LINEAR MAPS ON $M_n(\mathbb{C})$ PRESERVING INNER LOCAL SPECTRAL RADIUS ZERO

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Abstract. Let $M_n(\mathbb{C})$ be the algebra of all complex $n \times n$ matrices. Let x_0 be a nonzero vector in $\mathbb{C}^n$. We show that a linear unital map $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ preserves the inner local spectral radius zero at x_0, if and only if there exists an invertible matrix $A \in M_n(\mathbb{C})$ such that $\phi(T) = ATA^{-1}$.
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1. Introduction

Let $\mathcal{B}(X)$ be the algebra of all linear bounded operators on a complex Banach space X, and let $M_n(\mathbb{C})$ be the algebra of $n \times n$ complex matrices. The local resolvent set of an operator $T \in \mathcal{B}(X)$ at a point $x \in X$, denoted by $\rho_T(x)$, is the union of all open subsets $U \subseteq \mathbb{C}$ for which there exists an analytic function $f : U \to X$ such that $(T - \lambda)f(\lambda) = x$ for all $\lambda \in U$. The local spectrum of T at x is defined by $\sigma_T(x) := \mathbb{C} \setminus \rho_T(x)$ and is a (possibly empty) closed subset of $\sigma(T)$, the usual spectrum of $T \in \mathcal{B}(X)$. The local spectral radius of T at x is defined by :

$$r_T(x) := \limsup_{n\to\infty} \|T^n x\|^{\frac{1}{n}}.$$

An operator $T \in \mathcal{B}(X)$ is said to have the single-valued extension property (abbreviated SVEP) if for every open subset U of $\mathbb{C}$, the equation $(\lambda - T)f(\lambda) = 0$, $(\lambda \in U)$, has no nontrivial X-valued analytic solution f on U. Note that T has SVEP provided that its point spectrum has an empty interior. In particular, every matrix $T \in M_n(\mathbb{C})$ has SVEP and thus $\sigma_T(x) \neq \emptyset$ for all $T \in M_n(\mathbb{C})$ and all nonzero vectors $x \in \mathbb{C}^n$.

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For a closed subset F of $\mathbb{C}$, the glocal spectral subspace $\mathcal{X}_T(F)$ of T associated with F is defined by

$$\mathcal{X}_T(F) = \{x \in X : (\lambda - T)f(\lambda) = x \text{ has an analytic solution } f \text{ on } \mathbb{C}\backslash F\}.$$

The inner local spectral radius, $\iota_T(x)$, of T at $x \in X$ is defined by

$$\iota_T(x) := sup\{r \geq 0 : x \in \mathcal{X}_T(\mathbb{C} \setminus D(0,r))\},$$

where $D(0,r)$ is the open disc centered at the origin with the radius r. The inner local spectral radius coincide with the minimum modulus of $\sigma_T(x)$ provided that T has the single-valued extension property. For further information on spectral and local spectral theory we refer the reader to [1, 12].

An active research topic in matrix theory is the linear preserver problems (LPP) that deal with the characterization of linear operators on matrix spaces with some special properties such as leaving certain functions, subsets or relations invariant. Recently many authors have devoted their studies to maps that preserve local spectral quantities; see [1, 3, 4, 5, 6, 8, 9, 10].

In [3], Bourhim characterized continuous surjective linear maps on $\mathcal{B}(X)$ preserving the inner local spectral radius at a fixed nonzero vector in $\mathbb{C}^n$.

recently, M. Ech-chrif El Kettani and H. Benbouziane, in [7], described surjective additive maps on $\mathcal{B}(X)$ which preserve the operators of inner local spectral radius zero at all points of X. They showed that a surjective additive maps $\varphi : \mathcal{B}(X) \to \mathcal{B}(X)$ satisfying

$$\iota_{\varphi(T)}(x) = 0 \Leftrightarrow \iota_T(x) = 0 \ for\ all\ x \in X\ and\ T \in \mathcal{B}(X),$$

if and only if there is a nonzero scalar $c \in \mathbb{C}$ such that $\varphi(T) = cT$, for all $T \in \mathcal{B}(X)$. This result has been extended in [11] to the non additive setting where T. Jari gave the form all surjective not necessarily linear or additive $\varphi : \mathcal{B}(X) \to \mathcal{B}(X)$ satisfying $\iota_{\varphi(T)-\varphi(S)}(x) = 0 \Leftrightarrow \iota_{T-S}(x) = 0$ for all $T, S \in \mathcal{B}(X)$ and $x \in X$.

The aim of this paper is to study the linear unital map from $M_n(\mathbb{C})$ into its self which preserve operators of inner local spectral radius zero a fixed nonzero vector x_0 in $\mathbb{C}^n$. We show that a unitaly linear map $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ satisfies

$$\iota_T(x_0) = 0 \iff \iota_{\phi(T)}(x_0) = 0, (T \in M_n(\mathbb{C})), \tag{1.1}$$

if and only if there exists an invertible matrix $A \in M_n(\mathbb{C})$ such that $Ax_0 = x_0$ and $\phi(T) = ATA^{-1}$, for all $T \in M_n(\mathbb{C})$.

2. Notations and preliminaries

In this section, we introduce some notations and preliminary results which will be used to prove our main results in the next section. The duality between the

Banach space X and its dual, X^*, will be denoted by $\langle .,. \rangle$. For an $x \in X$ and $f \in X^*$, we denote as usual by $x \otimes f$ the rank one operator on X given by $z \mapsto \langle z, f \rangle x$.

The following lemma summarizes some basic properties of the local spectrum.

Lemma 2.1. *Let $T \in \mathcal{B}(X)$.*

(1) $\sigma_{su}(T) = \bigcup_{x \in X} \sigma_T(x)$.

(2) *If $x = y + z$, then $\sigma_T(x) \subseteq \sigma_T(y) \cup \sigma_T(z)$.*

Proof. See [1, 12]. □

An operator $T \in \mathcal{B}(X)$ is said to have the single-valued extension property at $\lambda_0 \in \mathbb{C}$, abbreviated T has the SVEP at λ_0, if for every neighbourhood U of λ_0 the only analytic function $f : U \to X$ which satisfies the equation

$$(\lambda - T)f(\lambda) = 0 \quad (\lambda \in U)$$

is the constant function $f \equiv 0$. The operator T is said to have the SVEP if T has the SVEP at every $\lambda \in \mathbb{C}$. Evidently, T has SVEP at every $\lambda \in \mathbb{C} \setminus int(\sigma_p(T))$, where $int(\sigma_p(T))$ denotes the interior of the point spectrum of T. In particular, if $\sigma_p(T)$ has empty interior, then T has SVEP. For example, every finite rank operator has SVEP.

In the finite dimensional case, every T belonging to $M_n(\mathbb{C})$ has the single-valued extension property. we can see, for example, the books [1, 12] for further information on the local spectral theory.

We shall make extensive use of the following result.

Lemma 2.2 ([1, Theorem 2.22])**.** *Let $T \in \mathcal{B}(X)$, let $\lambda \in \mathbb{C}$ and let $x \in \ker(T - \lambda)$. then $\sigma_T(x) \subset \{\lambda\}$. If, further, $x \neq 0$ and T has SVEP at λ, then $\sigma_T(x) = \{\lambda\}$.*

Given an operator $T \in \mathcal{B}(X)$, for every closed subset Ω of $\mathbb{C}$, the glocal spectral subspace $\mathcal{X}_T(\Omega)$ of T associated with Ω is defined by

$$\mathcal{X}_T(\Omega) = \{x \in X : (\lambda - T)f(\lambda) = x \text{ has an analytic solution } f \text{ on } \mathbb{C}\backslash\Omega\}.$$

It is a T-hyperinvariant subspace but not necessarily closed. Recall that the local spectral radius of T at x coincides with $\inf\{r \geq 0 : x \in \mathcal{X}_T(\overline{\mathrm{D}}(0, r))\}$; see [12, Proposition 3.3.13]. Here, $\overline{\mathrm{D}}(0, r)$ (resp. $\mathrm{D}(0, r)$) denotes the closed (resp. the open) disc centered at the origin with radius r. Analogously, the *inner local spectral radius*, $\iota_T(x)$, of T at $x \in X$ is defined by

$$\iota_T(x) := \sup\{r \geq 0 : x \in \mathcal{X}_T(\mathbb{C}\backslash\mathrm{D}(0, r))\},$$

and coincides with the minimum modulus of $\sigma_T(x)$ provided that T has the single-valued extension property; see [13]. For more details on the local spectral theory, we refer the reader to the remarkable books [1, 12].

Next lemma, quoted from [13], summarizes some elementary properties of the inner local spectral radius.

Lemma 2.3. *For an operator $T \in \mathcal{B}(X)$ and a vector $x \in X$, the following hold.*

(a) $x = 0$ *is the only vector for which* $\iota_T(x) = \infty$.

(b) $\iota_T(x) = 0$ *if and only if* $0 \in \sigma_T(x)$.

(c) $\iota_T(x) \leq \min\{|\lambda| : \lambda \in \sigma_T(x)\}$. *The equality holds if T has SVEP.*

(d) *For every bijective operator $A \in \mathcal{B}(X, Y)$, we have $\iota_{ATA^{-1}}(Ax) = \iota_T(x)$.*

Let $\mathcal{K}(C)$ denote the set of all nonempty compact subsets of $\mathbb{C}$, endowed with the Hausdorff metric. The following lemma proved by Aupetit [2].

Lemma 2.4. *Let X be a finite dimensional normed space. Then the map*

$$\sigma : T \in \mathcal{B}(X) \to \sigma(T) \in \mathcal{K}(C)$$

is continuous.

Proof. It is a consequence of [2, Corollary 3.4.5]. □

Let x_0 be a nonzero fixed vector in $\mathbb{C}^n$, we denote by $\mathcal{M}_{x_0}$ the set given by

$$\mathcal{M}_{x_0} := \{T \in M_n(\mathbb{C}) : \{Tx_0, T^2x_0, ..., T^nx_0\} \textit{ is a basis of } \mathbb{C}^n \textit{ and } |\sigma(T)| = n\},$$

where $|\sigma(T)|$ is the number of elements of $\sigma(T)$.

The following lemma that summarizes some important properties of $\mathcal{M}_{x_0}$, which plays an important role in the proof of the main theorem.

Lemma 2.5. *Let x_0 be a nonzero fixed vector in $\mathbb{C}^n$. The set $\mathcal{M}_{x_0}$ is dense in $M_n(\mathbb{C})$ and $\sigma(T) = \sigma_T(x_0)$ for all $T \in \mathcal{M}_{x_0}$.*

Proof. See [5, Lemma 2.5]. □

We close this section with this lemma in which we give a characterization of the zero matrix of $M_n(\mathbb{C})$)in terms of inner local spectral radius.

Lemma 2.6. *Let x_0 be a nonzero fixed vector in $\mathbb{C}^n$. Let A be a matrix in $M_n(\mathbb{C})$. The following conditions are equivalent:*

(1) $A = 0$.

(2) $\iota_{A+T}(x_0) = 0$ *for every nilpotent $T \in M_n(\mathbb{C})$.*

Proof. We only need to prove (2) ⇒ (1). indeed, let T be a nilpotent operator in $M_n(\mathbb{C})$ such that $\iota_{A+T}(x_0) = 0$. By Lemma 2.3 we have $0 \in \sigma_{A+T}(x_0) \subseteq \sigma(A + T)$, which implies, by [14, Propositio 5.2] , that $A = 0$; as desired. □

3. Main results

Theorem 3.1. *Fix a nonzero vector fixed x_0 in $\mathbb{C}^n$. A unitaly linear map $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ satisfies*

$$\iota_T(x_0) = 0 \iff \iota_{\phi(T)}(x_0) = 0, \text{ for all } T \in M_n(\mathbb{C}), \tag{3.1}$$

if and only if there exists an invertible matrix $A \in M_n(\mathbb{C})$ such that $Ax_0 = x_0$ and $\phi(T) = ATA^{-1}$, for all $T \in M_n(\mathbb{C})$.

Proof. The "if " part is obvious by Lemma 2.3, then we only need to check the necessity condition. So, assume that ϕ is a unitaly linear map from $M_n(\mathbb{C})$ into its self satisfying (3.1). Let us first show that ϕ is bijective. To do so, let $R \in \mathcal{M}_n$ such that $\phi(R) = 0$. For every nilpotent operator $T \in M_n(\mathbb{C})$, we have $\iota_T(x_0) = 0$ and $\iota_{\phi(T)}(x_0) = 0$. On the other hand $\iota_{\phi(T)+\phi(R)}(x_0) = \iota_{\phi(T)}(x_0) = 0$ therefore $\iota_{T+R)}(x_0) = 0$ for every nilpotent operator T. By [14, Propositio 5.2], we have $R = 0$ which implies that ϕ is injective. It is in fact a bijective map since $M_n(\mathbb{C})$ is a finite dimensional space.

Next, we prove that

$$\sigma(T) \subseteq \sigma(\phi(T)), \text{ for all } T \in M_n(\mathbb{C}). \tag{3.2}$$

Let $T \in \mathcal{M}_{x_0}$. By Lemma 2.1 there exists x such that $\sigma_{su}(T) = \sigma_T(x)$ and $x = \alpha_1 x_0 + ... + \alpha_{n-1}T^{n-1}x_0$. It follows from Lemma 2.5 that

$$\sigma(T) = \sigma_T(x) \subseteq \sigma_T(x_0) \cup ... \cup \sigma_T(T^{n-1}x_0) \subseteq \sigma_T(x_0).$$

Therefore $\sigma(T) = \sigma_T(x_0)$.

Let $\lambda \in \sigma(T) = \sigma_T(x_0)$ we have $0 \in \sigma_{T-\lambda}(x_0)$, thus $\iota_{T-\lambda}(x_0) = 0$. By hypothesis 3.1, $\iota_{\phi(T)-\lambda}(x_0) = 0$ and $0 \in \sigma_{\phi(T)-\lambda}(x_0)$. This implies that $\lambda \in \sigma(\phi(T))$ and therefore

$$\sigma(T) \subset \sigma(\phi(T)), \text{ for all } T \in \mathcal{M}_{x_0}.$$

Now, let $T \in M_n(\mathbb{C})$, by Lemma 2.5, there exists a sequence (T_p) in $\mathcal{M}_{x_0}$ such that $T_p \to T$. The continuity of ϕ and the continuity of the spectrum on $M_n(\mathbb{C})$ (Lemma 2.5) imply that

$$\sigma(T) = \lim \sigma(T_p) \subseteq \lim \sigma(\phi(T_p)) = \sigma(\phi(T)), \text{ for all } T \in M_n(\mathbb{C}).$$

Since ϕ is bijective and ϕ^{-1} satisfies 3.1, with similar reasoning, we get the reverse inclusion. And therefore

$$\sigma(\phi(T)) = \sigma(T), \text{ for all } T \in M_n(\mathbb{C}).$$

By [15, Theorem 2], there exists an invertible matrix $A \in M_n(\mathbb{C})$ such that either

$$\phi(T) = ATA^{-1}, \ (T \in M_n(\mathbb{C})) \tag{3.3}$$

or

$$\phi(T) = AT^tA^{-1}, \ (T \in M_n(\mathbb{C})), \tag{3.4}$$

where T^t denotes, as usual, the transpose of $T \in M_n(\mathbb{C})$.

The second form 3.4 is not possible. Indeed, assume for sake of contradiction that ϕ takes a such form. Let y be a non zero vector such that x_0 and y are linearly independent and $\langle y, A^{-1}x_0\rangle = 1$, and let f be a linear functional on $\mathbb{C}^n$ such that $f(x_0) = 0$ and $f(y) = 1$. Set $T := y \otimes f$. Note that $Tx_0 = 0$ and thus $i_T(x_0) = 0$. On the other hand $T^t\left(A^{-1}x_0\right) = f$ and $T^tf = f$. Then

$$\sigma_{T^t}\left(A^{-1}x_0\right) = \sigma_{T^t}\left(T^tA^{-1}x_0\right) = \sigma_{T^t}(f) = \{1\}.$$

Which proves, by Lemma 2.3, that

$$i_{\phi(T)}(x_0) = i_{AT^tA^{-1}}(x_0) = i_{T^t}(A^{-1}x_0) \neq 0.$$

This contradiction shows that φ takes the form (3.3).

It remains to prove that $Ax_0 = x_0$. Assume on the contrary that $A^{-1}x_0$ and x_0 are linearly independent. We could find a linear functional $f : \mathbb{C}^n \longrightarrow \mathbb{C}$ such that $f(A^{-1}x_0) = 0$ and $f(x_0) = 1$. Then for $T := x_0 \otimes f$ we have $Tx_0 = x_0$. Thus, $i_T(x_0) = 1$. On the other hand $ATA^{-1}x_0 = 0$ then $i_{\phi(T)}(x_0) = 0$, arriving to a contradiction. We conclude that $Ax_0 = \mu x_0$ for some nonzero scalar μ. Dividing A by μ if necessary, we may assume that $Ax_0 = x_0$. This finishes the proof. □

References

[1] P. Aiena, *Fredholm and local spectral theory, with application to multipliers*, Kluwer Acad. Publishers, 2004.

[2] B. Aupetit, *A primer on spectral theory*, Springer, New York, 1991.

[3] A. Bourhim, *Surjective linear maps preserving local spectra*, Linear Algebra Appl., 432 (2010), 383-393 .

[4] A. Bourhim, J. Mashreghi, *A survey on preservers of spectra and local spectra*, Contemp. Math, 638 (2015), 45-98.

[5] A. Bourhim and V.G. Miller, *Linear maps on $M_n(\mathbb{C})$ preserving the local spectral radius*, Studia Math., 188(1) (2008), 67-75.

[6] A. Bourhim and T.J. Ransford, *Additive maps preserving local spectrum*, Integr. Equ. Oper. Theory, 55 (2006), 377-385.

[7] M. Ech-chrif El Kettani, H. Benbouziane, *Additive maps preserving operators of inner local spectral radius zero*, Rendiconti del Circolo Matematico di Palermo, 63(2) (2014), 311-316.

[8] M. Ech-chrif El Kettani, H. Benbouziane and Y. Bouramdane, *Inner local spectral radius preservers of operator products*, Preprint.

[9] M. Elhodaibi and A. Jaatit, *Inner local spectral radius preservers*, Rendiconti del Circolo Matematico di Palermo, DOI: 10.1007/s12215-017-0308-8.

[10] M. González and M. Mbekhta, *Linear maps on $M_n(\mathbb{C})$ preserving the local spectrum*, Linear Algebra Appl., 427 (2007), 176-182.

[11] T. Jari, *Nonlinear maps preserving the inner local spectral radius*, Rendiconti del Circolo Matematico di Palermo, 64 (2015), 67-76.

[12] K.B. Laursen and M.M. Neumann, *An introduction to local spectral theory*, London Mathematical Society Monographs, New Series 20. The Clarendon Press,Oxford University Press, New York(2000).

[13] T.L. Miller, V.J. Miller and M. Neumann *Local Spectral properties of weighted shifts*, J. Oper. Theory, 51 (2004), 71-88.

[14] A. R. Sourour, *Inversibility preserving linear maps on L(X)*, Trans. Amer. Math. Soc., 348 (1996) 13-30

[15] A.R.Sourour, *The Gleason-Kahane-Zelazko theorem and its generalizations,* Banach Center Publ., 30 (1994) 327-331.

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