,

$$ {}_k^sW^{\alpha}\,[h(x)] = {}_k^sW^{\alpha}\,[f(x)] \;\Rightarrow\; h = f. \tag{21} $$

Proof. Using (6) and linearity of integrals, we get

$$\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_x^\infty (t^{s+1}-x^{s+1})^{\frac{\alpha}{k}-1}t^s[h(t)-f(t)]dt = 0,\ t>0. \tag{22}$$

By using theorem 2.12.5 of [3], we get

$$h(t)-f(t)=0 \ \Rightarrow \ h=f. \tag{23}$$

□

Corollary 2.5. *Let* $\alpha\in(0,1)$, $\lambda,\mu,\beta,k\in(0,\infty)$, $s>-1$. *Then* $\forall\, x\geqslant 0$,

$$ {}_k^sW^\alpha[\mu(x+a)^\beta+\lambda(x^{s+1})^{-\frac{\beta}{k}}] = \mu{}_k^sW^\alpha(x+a)^\beta+\lambda{}_k^sW^\alpha(x^{s+1})^{-\frac{\beta}{k}}. \tag{24}$$

Theorem 2.6. *Let* g *be continuous on* $[0,\infty)$ *and let* $\alpha\in(0,1)\,,k\in(0,\infty)$, $s>-1$. *Then* $\forall\, x\geqslant 0$,

$$M\left[{}^sW_k^\alpha(x^{s+1})^{-\frac{\alpha}{k}}g(x)\right]=(s+1)^{-\frac{\alpha}{k}}\frac{\Gamma_k\left(\frac{s_1k}{(s+1)}\right)}{\Gamma_k\left(\frac{s_1k}{(s+1)}+\alpha\right)}g^*\left(s_1\right),\ \mathrm{Re}(s_1)>0. \tag{25}$$

Proof. Using the results (6) and (7)

$$\begin{aligned} M\left[{}^sW_k^\alpha(x^{s+1})^{-\frac{\alpha}{k}}g(x)\right] &= \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_0^\infty x^{s_1-1} \\ &\quad\cdot[\int_x^\infty (t^{s+1}-x^{s+1})^{\frac{\alpha}{k}-1}(t^{s+1})^{-\frac{\alpha}{k}}t^s g(t)dt]dx. \end{aligned} \tag{26}$$

By Fubini's theorem

$$\begin{aligned} M\left[{}^sW_k^\alpha(x^{s+1})^{-\frac{\alpha}{k}}g(x)\right] &= \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_0^\infty (t^{s+1})^{-\frac{\alpha}{k}}g(t) \\ &\quad\cdot[\int_0^t (t^{s+1}-x^{s+1})^{\frac{\alpha}{k}-1}x^{s_1-1}t^s dx]dt. \end{aligned} \tag{27}$$

Substituting $x^{s+1}=t^{s+1}y$

$$\begin{aligned} M\left[{}^sW_k^\alpha(x^{s+1})^{-\frac{\alpha}{k}}g(x)\right] &= \frac{(s+1)^{-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_0^\infty t^{s_1-1}g(t) \\ &\quad\cdot[\int_0^1 (1-y)^{\frac{\alpha}{k}-1}y^{\frac{s_1k}{(s+1)k}-1}dy]dt. \end{aligned} \tag{28}$$

(2) and (6) leads to (25). □

Example 2.7. Let $\alpha \in (0,1), k \in (0,\infty), s > -1$. Then $\forall\, x \geqslant 0$, using theorem 2.6, we can get

$$M\left[{}^{s}W_k^{\alpha}(x^{s+1})^{-\frac{\alpha}{k}}(e^{-x})\right] = (s+1)^{-\frac{\alpha}{k}}\frac{\Gamma_k\left(\frac{s_1 k}{(s+1)}\right)}{\Gamma_k\left(\frac{s_1 k}{(s+1)}+\alpha\right)}\Gamma(s_1),\ \mathrm{Re}(s_1) > 0. \tag{29}$$

Theorem 2.8. *Let f be continuous on $[0,\infty)$ and $\alpha \in (0,1), k \in (0,\infty), s > -1$. Then $\forall\, x \geqslant 0$,*

$$M\left[{}^{s}W_k^{\alpha}f(x)\right] = (s+1)^{-\frac{\alpha}{k}}\frac{\Gamma_k\left(\frac{s_1 k}{(s+1)}\right)}{\Gamma_k\left(\frac{s_1 k}{(s+1)}+\alpha\right)}f^*\left(s_1+\frac{(s+1)\alpha}{k}\right),\ \mathrm{Re}(s_1)>0. \tag{30}$$

Proof. Using the results (6) and (7)

$$M\left[{}^{s}W_k^{\alpha}f(x)\right] = \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_0^\infty x^{s_1-1}[\int_x^\infty (t^{s+1}-x^{s+1})^{\frac{\alpha}{k}-1}t^s f(t)dt]dx. \tag{31}$$

By Fubini's theorem

$$M\left[{}^{s}W_k^{\alpha}f(x)\right] = \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_0^\infty f(t)[\int_0^t (t^{s+1}-x^{s+1})^{\frac{\alpha}{k}-1}x^{s_1-1}t^s dx]dt. \tag{32}$$

Substituting $x^{s+1} = t^{s+1}y$

$$\begin{aligned} M\left[{}^{s}W_k^{\alpha}f(x)\right] &= \frac{(s+1)^{-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_0^\infty t^{\frac{ks_1+(s+1)\alpha}{k}-1}f(t) \\ &\cdot [\int_0^1 (1-y)^{\frac{\alpha}{k}-1}y^{\frac{s_1 k}{(s+1)k}-1}dy]dt. \end{aligned} \tag{33}$$

(2) and (7) leads to (30). □

Example 2.9. Let $\alpha \in (0,1), k \in (0,\infty), s > -1$. Then $\forall\, x \geqslant 0$, using theorem 2.8, we can get

$$M\left[{}^{s}W_k^{\alpha}\ln x\, f(x)\right] = (s+1)^{-\frac{\alpha}{k}}\frac{\Gamma_k\left(\frac{s_1 k}{(s+1)}\right)}{\Gamma_k\left(\frac{s_1 k}{(s+1)}+\alpha\right)}\frac{d}{ds}\left[f^*(s_1+\frac{(s+1)\alpha}{k})\right], \tag{34}$$

$\mathrm{Re}(s_1) > 0$.

Some inequalities of extended $k-$Weyl fractional integral:

Theorem 2.10. *Let h, f are two synchronous on $[0,\infty)$ and $\alpha,\beta \in (0,1), k > 0, s > -1$. Then $\forall\ t \geqslant 0$, we have*

$$ {}^{s}_{k}W^{\alpha}(1)\,{}^{s}_{k}W^{\alpha}(hf(t)) \geq {}^{s}_{k}W^{\alpha}(h(t))\,{}^{s}_{k}W^{\alpha}(f(t)). \tag{35}$$

$$\begin{aligned} &{}^{s}_{k}W^{\alpha}(hf(t))\,{}^{s}_{k}W^{\beta}(1) + {}^{s}_{k}W^{\beta}(hf(t))\,{}^{s}_{k}W^{\alpha}(1) \geq \\ &{}^{s}_{k}W^{\alpha}(h(t))\,{}^{s}_{k}W^{\beta}(f(t)) + {}^{s}_{k}W^{\beta}(h(t))\,{}^{s}_{k}W^{\alpha}(f(t)). \end{aligned} \tag{36}$$

Proof. Since h, f are two synchronous on $[0,\infty)$, then $\forall\, v, w \geqslant 0$, we have

$$[h(v)-h(w)][f(v)-f(w)] \geq 0. \tag{37}$$

Therefore,

$$h(v)f(v)+h(w)f(w) \geq h(v)f(w)+h(w)f(v). \tag{38}$$

Multiplying this by $\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}(v^{s+1}-t^{s+1})^{\frac{\alpha}{k}-1}v^s$ and taking integral w.r.t. v over (t,∞), we get

$$\begin{aligned}
&\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_t^\infty (v^{s+1}-t^{s+1})^{\frac{\alpha}{k}-1}v^s h(v)f(v)dv\\
&+\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_t^\infty (v^{s+1}-t^{s+1})^{\frac{\alpha}{k}-1}v^s h(w)f(w)dv\\
&\geq \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_t^\infty (v^{s+1}-t^{s+1})^{\frac{\alpha}{k}-1}v^s h(v)f(w)dv\\
&+\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_t^\infty (v^{s+1}-t^{s+1})^{\frac{\alpha}{k}-1}v^s h(w)f(v)dv.
\end{aligned} \tag{39}$$

Using (6), we get

$${}^s_kW^\alpha(hf(t))+h(w)f(w){}^s_kW^\alpha(1) \geq f(w){}^s_kW^\alpha(h(t))+h(w){}^s_kW^\alpha(f(t)). \tag{40}$$

Multiplying this by $\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}(w^{s+1}-t^{s+1})^{\frac{\alpha}{k}-1}w^s$ and taking integral w.r.t. w over (t,∞), we get

$$\begin{aligned}
&\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\Big[\int_t^\infty (w^{s+1}-t^{s+1})^{\frac{\alpha}{k}-1}w^s\, {}^s_kW^\alpha(hf(t))\,dw\\
&+\int_t^\infty (w^{s+1}-t^{s+1})^{\frac{\alpha}{k}-1}w^s h(w)f(w){}^s_kW^\alpha(1)\,dw\\
&\geq \int_t^\infty (w^{s+1}-t^{s+1})^{\frac{\alpha}{k}-1}w^s f(w)\,{}^s_kW^\alpha(h(t))\,dw\\
&+\int_t^\infty (w^{s+1}-t^{s+1})^{\frac{\alpha}{k}-1}w^s h(w){}^s_kW^\alpha(f(t))\,dw\Big].
\end{aligned} \tag{41}$$

Using (6) and simplifying we obtained (35).

Again multiplying (40) by $\frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)}(w^{s+1}-t^{s+1})^{\frac{\beta}{k}-1}w^s$ and taking integral w.r.t. w over (t,∞), then using (6) and simplifying, we obtained (36) . □

Theorem 2.11. *Let h, f are two synchronous on $[0,\infty)$, $g \geqslant 0$ and $\alpha, \beta \in (0,1), k > 0, s > -1$. Then $\forall\ z \geqslant 0$, we have*

$$\begin{aligned} &{}_k^sW^{\alpha}hfg(z){}_k^sW^{\beta}(1) + {}_k^sW^{\alpha}(1){}_k^sW^{\beta}hfg(z) \\ &\geq {}_k^sW^{\alpha}hg(z){}_k^sW^{\beta}f(z) + {}_k^sW^{\alpha}fg(z){}_k^sW^{\beta}h(z) \\ &- {}_k^sW^{\alpha}g(z){}_k^sW^{\beta}hf(z) - {}_k^sW^{\alpha}hf(z){}_k^sW^{\beta}g(z) \\ &+ {}_k^sW^{\alpha}h(z){}_k^sW^{\beta}fg(z) + {}_k^sW^{\alpha}f(z){}_k^sW^{\beta}hg(z). \end{aligned} \tag{42}$$

Proof. Since h, f are synchronous on $[0,\infty)$, then $\forall\ r, t \geqslant 0$,

$$[h(r) - h(t)][f(r) - f(t)][g(r) + g(t)] \geq 0. \tag{43}$$

Multiplying and arranging, we get

$$\begin{aligned} &h(r)f(r)g(r) + h(t)f(t)g(t) \geq h(r)f(t)g(r) + h(t)f(r)g(r) \\ &-h(t)f(t)g(r) - h(r)f(r)g(t) + h(r)f(t)g(t) + h(t)f(r)g(t). \end{aligned} \tag{44}$$

Multiplying by $\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}(r^{s+1} - z^{s+1})^{\frac{\alpha}{k}-1}r^s$ and taking integral w.r.t. r over (z,∞)

$$\begin{aligned} &\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}[\int_z^{\infty} (r^{s+1} - z^{s+1})^{\frac{\alpha}{k}-1} r^s h(r)f(r)g(r)dr \\ &+ h(t)f(t)g(t) \int_z^{\infty} (r^{s+1} - z^{s+1})^{\frac{\alpha}{k}-1} r^s dr \\ &\geq f(t) \int_z^{\infty} (r^{s+1} - z^{s+1})^{\frac{\alpha}{k}-1} r^s h(r)g(r)dr \\ &+ h(t) \int_z^{\infty} (r^{s+1} - z^{s+1})^{\frac{\alpha}{k}-1} r^s f(r)g(r)dr \\ &- h(t)f(t) \int_z^{\infty} (r^{s+1} - z^{s+1})^{\frac{\alpha}{k}-1} r^s g(r)dr \\ &- g(t) \int_z^{\infty} (r^{s+1} - z^{s+1})^{\frac{\alpha}{k}-1} r^s h(r)f(r)dr \\ &+ f(t)g(t) \int_z^{\infty} (r^{s+1} - z^{s+1})^{\frac{\alpha}{k}-1} r^s h(r)dr \\ &+ h(t)g(t) \int_z^{\infty} (r^{s+1} - z^{s+1})^{\frac{\alpha}{k}-1} r^s f(r)dr]. \end{aligned} \tag{45}$$

Using (6), we get

$$\begin{aligned} &{}_k^sW^{\alpha}hfg(z) + h(t)f(t)g(t){}_k^sW^{\alpha}(1) \\ &\geq f(t){}_k^sW^{\alpha}hg(z) + h(t){}_k^sW^{\alpha}fg(z) \\ &- h(t)f(t){}_k^sW^{\alpha}g(z) - g(t){}_k^sW^{\alpha}hf(z) \\ &+ f(t)g(t){}_k^sW^{\alpha}h(z) + h(t)g(t){}_k^sW^{\alpha}f(z). \end{aligned} \tag{46}$$

Multiplying by $\frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)}(t^{s+1}-z^{s+1})^{\frac{\beta}{k}-1}t^s$ and taking integral w.r.t. t over (z,∞), then using (6), we get the required result. □

Corollary 2.12. *Let h,f are two synchronous on $[0,\infty)$, $g \geqslant 0$ and $\alpha \in (0,1), k>0, s>-1$. Then $\forall\ t \geqslant 0$, we have*

$$\begin{aligned} &{}^s_kW^{\alpha} fgh(t){}^s_kW^{\alpha}(1) \geq {}^s_kW^{\alpha} fh(t){}^s_kW^{\alpha} g(t) \\ &+{}^s_kW^{\alpha} gh(t){}^s_kW^{\alpha} f(t) - {}^s_kW^{\alpha} h(t){}^s_kW^{\alpha} fg(t). \end{aligned} \tag{47}$$

Theorem 2.13. *Let h,f are two synchronous on $(0,\infty)$, $g \geqslant 0$ and $\alpha,\beta \in (0,1), k>0, s>-1$. Then $\forall\ t \geqslant 0$, we have*

$$\begin{aligned} &{}^s_kW^{\alpha} fgh(t){}^s_kW^{\beta}(1) - {}^s_kW^{\alpha}(1){}^s_kW^{\beta} fgh(t) \\ &\geq {}^s_kW^{\alpha} fh(t){}^s_kW^{\beta} g(t) + {}^s_kW^{\alpha} gh(t){}^s_kW^{\beta} f(t) \\ &- {}^s_kW^{\alpha} h(t){}^s_kW^{\beta} fg(t) + {}^s_kW^{\alpha} fg(t){}^s_kW^{\beta} h(t) \\ &- {}^s_kW^{\alpha} f(t){}^s_kW^{\beta} gh(t) - {}^s_kW^{\alpha} g(t){}^s_kW^{\beta} fh(t). \end{aligned} \tag{48}$$

Proof. This can be proved by using the steps as done in Theorem 2.11. □

Theorem 2.14. *Let h,f are two synchronous on $[0,\infty)$ and $\alpha,\beta \in (0,1), k>0, s>-1$. Then $\forall\ z \geqslant 0$, we have*

$${}^s_kW^{\alpha}\left(h^2(z)\right){}^s_kW^{\beta}(1) + {}^s_kW^{\beta}\left(f^2(z)\right){}^s_kW^{\alpha}(1) \geq 2{}^s_kW^{\alpha}(h(z)){}^s_kW^{\beta}(f(z)). \tag{49}$$

and

$${}^s_kW^{\alpha} h^2(z){}^s_kW^{\beta} f^2(z) + {}^s_kW^{\beta} h^2(z){}^s_kW^{\alpha} f^2(z) \geq 2{}^s_kW^{\alpha} hf(z){}^s_kW^{\beta} hf(z). \tag{50}$$

Proof. Since h,f are synchronous on $(0,\infty)$ then $\forall\, r,t \geqslant 0$,,

$$[h(r)-f(t)]^2 \geq 0. \tag{51}$$

Therefore,

$$h^2(r)+f^2(t) \geq 2h(r)f(t). \tag{52}$$

Multiply by $\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}(r^{s+1}-z^{s+1})^{\frac{\alpha}{k}-1}r^s$ and taking integral w.r.t. r over (z,∞)

$$\begin{aligned} &\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_z^{\infty}(r^{s+1}-z^{s+1})^{\frac{\alpha}{k}-1}r^s h^2(r)dr \\ &+\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_z^{\infty}(r^{s+1}-z^{s+1})^{\frac{\alpha}{k}-1}r^s f^2(t)dr \\ &\geq \frac{2(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_z^{\infty}(r^{s+1}-z^{s+1})^{\frac{\alpha}{k}-1}r^s h(r)f(t)dr. \end{aligned} \tag{53}$$

Using (6), we get

$$ {}_k^sW^\alpha\left(h^2(z)\right)+f^2(t){}_k^sW^\alpha(1)\geq 2f(t){}_k^sW^\alpha(h(z)). \tag{54} $$

Multiply by $\frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)}(t^{s+1}-z^{s+1})^{\frac{\beta}{k}-1}t^s$ and taking integral w.r.t. t over (z,∞)

$$ \begin{aligned} &\frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_z^\infty (t^{s+1}-z^{s+1})^{\frac{\beta}{k}-1}t^s\,{}_k^sW^\alpha\left(h^2(z)\right)dt\\ &+\frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_z^\infty (t^{s+1}-z^{s+1})^{\frac{\beta}{k}-1}t^sf^2(t){}_k^sW^\alpha(1)\,dt\\ &\geq\frac{2(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_z^\infty (t^{s+1}-z^{s+1})^{\frac{\beta}{k}-1}t^s\,f(t){}_k^sW^\alpha(h(z))\,dt. \end{aligned} \tag{55} $$

Using the result (6), we get (49).

For (50), we take

$$ [h(r)f(t)-h(t)f(r)]^2\geq 0. \tag{56} $$

and get result as done in (49). □

Corollary 2.15. *Let h,f are two synchronous on $[0,\infty)$ and $\alpha,\beta\in(0,1),k>0,s>-1$. Then $\forall\ y\geqslant 0$, we have*

$$ {}_k^sW^\alpha h(1)[{}_k^sW^\alpha h^2(y)+{}_k^sW^\alpha f^2(y)]\geq 2{}_k^sW^\alpha h(y){}_k^sW^\alpha f(y), \tag{57} $$

and

$$ {}_k^sW^\alpha h^2(y){}_k^sW^\alpha f^2(y)\geq[{}_k^sW^\alpha hf(y)]^2. \tag{58} $$

Theorem 2.16. *Let $g:(-\infty,\infty)\to(-\infty,\infty)$ and we also have $\overline{g}(x)=\int_x^\infty g(u)u^sdu$, Then $\alpha\in(0,1),k>0,s>-1$,*

$$ {}_k^sW^\alpha\overline{g}(x)=k{}_k^sW^{\alpha+k}g(x). \tag{59} $$

Proof. Using the result (6) in the LHS of the equation (59), we have

$$ {}_k^sW^\alpha\overline{g}(x)=\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_x^\infty (t^{s+1}-x^{s+1})^{\frac{\alpha}{k}-1}t^s[\int_x^\infty g(u)u^sdu]dt. \tag{60} $$

Substituting the value of $\overline{g}(x)$

$$ {}_k^sW^\alpha\overline{g}(x)=\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_x^\infty (t^{s+1}-x^{s+1})^{\frac{\alpha}{k}-1}t^s[\int_x^\infty g(u)u^sdu]dt. \tag{61} $$

By Fubini's theorem

$$ {}_k^sW^\alpha\overline{g}(x)=\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_x^\infty g(u)u^s[\int_x^u (t^{s+1}-x^{s+1})^{\frac{\alpha}{k}-1}t^sdt]du. \tag{62} $$

Integrating and using (6), we get (59). □

Example 2.17. Let $\alpha \in (0,1), k \in (0,\infty), s > -1$. Then $\forall\, x \geqslant 0$ and $f(x) = e^{-\mu x^{s+1}}$, using the theorem 2.16, we can get

$$ {}^{s}_{k}W^{\alpha}\overline{f}(x) = \frac{ke^{-\mu x^{s+1}}}{[\mu k(s+1)]^{\frac{\alpha}{k}+1}}. \tag{63} $$

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