AN EQUIVALENT DEFINITION OF A $\mathcal{C}$-GROUP

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Abstract. In 1974, Fattahi [2] classified finite non-nilpotent groups in which every subgroup is either normal or abnormal. Surely, it is an interesting topic to define and classify a bigger class of groups containing those classified by Fattahi. In 2012, the first author etc defined a $\mathcal{C}$-group. A finite group G is said to be a $\mathcal{C}$-group if for each divisor d of the order of G, G always contains a subgroup H of order d such that H is either normal or abnormal in G. The class of $\mathcal{C}$-groups is really a class of groups bigger than those classified by Fattahi. During the first author etc investigate the structure of of $\mathcal{C}$-groups, they found that some property cannot hold if 'normal' is changed into 'subnormal'. So the authors of this paper is motivated to find an equivalent definition of a $\mathcal{C}$-group, which can be described by 'subnormal' and 'abnormal'. A $\mathcal{C}_1$-group is defined , some good properties of a $\mathcal{C}_1$-group are given, then it is proved that a $\mathcal{C}_1$-group is equivalent to a $\mathcal{C}$-group. At last, as an example of effectiveness of a $\mathcal{C}_1$-group, the necessary and sufficient conditions of a semi-product of a p-group and a p'-group to be a $\mathcal{C}$-group is given.
Keywords: normal, abnormal, subnormal, a $\mathcal{C}$-group, a $\mathcal{C}_1$-group.

1. Introduction

All groups considered in this paper are finite. Our notation and terminology are standard, see, for example, Robinson [6].

In 1974, Fattahi [2] classified finite non-nilpotent groups in which every subgroup is either normal or abnormal. Surely, it is an interesting topic to define and classify a bigger class of groups containing those classified by Fattahi. In 2012, Liu, Li and He [5] was motivated by this and introduced the following definition: A group G is called a $\mathcal{C}$-group if for each divisor d of the order of G, G contains a subgroup H of order d such that H is either normal or abnormal in G. Of course, a finite group with every subgroup is either normal or abnormal is a $\mathcal{C}$-group. So the class of $\mathcal{C}$-groups is really a class of groups bigger than those classified by Fattahi. In [5], it is investigated the structure of finite groups

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based on the assumption that some subgroups are $\mathcal{C}$-groups and the following equivalence is proved:

Theorem 1.1 ([5, Corollary 3.2]). *The following statements for a group G are equivalent:*

(1) *G is a $\mathcal{C}$-group.*

(2) *For each subgroup B of G, there exists a subgroup H of G of order $|B|$ in G such that H is either normal or abnormal in G.*

Notice that the above theorem doesn't hold if "normal" is replaced by "subnormal" in Statement (2). For example, the alternating group of degree 4 is a counterexample. This motivates us to investigate if there exists a kind of equivalent definition of $\mathcal{C}$-group, which is described just by using "subnormal" and "abnormal". To reach this target, we give the following definition:

Definition 1.2. *A group G is called a $\mathcal{C}_1$-group if for each subgroup B of a group G, there exists a subgroup H of order $|G : B|$ in G such that H is either subnormal or abnormal in G.*

By using the definition of a $\mathcal{C}_1$-group, some good properties of a $\mathcal{C}_1$-group are given as follows:

Theorem 1.3. *The following statements for a group G are equivalent:*

(a) *G is a $\mathcal{C}_1$-group.*

(b) *Either G is nilpotent or G satisfies the following three conditions:*

$(b-1)$ *G is supersolvable.*

$(b-2)$ *$G/F(G)$ is cyclic of order p, where p is the smallest prime diving the order of G.*

$(b-3)$ *$G/O_p(G)$ is a Frobenius group with Frobenius complement $P/O_p(G)$, where P is a Sylow p-subgroup of G and $P/O_p(G)$ is cyclic of order p.*

Then, we get the following equivalence by using above theorem:

Theorem 1.4. *A group G is a $\mathcal{C}$-group if and only if G is a $\mathcal{C}_1$-group.*

Although a $\mathcal{C}_1$-group and a $\mathcal{C}$-group are equivalent, we find that it is easier to get structure of a $\mathcal{C}$-group by using definition of $\mathcal{C}_1$-group. At last, as an example of effectiveness of a $\mathcal{C}_1$-group, the necessary and sufficient conditions of a semi-product of a p-group and a p'-group to be a $\mathcal{C}_1$-group is given as follows:

Theorem 1.5. *Let P be a p-group that acts non-trivially on a p'-group M. Then the semidirect product $P \ltimes M$ is a $\mathcal{C}$-group if the following conditions hold:*

(1) *The automorphism group induced by P on M is a cyclic group $\langle\alpha\rangle$ of order p.*

(2) *α is a fixed-point-free automorphism.*

(3) *α is a semi-power-automorphism.*

The converse is true for a non-nilpotent $\mathcal{C}_1$-group.

2. Preliminaries

In this section we present some lemmas which are required in Section 3. A subgroup H of a group G is said to be abnormal in G if $g \in \langle H, H^g\rangle$ for all g in G. A subgroup H is said to be pronormal in a group G if for all g in G the subgroups H and H^g are conjugate in $\langle H, H^g\rangle$.

Lemma 2.1. *Let H be a subgroup of a group G. Then the following statements are true:*

(a) *If H is abnormal in G and $H \leq K \leq G$, then K is also abnormal in G.*

(b) *H is abnormal in G if and only if H is self-normalizing and pronormal in G.*

(c) *Let $N \trianglelefteq G$ and $N \leq H$. Then H is abnormal in G if and only if H/N is abnormal in G/N.*

(d) *If $A \leq B < G$ and B is subnormal in G, then A is not abnormal in G.*

(e) *A subnormal Sylow p-subgroup is normal and in a solvable group a subnormal Hall subgroup is normal.*

Proof. Statements (a) and (b) hold by [1, Chapter 1, 6.20 and 6.21]. Statement (c) follows by directly verifying the definition. Statement (d) also follows from Statements (a) and (b). $\square$

Lemma 2.2. *Let $G = H \ltimes M$ be the semidirect product of the non-trivial subgroup H with the normal subgroup M. Then the following statements are equivalent:*

(a) *G is a Frobenius group with Frobenius kernel M.*

(b) *H is a group of fixed-point-free automorphisms of M.*

Proof. See [4, V, 8.5 Satz]. $\square$

Lemma 2.3 ([5, Theorem 3.1])**.** *The following statements for a group G are equivalent:*

(a) *G is a $\mathcal{C}$-group.*

(b) *Either G is nilpotent or G satisfies the following three conditions:*

($b-1$) *G is supersolvable,*

($b-2$) *$G/F(G)$ is cyclic of order p, where p is the smallest prime divisor of the order of G, and*

($b-3$) *$G/O_p(G)$ is a Frobenius group whose Frobenius complement $P/O_p(G)$ is cyclic of order p, where P is a Sylow p-subgroup of G.*

Lemma 2.4 ([3, Theorem 1])**.** *A group G is nilpotent if and only if for each divisor d of the order of G there exists a normal subgroup of order d.*

Note that the above lemma doesn't hold if the word "nilpotent" is replaced by "supersolvable". But we have:

Lemma 2.5 ([5, Lemma 2.5])**.** *Let M be a normal nilpotent subgroup of a group G. If G is supersolvable, then for every divisor d of the order of M there exists a subgroup H of M of order d such that H is normal in G.*

Let α be an automorphism of a group G. Then we say that α is a semi-power-automorphism of G if there exist elements $a_1, a_2, ..., a_n$ which generate G such that α maps a_i to a power of a_i for all $i \in \{1, 2, \dots, n\}$ (see [5, Definition 1.4]).

Lemma 2.6 ([5, Lemma 2.2])**.** *The following statements for a group G are true:*

(a) *Suppose that $G = H\langle x\rangle$, where H is a normal nilpotent subgroup of G. If x induces a semi-power-automorphism on $H/\Phi(H)$, then G is supersolvable.*

(b) *If N is a normal nilpotent subgroup of a supersolvable group G, then each element x of G induces a semi-power-automorphism on $N/\Phi(N)$.*

3. Proofs of theorems

In this section, we prove Theorems 1.3, 1.4, and 1.5:

The proof of the Theorem 1.3: (a) $\Rightarrow$ (b): Suppose that G is not nilpotent. Take the primary decomposition of $|G|$ as

$$|G| = p_1^{m_1} p_2^{m_2} \cdots p_s^{m_s}, m_i \geq 1.$$

where $p_1 < p_2 < \cdots < p_s$. We split the proof into the following steps:

(1) G is solvable.

Let P_i be a Sylow p_i-subgroup of G, where $i \in \{1, \dots, s\}$. Then from our assumption there exists a subgroup H of G with order $|G : P_i|$. This implies that H is a Hall p_i'-subgroup of G. By a well-known theorem of P. Hall [6, 9.1.8], we can see that G is solvable.

(2) G is p_1-nilpotent and the normal p_1-complement is nilpotent.

Since G has a subgroup of order p_1, G has a subgroup K of index p_1 which is either subnormal or abnormal. By using the permutation representation of G on K, we see that G/K_G is isomorphic to a subgroup of the symmetric group S_{p_1} of degree p_1. The minimality of p_1 forces $K = K_G$. It follows that K is normal in G. Let $q \in \{p_2, \dots, p_s\}$ and H be a Hall q'-subgroup of G. Then, by assumption, there exists a Sylow q-subgroup S of G which is either abnormal or subnormal in G. Since K is normal in G, Lemma 2.1 (d) implies that S is subnormal. Since S is a Sylow q-subgroup of G, Lemma 2.1 (e) gives S is normal. Hence the product of all the Sylow q-subgroups, $q \in \{p_2, \dots, p_s\}$ is a normal p_1-complement and this normal p-complement is nilpotent.

(3) Let P be a Sylow p_1-subgroup of G, then P is abnormal in G.

Let B be a Hall ${p_1}'$-subgroup of G. Then from our assumption there exists a subgroup H of order $|G : B|$ in G such that H is either subnormal or abnormal in G. If H is subnormal in G, then it follows from Lemma 2.1 (e) that H is normal in G, whence $H = P$. By Statement (2), we have that G is nilpotent. This contradiction shows that H is abnormal in G. By Lemma 2.1, we obtain that P is abnormal in G.

(4) $|G/F(G)| = |P/O_{p_1}(G)| = p_1$.

If $m_1 = 1$, then Statement (4) follows from Statements (2) and (3). Hence we may assume that $m_1 \geq 2$. Let B_1 be a subgroup of G of order p_1 and M a normal p_1-complement of G. Then B_1M is a subgroup of G. Set $B = B_1M$. By hypothesis, there exists a subgroup H of G of order $|G : B| = {p_1}^{m_1-1}$ such that H is either subnormal or abnormal in G. By Sylow's theorem, we have that $H < P^x$ for some $x \in G$, whence $H < N_G(H)$. It follows from Lemma 2.1 (b) that H is subnormal in G and so $H \leq O_{p_1}(G)$. On the other hand, by Statement (3), P is abnormal in G, which means that $O_{p_1}(G) < P$. Therefore $H = O_{p_1}(G)$ by comparing the orders and hence $P/O_{p_1}(G)$ is a cyclic group of order p_1. The above proofs and Statement (2) are combined to give that $G/F(G)$ is also a cyclic group of order p_1.

(5) $G/O_{p_1}(G)$ is a Frobenius group with Frobenius complement $P/O_{p_1}(G)$.

Set $\bar{G} = G/O_{p_1}(G)$. Then $\bar{G} = \bar{P} \ltimes F(\bar{G})$. If $\bar{G}$ is not a Frobenius group, then there is a non-trivial element $\bar{y}$ in $\bar{P}$ such that $C_{F(\bar{G})}(\bar{y}) \neq \bar{1}$. We can select a non-trivial element $\bar{x} \in C_{F(\bar{G})}(\bar{y})$, then $[\bar{x}, \bar{y}] = \bar{1}$ and so $[\bar{x}, \bar{P}] = \bar{1}$ as $|\bar{P}| = p_1$. On the other hand, by Statement (3) and Lemma 2.1 (c), we can see that $\bar{P}$ is abnormal in $\bar{G}$. This contradiction shows that $C_{F(\bar{G})}(\bar{y}) = \bar{1}$ and hence $\bar{P}$ induces a fixed-point-free automorphism on $F(\bar{G})$. It follows from Lemma 2.2 that $\bar{G}$ is a Frobenius group with Frobenius complement $\bar{P}$.

(6) G is supersolvable.

In view of Statement (2), we can see that $G/M \cong P$, where M is the normal p_1-complement of G. Hence we only need to show that every chief factor of G below M is cyclic in order to conclude that G is supersolvable. Now, let K/L be a chief factor of G such that $K \leq M$. Then K/L is an elementary abelian q-group for some prime $q \in \pi(M)$. By [4, VI, 5.4 Satz], we can see that $F(G) \leq C_G(K/L)$. Then the automorphism group induced by G on K/L is of order p_1. We may assume without loss generality that $q = p_2$. By Statement (2), M is nilpotent and so M contains a subgroup B of order $p_2^{m_2-1}p_3^{m_3}\cdots p_s^{m_s}$. By hypothesis, there exists a subgroup H of order $|G:B| = p_1^{m_1}p_2 = p_1^{m_1}q$ in G such that H is either subnormal or abnormal in G. Then H contains a Sylow p_1-subgroup of G, say P. Applying Statement (3), P is abnormal in G and hence H is abnormal in G by Lemma 2.1 (a) too. Let Q be a Sylow q-subgroup of H. Then Q is of order q and is contained in M by Statement (2). It follows that $Q = H \cap M \trianglelefteq H$. If $C_H(Q) = H$, then Q centralizes P, in contradiction to the fact that P is abnormal in G. Thus $C_H(Q) < H$. Since $[O_{p_1}(G), Q] = 1$, we obtain that $H/C_H(Q)$ is cyclic of order p_1. It follows that $H/C_H(Q)$ is isomorphic to a subgroup of $Aut(Q)$, a cyclic group of order dividing $q-1$. This means that $p_1\big|(q-1)$. According to [7, Chapter 1, Lemma 1.3], we conclude that K/L is cyclic of order q, as desired.

(b)$\Rightarrow$ (a): If G is nilpotent, then G is a $\mathcal{C}_1$-group by Lemma 2.4. Hence we may assume that G is not nilpotent. Let B be a subgroup of G of index $p_1^m t$, where $(p_1, t) = 1$ and $m \geq 0$. We proceed in two cases:

Case 1: $m \leq m_1$.

By Statement (b-2), we can see that $p_1^m t$ is a divisor of the order of $F(G)$. In view of Statement (b-1) and Lemma 2.5, there exists a subgroup of order $p_1^m t$ in $F(G)$ such that it is normal in G, as desired.

Case 2: $m = m_1$.

It is clear that t is a divisor of the order of $F(G)$. By (b-1) and Lemma 2.5, there exists a subgroup H of order t in $F(G)$ such that H is normal in G. Then the subgroup PH is of order $p_1^m t$. Set $\bar{G} = G/O_{p_1}(G)$, then $\bar{G} = \bar{P} \ltimes F(\bar{G})$. If $\bar{P}$ is not self-normalizing, then there is a non-trivial element $\bar{x}$ in $N_{\bar{G}}(\bar{P})$ but not in $\bar{P}$. It follows from the theorem of Burnside [4, IV, 2.6 Hauptsatz and 2.7 Satz] that $\langle \bar{x} \rangle \bar{P}$ is p-nilpotent and so $[\bar{x}, \bar{P}] = \bar{1}$. The Statement (b-3) provides a contradiction and thus $N_{\bar{G}}(\bar{P}) = \bar{P}$. We have that $\bar{P}$ is pronormal in $\bar{G}$ since $\bar{P}$ is a Sylow subgroup of $\bar{G}$. By Lemma 2.1, we obtain that $\bar{P}$ is abnormal in $\bar{G}$. According to Lemma 2.1 (c), we can see that P is abnormal in G and so is PH by Lemma 2.1 (a), as desired.

The Statement (a) is now completely proved by means of the previous cases.

This completes the proof of the theorem. $\square$

The proof of the Theorem 1.4: This is transparent from Theorem 1.3 and Lemma 2.3. $\square$

The proof of the Theorem 1.5: Let $G = P \ltimes M$. We assume that G is a $\mathcal{C}_1$-group. We can assume that G is not nilpotent and G satisfies Statements (b-1), (b-2) and (b-3) of Theorem 1.3. By (b-2), we have $M \leq F(G)$ and so M is nilpotent.

Now, Statement (b-1) and Lemma 2.6 imply Statements (1) and (3). Statement (2) follows from Statement (b-3).

Conversely, by the theorem of Thompson [6, Theorem 10.5.4], we can see that M is nilpotent. Let $K = \langle x \in P \mid a^x = a,\ a \in M\rangle$. Then P/K is of order p by hypothesis and so $K = O_p(G)$. Thus $F(G) = KM$ and $G/F(G)$ is cyclic of order p. It follows that $G/O_p(G)$ is a Frobenius group, whose Frobenius complement $P/O_p(G)$ is cyclic of order p. Applying Lemma 2.6, we can see that G is supersolvable. Therefore G is a $\mathcal{C}_1$-group by Theorem 1.3. $\square$

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