ALGEBRAIC HYPERSTRUCTURES AND SOCIAL RELATIONS

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Abstract. The relations between the people of a certain set of individuals can be described, from a static point of view, through the arrays of Moreno. From a dynamic point of view, however, account must be taken of the coalitions formed between people. These coalitions can be specified or one or more alternatives can be selected while making decisions involving more decision-makers; further there are presented coordination strategies for more people in order to ensure the maximum utility in social problems modelling using cooperative games. This paper presents how algebraic hyperstructures can be a useful mathematical tool both for the study of social relations from a static point of view and for the study of the social dynamics leading to the formation of coalitions. It shows that, surprisingly, many properties deemed significant only from an algebraic or geometric viewpoints in a set of individuals, deep meanings from the social point of view.

Keywords: social relations, social aggregations, hyperoperations, multiperson decision making, cooperative games.

1. Introduction

The relations between the people of a certain set of individuals can be described, from a "static" point of view, through the socio-matrices of Moreno. The Moreno technique was developed by Moreno in 1946, published in [24, 25]. The line of Moreno research was developed and used by various authors. The mathematical tool used by Moreno et al. is the *theory of relations*. We consider a set $U = \{x_1, x_2, \ldots, x_n\}$ of individuals that we assume forms a "social group" (e.g. the students in a school, the professors in a university, etc.). A relation R on U is represented by a matrix $M_R = (m_{rs}), r, s \in \{1, 2, \ldots, n\}$, where $m_{rs} = 1$ if

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$x_r R x_s$ and $m_{rs} = 0$ if x_r is not in the relation R with x_s. We use also the notation $x_r R x_s = m_{rs}$, i.e. $x_r R x_s = 1$ means that x_r is in relation with x_s, and $x_r R x_s = 0$ that x_r is not in relation with x_s.

The objective of the Moreno et al. research is to study, from the relation R, as, and from some particular points of view, social subgroups can be formed, using suitable algorithms based on statistical theories.

In this paper, in addition to above mentioned Moreno research, we propose the construction of social subgroups with algebraic algorithms based on the theory of algebraic hyperstructures while looking for and attributing social meanings to hyperoperations considered. In particular, a hyperproduct is seen as the result of the interaction of the components of an ordered pair of elements of U.

From a "dynamic" point of view, coalitions formed between people should be considered. These coalitions can be specified or one or more alternatives can be selected while making decisions involving more decision-makers; further there are presented coordination strategies for more people in order to ensure the maximum utility in social problems modelling using cooperative games.

In this paper we show how algebraic hyperstuctures can be useful mathematical tool, both for the study of social relations from a static point of view and for the study of the social dynamics leading to the formation of coalitions. Our research consists in interpreting, in some social contexts, ideas and results obtained by theories on multivalued operations, and to develop, from these, algorithms for the construction or the recognition of social groups, that are often latent, i.e. not detectable by traditional survey tools.

2. A "social approach" to the algebraic hyperstructures

Let us recall some fundamental concepts on algebraic hyperstructures (see e.g. [1-5, 10, 29]. We denote with U a given nonempty set (e.g. of numbers, individuals, media, etc.), with $P(U)$ the family of subsets of U. Moreover let $P^*(U) = P(U) - \{\emptyset\}$.

Definition 2.1. A *partial hyperoperation* (or *multivalued operation*) on U is a function $\alpha : U \times U \to P(U)$ that to every ordered pair (x, y) of elements of U associates a subset of U, denoted with $x\alpha y$. The pair (U, α) is called a *partial hypergroupoid.*

If $x\alpha y \neq \emptyset, \forall x, y \in U$, then α is said to be a *hyperoperation* on U and (U, α) is said *hypergroupoid.*

Usually, if there is no possibility of misunderstanding, it is used multiplicative notation and $x\alpha y$ is called "*hyperproduct* of x and y"

The partial hyperoperation α is said to be:

- *commutative* if $x\alpha y = y\alpha x, \forall x, y \in U$;

- *weak commutative* if $x\alpha y \neq \emptyset \Rightarrow x\alpha y \cap y\alpha x \neq \emptyset$, $\forall x, y \in U$ ([29]).

Social interpretation 2.2. Let U be a social group. Then the hyperproduct $x\alpha y$ can be interpreted, in a suitable social context, as the set of the individuals chosen *for a particular activity* or *from a suitable viewpoint* by x and y, with the condition that first expresses his/her opinion x and then y. The condition $x\alpha y \neq \emptyset$, $\forall x, y \in U$ means that at least one individual must be selected; the commutativity means that it is not important who is expressed first; the weak commutativity means that you can agree on a compromise, whatever the individual who expresses the ideas first. In other words, the weak commutativity can be considered a condition for activating mediation or consensus strategies.

Definition 2.3. Let (U, α) be a *partial hypergroupoid.* For every A, B, subsets of U, we define $A\alpha B$ as the union of the sets $x\alpha y$, with $x \in A$, $y \in B$. Moreover we write $x\alpha B$ and $A\alpha y$ to denote $\{x\}\alpha B$ and $A\alpha\{y\}$, respectively.

Definition 2.4. A partial hypergroupoid (U, α) is said to be:
- *associative* (or *partial semi-hypergroup*) if $(x\alpha y)\alpha z{=}x\alpha(y\alpha z), \forall x, y, z \in U$;
- *weak associative* if

$$[(x\alpha y)\alpha z \neq \emptyset \text{ or } x\alpha(y\alpha z) \neq \emptyset] \Rightarrow (x\alpha y)\alpha z \cap x\alpha(y\alpha z) \neq \emptyset, \forall x, y, z \in U;$$

- *strictly weak associative* if, for every non empty subsets A, B, C of U:

$$[(A\alpha B)\alpha C \neq \emptyset \text{ or } A\alpha(B\alpha C) \neq \emptyset] \Rightarrow (A\alpha B)\alpha C \cap A\alpha(B\alpha C) \neq \emptyset. \tag{2.1}$$

Social interpretation 2.5. Let U be a set of individuals. Then, for every nonempty subsets A, B, of $U, A\alpha B$ is the set of the individuals chosen for a particular activity or from a suitable viewpoint by at least an x in A and a y in B, with the condition that first expresses his/her opinion x and then y. The associativity means that, if three social groups A, B, C, in this order, aggregate their opinions, is not important if aggregate first A and B or B and C. The strictly weak associativity means that you can agree on a compromise, whatever the social groups aggregate first. In other words, the strictly weak associativity can be considered a condition for activating mediation or consensus among groups ([12, 15]).

Unfortunately, in a political or social context, associativity, and always also the weak associativity, usually does not apply. The success of political action is often determined by the order in which the various groups aggregate to put forward their opinions and interests.

Definition 2.6. A partial hypergroupoid (U, α) is said to be a *partial quasi-hypergroup* if, for every x in U, $x\alpha U = U = U\alpha x$.

Social interpretation 2.7. Let U be a set of individuals. In a social context let us say that x marginalizes y as first decision making if $y \notin x\alpha U$. Similarly, we say that y marginalizes x as second decision maker if $x \notin U\alpha y$.

If (U, α) is a partial quasi-hypergroup we have as social interpretation that no individual can be marginalized by another, whatever the order in which the opinions are expressed.

Definition 2.8. A hypergroupoid (U, α) is said to be a *hypergroup* if it is a quasi-hypergroup and a semi-hypergroup.

Social interpretation 2.9. Let U be a set of individuals. If (U, α) is a hypergroup, then:

- any ordered pair (x, y) of $U \times U$, choose at least an individual belonging to U;
- no individual can be marginalized by another;
- the choices made by the three groups A, B, C, of individuals, in order, that aggregate their opinions, do not depend on the fact that are aggregated before A and B or B and C before;
- if (U, α) is commutative, the result of the aggregation of three groups A, B, C, (and, in general, also n groups) is not dependent on the order in which the groups are aggregated ([13, 14]).

So, if you have a commutative hypergroup, the choices are, in a sense, objective, democratic and cannot be altered significantly by political skills.

Social example 2.10. In [17, 28] has been described a research on *self-organizing socialization* carried out in schools of Abruzzo and were built socio-matrices, addressing students of various schools questions like:

- Q1 = "Who would you invite to a party?",
- Q2 = "Who do you think can help you in case of trouble ?",
- Q3 = "Who do you think can be invited, as a friend, for his skills?",
- Q4 = "With whom would you go on vacation?".

Each question Qj induces a relationship Rj in the set $U = \{x_1, x_2, \dots, x_n\}$ of students (for a class or school) interviewed, placing $xRjy$ if y is one of the students indicated by x in question Qj. So each question Qj corresponds a Moreno socio-matrix $Mj = (mj_{rs})$, with n rows and n columns.

Using an approach based on algebraic hyperstructures the same questions should be targeted to the ordered pairs (x, y) of students making sure first x expresses his/her opinion, then y and finally you find an agreement between x and y. The questions to be put to pairs of students would be:

- D1 = "Who do you agree on to invite to a party?",
- D2 = "Who do you both think can help you in case of difficulties (troubles)?",
- D3 = "Who would you consider to be your friends for his/her skills?",
- D4 = "With whom do you both think is pleasant to go on vacation?".

From this point of view any Moreno socio-matrix $M_{Rj} = (mj(x, z)), x, z \in U$, is replaced by a "hyperoperation socio-matrices" $Hj = (hj((x, y), z)), (x, y) \in U \times U$, $z \in U$, with n^2 rows and n columns.

Obtain directly the hyperoperation socio-matrices Hj from the interviews Moreno can be very difficult for the time needed to interview the couples. There are, however, many criteria on which these matrices are obtained as a processing of those of Moreno [24, 25].

Two simple criteria are the following "or" and "and" criteria.

Criterion "or": we assume, for every $(x, y) \in U \times U, z \in U$:

$$hj^{or}((x,y),z) = \max\{mj(x,z), mj(y,z)\}; \tag{2.2}$$

Criterion "and": we assume, for every $(x, y) \in U \times U, z \in U$:

$$hj^{and}((x,y),z) = \min\{mj(x,z), mj(y,z)\}. \tag{2.3}$$

The applications:

- $\alpha_{or} : (x, y) \in U \times U \to \{z : hj^{or}((x, y), z) = 1\}$;
- $\alpha_{and} : (x, y) \in U \times U \to \{z : hj^{and}((x, y), z) = 1\}$;

are two algebraic partial hyperoperations, associated to the relation Rj, and, for every $(x, y) \in$U×U, $\alpha_{and}(x, y) \subseteq \alpha_{or}(x, y)$.

Some other partial hyperoperations, associated to the relation Rj, are:

- α_{nor}: $(x, y) \in U \times U \to \{z : hj^{or}((x, y), z) = 0\}$;
- α_{nand}: $(x, y) \in U \times U \to \{z : hj^{and}((x, y), z) = 0\}$;
- $\alpha_d : (x, y) \in U \times U \to \{z : hj^{or}((x, y), z) - hj^{and}((x, y), z) = 1\}$.

3. A "geometric and social" approach to coalition formation

Let us assume the following definition:

Definition 3.1. A hypergroupoid (U, α) is said to be a *coalition forming structure* if, for every x, y in U, $\{x, y\} \subseteq x\alpha y$.

Social interpretation 3.2. Let U be a set of individuals. If (U, α) is a *coalition forming structure*, then individuals who decide who can add to a given task or a given job fit always even themselves. That is, for successive aggregations, they are formed coalitions ever more numerous, able to achieve certain goals. This can be very important in the case of decisions made by most decision-makers, in which it is crucial to reach a majority or in cooperative games ([15, 27]).

Geometric interpretation 3.3. From a geometric point of view the individuals of U can be regarded as points of a particular geometric space. U is represented by a "point cloud" and the hyperproducts xαy are particular *clusters* of such cloud. Two particular cases are obtained from the socio-matrices of Moreno and from multi-criteria analysis.

A geometric representation of Moreno matrices. Let $U = \{x_1, x_2, \ldots, x_n\}$. If $M_R = (m_{rs}), r, s \in \{1, 2, \ldots, n\}$ is the socio-matrix of a relation R on

U, we can introduce two geometric (Euclidean) representations of U, in the Euclidean n-dimensional space E^n. The first is considering x_i as the point P_i of E^n with coordinates m_{is}, let us call the "*active or direct representation*" with respect to R; the second is considering x_i as the point Q_i of E^n with coordinates m_{ri}, let us call the "*passive or inverse representation*" with respect to R. Then we obtain the "*active cloud*", $\{P_i, i = 1, 2, \dots, n\}$ and the "*passive cloud*" $\{P_i, i = 1, 2, \dots, n\}$.

In each of these clouds we can introduce a metric, such as the Euclidean metric d_E or the Hamming distance d_H, on which to base the formation of coalitions. For example, we can introduce a partial hyperoperation α setting a positive number ϵ and placing:

$$\forall x, y \in U, x\alpha y = \{z \in U : d_E(x, z) \diamondsuit d_E(y, z)\} < \epsilon, \tag{3.1}$$

where "$\diamondsuit$" is a suitable algebraic operation in the set of nonnegative real numbers, e.g. addition, multiplication, maximum, minimum, etc.

A geometric representation of multi-criteria analysis. We analyze the elements of U on the basis of a set of criteria $K = \{C_1, C_2, \dots, C_m\}$, aimed at a certain objective. We attribute to each individual x_i a score s_{ij} against each criterion C_j. This score is a positive real number, usually belonging to $[0, 1]$ which measures the extent to which individual satisfies the criterion. In such a case each individual x_i is represented, in the Euclidean space E^m by the point $P_i = (s_{i1}, s_{i2}, \dots, s_{im})$. Also in this case we can introduce partial hyperoperations according to the above formula (3.1).

4. Conclusions and perspective of research

We have shown that the use of partial hyperstructures can be a useful tool for analyzing social relationships, and allows us to discover latent social subgroups, evaluate the formation of coalitions, and see aspects of social groups that do not appear in the classical analysis. In addition, each of the algebraic properties of hyperstructures can be translated to properties and behaviours of the social group; therefore the social analysis with the algebraic hyperstructures looks as a very powerful analysing tool.

One line of research is the application of the theories presented and hyperstructures introduced to case studies, statistical analyses to be made, or case studies already analysed by a statistical point of view, for which you want to obtain further processing.

A second research direction is achieved by replacing the relations with fuzzy relations, obtaining fuzzy Moreno socio-matrices. In this case, we can consider many possible hyperstructures associated with the α-cuts of the fuzzy sets obtained. Another generalization of the Moreno models is obtained by considering fuzzy relations, that appear to be more adequate than the crisp one to represent human perceptions and communication [9, 16, 26, 30, 31], for statistical inferences with imprecise data [17-20], and for decision making [22]. The connections between hyperstrucures and fuzzy sets are described in [6-8, 14, 21, 23].

Every fuzzy relation $R^{\sim}$ in the set $\mathrm{U} = \{x_1, x_2, \ldots, x_n\}$ of individuals is represented by a fuzzy socio-matrix $M_R^{\sim} = (m_{rs})$ where $m_{rs} \in [0, 1]$ is the degree in which the fuzzy relation $R^{\sim}$ holds. We write $x_r\ R^{\sim}\ x_s = m_{rs}$.

The sociological results known for the crisp socio-matrices can be, in many ways, extended to fuzzy socio-matrices. This approach is described in [11].

Finally, many results can be obtained considering hyperstructures associated to relations composed by those presented, in particular the powers of these relations.

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