HERMITE-HADAMARD TYPE INEQUALITIES FOR GENERALIZED (s, m, φ)-PREINVEX GODUNOVA-LEVIN FUNCTIONS

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Abstract. In the present paper, the notion of (m, φ)-invex set and generalized (s, m, φ)-preinvex Godunova-Levin function of second kind are introduced and some new integral inequalities involving generalized (s, m, φ)-preinvex Godunova-Levin function of second kind along with beta function are given. By using new identities for fractional integrals some new estimates on generalizations of Hermite-Hadamard type inequalities for generalized (s, m, φ)-preinvex Godunova-Levin functions of second kind via Riemann-Liouville fractional integral are established. At the end, some applications to special means are given.

Keywords: Hermite-Hadamard inequality, Hölder's inequality, Minkowski's inequality, power mean inequality, Riemann-Liouville fractional integral, m-invex, P-function.

1. Introduction and preliminaries

The following notation are used throughout this paper. We use I to denote an interval on the real line $\mathbb{R} = (-\infty, +\infty)$ and I° to denote the interior of I. For any subset $K \subseteq \mathbb{R}^n, K^\circ$ is used to denote the interior of K. $\mathbb{R}^n$ is used to denote a n-dimensional vector space. The nonnegative real numbers are denoted by $\mathbb{R}_\circ = [0, +\infty)$. The set of integrable functions on the interval $[a, b]$ is denoted by $L_1[a, b]$.

The following inequality, named Hermite-Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

Theorem 1.1. *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function on I and $a, b \in I$ with $a < b$. Then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

In recent years, various generalizations, extensions and variants of such inequalities have been obtained. For other recent results concerning Hermite-Hadamard type inequalities through various classes of convex functions, (see [18]) and the references cited therein, also (see [17]) and the references cited therein.

Fractional calculus (see [18]) and the references cited therein, was introduced at the end of the nineteenth century by Liouville and Riemann, the subject of which has become a rapidly growing area and has found applications in diverse fields ranging from physical sciences and engineering to biological sciences and economics.

Definition 1.2. *Let $f \in L_1[a, b]$. The Riemann-Liouville integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, \quad x > a$$

and

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, \quad b > x,$$

where $\Gamma(\alpha) = \int_0^{+\infty} e^{-u}u^{\alpha-1}du$. Here $J^0_{a+}f(x) = J^0_{b-}f(x) = f(x)$.

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

Due to the wide application of fractional integrals, some authors extended to study fractional Hermite-Hadamard type inequalities for functions of different classes (see [18]-[19]) and the references cited therein.

Definition 1.3 (see [2]). *A nonnegative function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}_{\circ}$ is said to be P-function or P-convex, if*

$$f(tx + (1-t)y) \leq f(x) + f(y), \quad \forall x, y \in I,\ t \in [0, 1].$$

Definition 1.4 (see [3]). *A function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}_{\circ}$ is said to be a Godunova-Levin function or $f \in Q(I)$, if f is nonnegative and for all $x, y \in I$, $t \in (0, 1)$, we have that*

$$f(tx + (1-t)y) \leq \frac{f(x)}{t} + \frac{f(y)}{1-t}.$$

The class $Q(I)$ was firstly described in (see [3]) by Godunova and Levin. Some further properties of it are given in (see [2],[4],[5]). Among others, it is noted that nonnegative monotone and nonnegative convex functions belong to this class of functions.

Definition 1.5 (see [6]). *A function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}_\circ$ is said to be (s,m)-Godunova-Levin functions of first kind or $f \in Q^1_{(s,m)}$, if $\forall s, m \in (0,1]$, we have*

$$f(tx + m(1-t)y) \leq \frac{1}{t^s} f(x) + m\left(\frac{1}{1-t^s}\right) f(y), \quad \forall x, y \in I, \ t \in (0,1).$$

We would like to mention that Definition 1.5 is also introduced and studied by Li et al. (see [7]) independently. For $m = 1$ in Definition 1.5 we have the definition of s-Godunova-Levin functions of first kind, which is introduced and investigated by Noor et al. (see [8]).

Definition 1.6 (see [6]). *A function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}_\circ$ is said to be (s,m)-Godunova-Levin functions of second kind or $f \in Q^2_{(s,m)}$, if $s \in [0,1]$, $m \in (0,1]$, we have*

$$f(tx + m(1-t)y) \leq \frac{1}{t^s} f(x) + m\left(\frac{1}{(1-t)^s}\right) f(y), \quad \forall x, y \in I, \ t \in (0,1).$$

It is obvious that for $s = 0, m = 1, (s,m)$-Godunova-Levin functions of second kind reduces to Definition 1.3 of P-functions. If $s = 1, m = 1$, it then reduces to Godunova-Levin functions. For $m = 1$, we have the definition of s-Godunova-Levin function of second kind introduced and studied by Dragomir (see [9],[10]).

Definition 1.7 (see [14]). *A set $K \subseteq \mathbb{R}^n$ is said to be invex with respect to the mapping $\eta : K \times K \longrightarrow \mathbb{R}^n$, if $x + t\eta(y,x) \in K$ for every $x, y \in K$ and $t \in [0,1]$.*

Notice that every convex set is invex with respect to the mapping $\eta(y,x) = y - x$, but the converse is not necessarily true (see [14],[15]) and the references therein.

Definition 1.8 (see [16]). *The function f defined on the invex set $K \subseteq \mathbb{R}^n$ is said to be preinvex with respect η, if for every $x, y \in K$ and $t \in [0,1]$, we have that*

$$f\left(x + t\eta(y,x)\right) \leq (1-t)f(x) + tf(y).$$

The concept of preinvexity is more general than convexity since every convex function is preinvex with respect to the mapping $\eta(y,x) = y - x$, but the converse is not true.

The Gauss-Jacobi type quadrature formula has the following

$$\int_a^b (x-a)^p (b-x)^q f(x)dx = \sum_{k=0}^{+\infty} B_{m,k} f(\gamma_k) + R^\star_m |f|, \tag{1.2}$$

for certain $B_{m,k}, \gamma_k$ and rest $R^\star_m|f|$ (see [11]).

Recently, Liu (see [12]) obtained several integral inequalities for the left hand side of (1.2) under the Definition 1.3 of P-function.

Also in (see [13]), Özdemir et al. established several integral inequalities concerning the left-hand side of (1.2) via some kinds of convexity.

Motivated by these results, the aim of this paper is to establish some generalizations of Hermite-Hadamard type inequalities using new identities given in Section 3 for generalized (s, m, φ)-preinvex Godunova-Levin functions of second kind via Riemann-Liouville fractional integral.

The paper is organized as follows: In Section 2, the notion of (m, φ)-invex set is introduced and an interesting property is derived. Also the notion of generalized (s, m, φ)-preinvex Godunova-Levin functions of second kind is introduced and some new integral inequalities involving generalized (s, m, φ)-preinvex Godunova-Levin functions of second kind along with beta function are given. In Section 3, some generalized integral inequalities of Hermite-Hadamard via fractional integrals for generalized (s, m, φ)-preinvex Godunova-Levin functions of second kind are given. In Section 4, some applications to special means are given. These results provide new estimates on these Hermite-Hadamard types.

2. New integral inequalities

Definition 2.1 (see [1])**.** *A set $K \subseteq \mathbb{R}^n$ is said to be m-invex with respect to the mapping $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}^n$ for some fixed $m \in (0,1]$, if $mx + t\eta(y, mx) \in K$ holds for each $x, y \in K$ and any $t \in [0,1]$.*

Remark 2.2. In Definition 2.1, under certain conditions, the mapping $\eta(y, mx)$ could reduce to $\eta(y, x)$. For example when $m = 1$, then the m-invex set degenerates an invex set on K.

First we give new definition, to be referred as (m, φ)-invex set.

Definition 2.3. *Let $\varphi : I \longrightarrow K$ be an arbitrary function. A set $K \subseteq \mathbb{R}^n$ is said to be (m, φ)-invex with respect to the mapping $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}^n$ for some fixed $m \in (0,1]$, if $m\varphi(y) + t\eta(\varphi(x), \varphi(y), m) \in K$ holds for each $x, y \in I$ and any $t \in [0,1]$.*

Example 2.4. Let $m = \frac{1}{4}$ and $X = (0, \frac{\pi}{2}] \bigcup (\pi, \frac{3\pi}{2}]$

$$\eta(x,y,m) = \begin{cases} m\sin(y-x), & \text{if } x \in (0, \frac{\pi}{2}],\ y \in (0, \frac{\pi}{2}]; \\ -m\sin(y-x), & \text{if } x \in (\pi, \frac{3\pi}{2}],\ y \in (\pi, \frac{3\pi}{2}]; \\ m\sin x, & \text{if } x \in (0, \frac{\pi}{2}],\ y \in (\pi, \frac{3\pi}{2}]; \\ -m\sin x, & \text{if } x \in (\pi, \frac{3\pi}{2}],\ y \in (0, \frac{\pi}{2}]. \end{cases}$$

Then X is an (m, φ)-invex set with respect to η for function $\varphi(x) = x$, $\forall x \in I$, $\forall t \in [0,1]$ and $m = \frac{1}{4}$. It is obvious that X is not a convex set.

According to the above definition, we derive an interesting property of the (m, φ)-invex set. The proof of the following proposition 2.5 is straightforward.

Proposition 2.5. *If K_i, $i \in I = \{1, 2, \ldots, n\}$ is a family of (m, φ)-invex sets in $\mathbb{R}^n$ with respect to the same $\eta : \mathbb{R}^n \times \mathbb{R}^n \times (0, 1] \longrightarrow \mathbb{R}^n$ for some fixed $m \in (0, 1]$ and same arbitrary function φ, then the intersection $\bigcap_{i \in I} K_i$ is an (m, φ)-invex set.*

We next give new definition, to be referred as generalized (s, m, φ)-preinvex Godunova-Levin function of second kind.

Definition 2.6. *Let $K \subseteq \mathbb{R}$ be an open nonempty m-invex set with respect to $\eta : K \times K \times (0, 1] \longrightarrow \mathbb{R}$ and $\varphi : I \longrightarrow K$ a continuous function. For $f : K \longrightarrow \mathbb{R}$, any fixed $s \in [0, 1]$ and some fixed $m \in (0, 1]$, if*

$$f(m\varphi(y) + t\eta(\varphi(x), \varphi(y), m)) \leq \frac{f(\varphi(x))}{t^s} + m\frac{f(\varphi(y))}{(1-t)^s}, \tag{2.1}$$

is valid for all $x, y \in I$, $t \in (0, 1)$, then we say that $f(x)$ is a generalized (s, m, φ)-preinvex Godunova-Levin function of second kind with respect to η or $f \in Q^{\star 2}_{(s,m,\varphi)}$.

Remark 2.7. In Definition 2.6, it is worthwhile to note that generalized (s, m, φ)-preinvex Godunova-Levin function of second kind is an (s, m)-Godunova-Levin functions of second kind on $K = I$ with respect to $\eta(\varphi(x), \varphi(y), m) = \varphi(x) - m\varphi(y)$ and $\varphi(x) = x$, for all $x, y \in I$.

In this section, in order to prove our main results regarding some new integral inequalities involving generalized (s, m, φ)-preinvex Godunova-Levin functions of second kind along with beta function, we need the following new lemma:

Lemma 2.8. *Let $\varphi : I \longrightarrow K$ be a continuous function and $\eta : K \times K \times (0, 1] \longrightarrow \mathbb{R}$. Assume that $f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \longrightarrow \mathbb{R}$ be a continuous function on K° and $\eta(\varphi(b), \varphi(a), m) > 0$. Then for some fixed $m \in (0, 1]$ and any fixed $p, q > 0$, we have*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx$$
$$= \eta^{p+q+1}(\varphi(b), \varphi(a), m) \int_0^1 t^p(1-t)^q f(m\varphi(a) + t\eta(\varphi(b), \varphi(a), m))dt.$$

Proof. It is easy to observe that

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx$$
$$= \eta(\varphi(b), \varphi(a), m) \int_0^1 (m\varphi(a) + t\eta(\varphi(b), \varphi(a), m) - m\varphi(a))^p$$
$$\times (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - m\varphi(a) - t\eta(\varphi(b), \varphi(a), m))^q$$
$$\times f(m\varphi(a) + t\eta(\varphi(b), \varphi(a), m))dt$$

$$= \eta^{p+q+1}(\varphi(b), \varphi(a), m) \int_0^1 t^p (1-t)^q f(m\varphi(a) + t\eta(\varphi(b), \varphi(a), m))dt.$$

□

The following definition will be used in the sequel.

Definition 2.9. *The Euler beta function is defined for $x, y > 0$ as*

$$\beta(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Theorem 2.10. *Let $\varphi : I \longrightarrow K$ be a continuous function and $\eta : K \times K \times (0, 1] \longrightarrow \mathbb{R}$. Assume that $f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \longrightarrow \mathbb{R}$ be a continuous function on K° and $\eta(\varphi(b), \varphi(a), m) > 0$. If $k > 1$ and $|f|^{\frac{k}{k-1}}$ is a generalized (s, m, φ)-preinvex Godunova-Levin function of second kind on K for some fixed $m \in (0, 1]$ and any fixed $s \in [0, 1)$, then for any fixed $p, q > 0$,*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx$$
$$\leq \frac{\eta^{p+q+1}(\varphi(b), \varphi(a), m)}{(1-s)^{\frac{k-1}{k}}} \Big[\beta(kp+1, kq+1)\Big]^{\frac{1}{k}} \Big(m|f(\varphi(a))|^{\frac{k}{k-1}} + |f(\varphi(b))|^{\frac{k}{k-1}}\Big)^{\frac{k-1}{k}}.$$

Proof. Since $|f|^{\frac{k}{k-1}}$ is a generalized (s, m, φ)-preinvex Godunova-Levin function of second kind on K, combining with Lemma 2.8, Definition 2.9, Hölder inequality and properties of the modulus, we get

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx$$
$$\leq |\eta(\varphi(b), \varphi(a), m)|^{p+q+1} \left[\int_0^1 t^{kp}(1-t)^{kq}dt\right]^{\frac{1}{k}}$$
$$\times \left[\int_0^1 \left|f(m\varphi(a) + t\eta(\varphi(b), \varphi(a), m))\right|^{\frac{k}{k-1}} dt\right]^{\frac{k-1}{k}}$$
$$\leq \eta^{p+q+1}(\varphi(b), \varphi(a), m) \Big[\beta(kp+1, kq+1)\Big]^{\frac{1}{k}}$$
$$\times \left[\int_0^1 \left(\frac{m|f(\varphi(a))|^{\frac{k}{k-1}}}{(1-t)^s} + \frac{|f(\varphi(b))|^{\frac{k}{k-1}}}{t^s}\right) dt\right]^{\frac{k-1}{k}}$$
$$= \frac{\eta^{p+q+1}(\varphi(b), \varphi(a), m)}{(1-s)^{\frac{k-1}{k}}} \Big[\beta(kp+1, kq+1)\Big]^{\frac{1}{k}} \Big(m|f(\varphi(a))|^{\frac{k}{k-1}} + |f(\varphi(b))|^{\frac{k}{k-1}}\Big)^{\frac{k-1}{k}}.$$

The proof of Theorem 2.10 is completed. □

Theorem 2.11. *Let $\varphi : I \longrightarrow K$ be a continuous function and $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$. Assume that $f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \longrightarrow \mathbb{R}$ be a continuous function on K° and $\eta(\varphi(b), \varphi(a), m) > 0$. If $l \geq 1$ and $|f|^l$ is a generalized (s, m, φ)-preinvex Godunova-Levin function of second kind on K for some fixed $m \in (0,1]$ and any fixed $s \in [0,1]$, then for any fixed $p, q > 0$,*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx$$
$$\leq \eta^{p+q+1}(\varphi(b), \varphi(a), m)\Big[\beta(p+1, q+1)\Big]^{\frac{l-1}{l}}$$
$$\times \Big[m|f(\varphi(a))|^l \beta\left(p+1, q-s+1\right) + |f(\varphi(b))|^l \beta\left(p-s+1, q+1\right)\Big]^{\frac{1}{l}}.$$

Proof. Since $|f|^l$ is a generalized (s, m, φ)-preinvex Godunova-Levin function of second kind on K, combining with Lemma 2.8, Definition 2.9, the well-known power mean inequality and properties of the modulus, we get

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx$$
$$= \eta^{p+q+1}(\varphi(b), \varphi(a), m)$$
$$\times \int_0^1 \Big[t^p(1-t)^q\Big]^{\frac{l-1}{l}} \Big[t^p(1-t)^q\Big]^{\frac{1}{l}} f(m\varphi(a) + t\eta(\varphi(b), \varphi(a), m))dt$$
$$\leq |\eta(\varphi(b), \varphi(a), m)|^{p+q+1} \left[\int_0^1 t^p(1-t)^q dt\right]^{\frac{l-1}{l}}$$
$$\times \left[\int_0^1 t^p(1-t)^q \big|f(m\varphi(a) + t\eta(\varphi(b), \varphi(a), m))\big|^l dt\right]^{\frac{1}{l}}$$
$$\leq \eta^{p+q+1}(\varphi(b), \varphi(a), m)\Big[\beta(p+1, q+1)\Big]^{\frac{l-1}{l}}$$
$$\times \left[\int_0^1 t^p(1-t)^q \left(\frac{m|f(\varphi(a))|^l}{(1-t)^s} + \frac{|f(\varphi(b))|^l}{t^s}\right) dt\right]^{\frac{1}{l}}$$
$$= \eta^{p+q+1}(\varphi(b), \varphi(a), m)\Big[\beta(p+1, q+1)\Big]^{\frac{l-1}{l}}$$
$$\times \Big[m|f(\varphi(a))|^l \beta\left(p+1, q-s+1\right) + |f(\varphi(b))|^l \beta\left(p-s+1, q+1\right)\Big]^{\frac{1}{l}}.$$

The proof of Theorem 2.11 is completed. □

3. Generalized integral inequalities via fractional integrals

In this section, in order to prove our main results regarding some Hermite-Hadamard type inequalities for generalized (s, m, φ)-preinvex Godunova-Levin function of second kind via fractional integrals, we need the following two new lemmas:

Lemma 3.1. *Let $\varphi : I \longrightarrow K$ be a continuous function. Suppose $K \subseteq \mathbb{R}$ be an open nonempty m-invex subset with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ for some fixed $m \in (0,1]$ and let $a < b$ with $\eta(\varphi(a),\varphi(b),m) > 0$. Assume that $f : K \longrightarrow \mathbb{R}$ is a differentiable function on K° and $f' \in L_1[m\varphi(b), m\varphi(b)+\eta(\varphi(a),\varphi(b),m)]$. Then, for each $x \in [m\varphi(b), m\varphi(b)+\eta(\varphi(a),\varphi(b),m)]$ and $\alpha, \lambda \in (0,1]$, we have*

$$\begin{aligned}
&\frac{(1+\alpha(1-\lambda))f(m\varphi(b)+\eta(\varphi(a),\varphi(b),m))+(1-\alpha(1-\lambda))f(m\varphi(b))}{2}\\
&-\frac{\Gamma(\alpha+1)}{2\eta(\varphi(a),\varphi(b),m)^\alpha}\\
&\times\Big[J^\alpha_{(m\varphi(b))+}f(m\varphi(b)+\eta(\varphi(a),\varphi(b),m))\\
&+J^\alpha_{(m\varphi(b)+\eta(\varphi(a),\varphi(b),m))-}f(m\varphi(b))\Big]\\
&=\frac{\eta(\varphi(a),\varphi(b),m)}{2}\int_0^1 (t^\alpha+\alpha(1-\lambda)\\
&-(1-t)^\alpha)\, f'(m\varphi(b)+t\eta(\varphi(a),\varphi(b),m))dt.
\end{aligned} \tag{3.1}$$

Proof. A simple proof of the equality (3.1) can be done by performing an integration by parts in the integrals from the right side and changing the variable. The details are left to the interested reader. □

Lemma 3.2. *Let $\varphi : I \longrightarrow K$ be a continuous function. Suppose $K \subseteq \mathbb{R}$ be an open nonempty m-invex subset with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ for some fixed $m \in (0,1]$ and let $a < b$ with $\eta(\varphi(a),\varphi(b),m) > 0$ where $\eta(\varphi(a),\varphi(b),m) \neq 0$. Assume that $f : K \longrightarrow \mathbb{R}$ be a twice differentiable function on K° and $f'' \in L_1[m\varphi(b), m\varphi(b)+\eta(\varphi(a),\varphi(b),m)]$. Then, for each $x \in [m\varphi(b), m\varphi(b)+\eta(\varphi(a),\varphi(b),m)]$ and $\alpha \in (0,1]$, we have*

$$\begin{aligned}
&\frac{f(m\varphi(b)+\eta(\varphi(a),\varphi(b),m))+f(m\varphi(b))}{2}-\frac{\Gamma(\alpha+1)}{2\eta(\varphi(a),\varphi(b),m)^\alpha}\\
&\times\Big[J^\alpha_{(m\varphi(b))+}f(m\varphi(b)+\eta(\varphi(a),\varphi(b),m))\\
&+J^\alpha_{(m\varphi(b)+\eta(\varphi(a),\varphi(b),m))-}f(m\varphi(b))\Big]\\
&=\frac{\eta^2(\varphi(a),\varphi(b),m)}{2(\alpha+1)}\int_0^1\left(1-(1-t)^{\alpha+1}-t^{\alpha+1}\right)\\
&\cdot f''(m\varphi(b)+t\eta(\varphi(a),\varphi(b),m))dt.
\end{aligned} \tag{3.2}$$

Proof. A simple proof of the equality (3.2) can be done by performing an integration by parts in the integrals from the right side and using Lemma 3.1 with $\lambda = 1$. The details are left to the interested reader. □

Using Lemma 3.1, the following results can be obtained for the corresponding version for power of the absolute value of the first derivative.

Theorem 3.3. *Let $\varphi : I \longrightarrow A$ be a continuous function. Suppose $A \subseteq \mathbb{R}$ be an open nonempty m-invex subset with respect to $\eta : A \times A \times (0,1] \longrightarrow \mathbb{R}$ for any fixed $s \in [0,1)$, some fixed $m \in (0,1]$ and let $a < b$ with $\eta(\varphi(a), \varphi(b), m) > 0$. Assume that $f : A \longrightarrow \mathbb{R}$ is a differentiable function on A°. If $|f'|^q$ is a generalized (s, m, φ)-preinvex Godunova-Levin function of second kind on $[m\varphi(b), m\varphi(b) + \eta(\varphi(a), \varphi(b), m)]$, $q > 1$, $p^{-1} + q^{-1} = 1$, then for each $x \in [m\varphi(b), m\varphi(b) + \eta(\varphi(a), \varphi(b), m)]$ with $\alpha, \lambda \in (0,1]$, we have*

$$\begin{aligned} &\Bigg| \frac{(1+\alpha(1-\lambda))f(m\varphi(b)+\eta(\varphi(a),\varphi(b),m)) + (1-\alpha(1-\lambda))f(m\varphi(b))}{2} \\ &- \frac{\Gamma(\alpha+1)}{2\eta(\varphi(a),\varphi(b),m)^{\alpha}} \\ &\times \Big[J^{\alpha}_{(m\varphi(b))+} f(m\varphi(b)+\eta(\varphi(a),\varphi(b),m)) \\ &+ J^{\alpha}_{(m\varphi(b)+\eta(\varphi(a),\varphi(b),m))-} f(m\varphi(b)) \Big] \Bigg| \\ &\le \frac{\eta(\varphi(a),\varphi(b),m)}{2} \left[\alpha(1-\lambda) + 2\left(\frac{1}{p\alpha+1}\right)^{\frac{1}{p}} \right] \\ &\cdot \left(\frac{|f'(\varphi(a))|^q + m|f'(\varphi(b))|^q}{1-s} \right)^{\frac{1}{q}}. \end{aligned} \tag{3.3}$$

Proof. Denote

$$\begin{aligned} S_{f,\eta,\varphi}(\alpha,\lambda,m,a,b) &= \frac{\eta(\varphi(a),\varphi(b),m)}{2} \\ &\times \int_0^1 \left(t^{\alpha} + \alpha(1-\lambda) - (1-t)^{\alpha}\right) f'(m\varphi(b) + t\eta(\varphi(a),\varphi(b),m))dt. \end{aligned} \tag{3.4}$$

Suppose that $q > 1$. Using Lemma 3.1, relation (3.4), the fact that $|f'|^q$ is a generalized (s, m, φ)-preinvex Godunova-Levin function of second kind, property of the modulus, Hölder's inequality, Minkowski's inequality and properties of the modulus, we have

$$\begin{aligned} &|S_{f,\eta,\varphi}(\alpha,\lambda,m,a,b)| \\ &\le \frac{|\eta(\varphi(a),\varphi(b),m)|}{2} \left[\int_0^1 \left(t^{\alpha} + \alpha(1-\lambda)\right) |f'(m\varphi(b) + t\eta(\varphi(a),\varphi(b),m))| dt \right. \\ &\left. + \int_0^1 (1-t)^{\alpha} |f'(m\varphi(b) + t\eta(\varphi(a),\varphi(b),m))| dt \right] \\ &\le \frac{\eta(\varphi(a),\varphi(b),m)}{2} \end{aligned}$$

$$\times\left[\left(\int_0^1 (t^\alpha+\alpha(1-\lambda))^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'(m\varphi(b)+t\eta(\varphi(a),\varphi(b),m))|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1 (1-t)^{p\alpha}dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'(m\varphi(b)+t\eta(\varphi(a),\varphi(b),m))|^q dt\right)^{\frac{1}{q}}\right]$$
$$\leq \frac{\eta(\varphi(a),\varphi(b),m)}{2}\left[\left(\int_0^1 t^{p\alpha}dt\right)^{\frac{1}{p}}+\left(\int_0^1 \alpha^p(1-\lambda)^p dt\right)^{\frac{1}{p}}+\left(\int_0^1 (1-t)^{p\alpha}dt\right)^{\frac{1}{p}}\right]$$
$$\times\left(\int_0^1\left(\frac{|f'(\varphi(a))|^q}{t^s}+\frac{m|f'(\varphi(b))|^q}{(1-t)^s}\right)dt\right)^{\frac{1}{q}}$$
$$=\frac{\eta(\varphi(a),\varphi(b),m)}{2}\left[\alpha(1-\lambda)+2\left(\frac{1}{p\alpha+1}\right)^{\frac{1}{p}}\right]\left(\frac{|f'(\varphi(a))|^q+m|f'(\varphi(b))|^q}{1-s}\right)^{\frac{1}{q}}.$$

The proof of Theorem 3.3 is completed. □

Corollary 3.4. *Under the conditions of Theorem 3.3, if we choose $\lambda=m=1$ and $\eta(\varphi(a),\varphi(b),1)=\varphi(a)-\varphi(b)$, we get the following generalized Hermite-Hadamard type inequality for fractional integrals:*

$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{2}-\frac{\Gamma(\alpha+1)}{2(\varphi(a)-\varphi(b))^\alpha}\left[J^\alpha_{\varphi(b)+}f(\varphi(a))+J^\alpha_{\varphi(a)-}f(\varphi(b))\right]\right|$$
$$\leq(\varphi(b)-\varphi(a))\left(\frac{1}{p\alpha+1}\right)^{\frac{1}{p}}\left(\frac{|f'(\varphi(a))|^q+|f'(\varphi(b))|^q}{1-s}\right)^{\frac{1}{q}}. \tag{3.5}$$

Theorem 3.5. *Let $\varphi: I\longrightarrow A$ be a continuous function. Suppose $A\subseteq\mathbb{R}$ be an open nonempty m-invex subset with respect to $\eta: A\times A\times(0,1]\longrightarrow\mathbb{R}$ for any fixed $s\in[0,1)$, some fixed $m\in(0,1]$ and let $a<b$ with $\eta(\varphi(a),\varphi(b),m)>0$. Assume that $f: A\longrightarrow\mathbb{R}$ is a differentiable function on A°. If $|f'|^q$ is a generalized (s,m,φ)-preinvex Godunova-Levin function of second kind on $[m\varphi(b),m\varphi(b)+\eta(\varphi(a),\varphi(b),m)]$, $q\geq 1$, then for each $x\in[m\varphi(b),m\varphi(b)+\eta(\varphi(a),\varphi(b),m)]$ with $\alpha,\lambda\in(0,1]$, we have*

$$\left|\frac{(1+\alpha(1-\lambda))f(m\varphi(b)+\eta(\varphi(a),\varphi(b),m))+(1-\alpha(1-\lambda))f(m\varphi(b))}{2}\right.$$
$$-\frac{\Gamma(\alpha+1)}{2\eta(\varphi(a),\varphi(b),m)^\alpha}$$
$$\left.\times\left[J^\alpha_{(m\varphi(b))+}f(m\varphi(b)+\eta(\varphi(a),\varphi(b),m))+J^\alpha_{(m\varphi(b)+\eta(\varphi(a),\varphi(b),m))-}f(m\varphi(b))\right]\right|$$
$$\leq\frac{\eta(\varphi(a),\varphi(b),m)}{2}$$

$$\times\left[\left(\frac{1}{\alpha+1}+\alpha(1-\lambda)\right)^{1-\frac{1}{q}}\left[|f'(\varphi(a))|^q\left(\frac{1}{\alpha-s+1}+\frac{\alpha(1-\lambda)}{1-s}\right)\right.\right.$$

$$+|f'(\varphi(b))|^q\left(m\beta(\alpha+1,1-s)+\frac{m\alpha(1-\lambda)}{1-s}\right)\Bigg]^{\frac{1}{q}} \tag{3.6}$$

$$\left.+\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left[|f'(\varphi(a))|^q\beta(\alpha+1,1-s)+\frac{m|f'(\varphi(b))|^q}{\alpha-s+1}\right]^{\frac{1}{q}}\right].$$

Proof. Suppose that $q \geq 1$. Using Lemma 3.1, relation (3.4), the fact that $|f'|^q$ is a generalized (s, m, φ)-preinvex Godunova-Levin function of second kind, the well-known power mean inequality and properties of the modulus, we have

$$|S_{f,\eta,\varphi}(\alpha,\lambda,m,a,b)|$$
$$\leq \frac{|\eta(\varphi(a),\varphi(b),m)|}{2}\left[\int_0^1 (t^\alpha+\alpha(1-\lambda))\,|f'(m\varphi(b)+t\eta(\varphi(a),\varphi(b),m))|dt\right.$$
$$\left.+\int_0^1(1-t)^\alpha|f'(m\varphi(b)+t\eta(\varphi(a),\varphi(b),m))|dt\right]$$
$$\leq \frac{\eta(\varphi(a),\varphi(b),m)}{2}\left[\left(\int_0^1(t^\alpha+\alpha(1-\lambda))dt\right)^{1-\frac{1}{q}}\right.$$
$$\times\left(\int_0^1(t^\alpha+\alpha(1-\lambda))|f'(m\varphi(b)+t\eta(\varphi(a),\varphi(b),m))|^q dt\right)^{\frac{1}{q}}$$
$$\left.+\left(\int_0^1(1-t)^\alpha dt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t)^\alpha|f'(m\varphi(b)+t\eta(\varphi(a),\varphi(b),m))|^q dt\right)^{\frac{1}{q}}\right]$$
$$\leq \frac{\eta(\varphi(a),\varphi(b),m)}{2}\left[\left(\frac{1}{\alpha+1}+\alpha(1-\lambda)\right)^{1-\frac{1}{q}}\right.$$
$$\times\left(\int_0^1(t^\alpha+\alpha(1-\lambda))\left(\frac{|f'(\varphi(a))|^q}{t^s}+\frac{m|f'(\varphi(b))|^q}{(1-t)^s}\right)dt\right)^{\frac{1}{q}}$$
$$\left.+\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t)^\alpha\left(\frac{|f'(\varphi(a))|^q}{t^s}+\frac{m|f'(\varphi(b))|^q}{(1-t)^s}\right)dt\right)^{\frac{1}{q}}\right]$$
$$=\frac{\eta(\varphi(a),\varphi(b),m)}{2}\left[\left(\frac{1}{\alpha+1}+\alpha(1-\lambda)\right)^{1-\frac{1}{q}}\left[|f'(\varphi(a))|^q\left(\frac{1}{\alpha-s+1}+\frac{\alpha(1-\lambda)}{1-s}\right)\right.\right.$$
$$+|f'(\varphi(b))|^q\left(m\beta(\alpha+1,1-s)+\frac{m\alpha(1-\lambda)}{1-s}\right)\Bigg]^{\frac{1}{q}}$$

$$+\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left[\left[|f'(\varphi(a))|^q\beta(\alpha+1,1-s)+\frac{m|f'(\varphi(b))|^q}{\alpha-s+1}\right]^{\frac{1}{q}}\right].$$

The proof of Theorem 3.5 is completed. □

Corollary 3.6. *Under the conditions of Theorem 3.5, if we choose $\lambda = m = 1$ and $\eta(\varphi(a), \varphi(b), 1) = \varphi(a) - \varphi(b)$, we get the following generalized Hermite-Hadamard type inequality for fractional integrals:*

$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{2}-\frac{\Gamma(\alpha+1)}{2(\varphi(a)-\varphi(b))^{\alpha}}\left[J^{\alpha}_{\varphi(b)+}f(\varphi(a))+J^{\alpha}_{\varphi(a)-}f(\varphi(b))\right]\right|$$

$$\leq\frac{(\varphi(b)-\varphi(a))}{2}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left[\left(\frac{|f'(\varphi(a))|^q}{\alpha-s+1}+|f'(\varphi(b))|^q\beta(\alpha+1,1-s)\right)^{\frac{1}{q}}\right. \tag{3.7}$$

$$\left.+\left(|f'(\varphi(a))|^q\beta(\alpha+1,1-s)+\frac{|f'(\varphi(b))|^q}{\alpha-s+1}\right)^{\frac{1}{q}}\right].$$

Using Lemma 3.2, the following results can be obtained for the corresponding version for power of the absolute value of the second derivative.

Theorem 3.7. *Let $\varphi : I \longrightarrow A$ be a continuous function. Suppose $A \subseteq \mathbb{R}$ be an open nonempty m-invex subset with respect to $\eta : A \times A \times (0,1] \longrightarrow \mathbb{R}$ for any fixed $s \in [0,1)$, some fixed $m \in (0,1]$ and let $a < b$ with $\eta(\varphi(a), \varphi(b), m) > 0$. Assume that $f : A \longrightarrow \mathbb{R}$ be a twice differentiable function on A°. If $|f''|^q$ is a generalized (s, m, φ)-preinvex Godunova-Levin function of second kind on $[m\varphi(b), m\varphi(b) + \eta(\varphi(a), \varphi(b), m)]$, $q > 1$, $p^{-1} + q^{-1} = 1$, then for each $x \in [m\varphi(b), m\varphi(b) + \eta(\varphi(a), \varphi(b), m)]$ and $\alpha \in (0, 1]$, we have*

$$\left|\frac{f(m\varphi(b)+\eta(\varphi(a),\varphi(b),m))+f(m\varphi(b))}{2}-\frac{\Gamma(\alpha+1)}{2\eta(\varphi(a),\varphi(b),m)^{\alpha}}\right.$$

$$\times\left[J^{\alpha}_{(m\varphi(b))+}f(m\varphi(b)+\eta(\varphi(a),\varphi(b),m))\right.$$

$$\left.\left.+J^{\alpha}_{(m\varphi(b)+\eta(\varphi(a),\varphi(b),m))-}f(m\varphi(b))\right]\right| \tag{3.8}$$

$$\leq\frac{\eta^2(\varphi(a),\varphi(b),m)}{2(\alpha+1)}\left(\frac{p(\alpha+1)-1}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\frac{|f''(\varphi(a))|^q+m|f''(\varphi(b))|^q}{1-s}\right)^{\frac{1}{q}}.$$

Proof. Denote

$$R_{f,\eta,\varphi}(\alpha,m,a,b)=\frac{\eta^2(\varphi(a),\varphi(b),m)}{2(\alpha+1)}$$

$$\times\int_0^1\left(1-(1-t)^{\alpha+1}-t^{\alpha+1}\right)f''(m\varphi(b)+t\eta(\varphi(a),\varphi(b),m))dt. \tag{3.9}$$

Suppose that $q > 1$. Using Lemma 3.2, relation (3.9), the fact that $|f''|^q$ is a generalized (s,m,φ)-preinvex Godunova-Levin function of second kind, Hölder's inequality and properties of the modulus, we have

$$|R_{f,\eta,\varphi}(\alpha,m,a,b)| \leq \frac{\eta^2(\varphi(a),\varphi(b),m)}{2(\alpha+1)}$$
$$\times \int_0^1 \left(1-(1-t)^{\alpha+1}-t^{\alpha+1}\right)|f''(m\varphi(b)+t\eta(\varphi(a),\varphi(b),m))|dt$$
$$\leq \frac{\eta^2(\varphi(a),\varphi(b),m)}{2(\alpha+1)}\left(\int_0^1 \left(1-(1-t)^{\alpha+1}-t^{\alpha+1}\right)^p dt\right)^{\frac{1}{p}}$$
$$\times \left(\int_0^1 |f''(m\varphi(b)+t\eta(\varphi(a),\varphi(b),m))|^q dt\right)^{\frac{1}{q}}$$
$$\leq \frac{\eta^2(\varphi(a),\varphi(b),m)}{2(\alpha+1)}\left(\int_0^1 \left(1-(1-t)^{p(\alpha+1)}-t^{p(\alpha+1)}\right) dt\right)^{\frac{1}{p}}$$
$$\times \left(\int_0^1 \left(\frac{|f''(\varphi(a))|^q}{t^s}+\frac{m|f''(\varphi(b))|^q}{(1-t)^s}\right)dt\right)^{\frac{1}{q}}$$
$$= \frac{\eta^2(\varphi(a),\varphi(b),m)}{2(\alpha+1)}\left(\frac{p(\alpha+1)-1}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\frac{|f''(\varphi(a))|^q+m|f''(\varphi(b))|^q}{1-s}\right)^{\frac{1}{q}}.$$

The proof of Theorem 3.7 is completed. □

Corollary 3.8. *Under the conditions of Theorem 3.7, if we choose $m = 1$ and $\eta(\varphi(a),\varphi(b),1) = \varphi(a) - \varphi(b)$, we get the following generalized Hermite-Hadamard type inequality for fractional integrals:*

$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{2}-\frac{\Gamma(\alpha+1)}{2(\varphi(a)-\varphi(b))^\alpha}\left[J^\alpha_{\varphi(b)+}f(\varphi(a))+J^\alpha_{\varphi(a)-}f(\varphi(b))\right]\right|$$
$$\leq \frac{(\varphi(b)-\varphi(a))^2}{2(\alpha+1)}\left(\frac{p(\alpha+1)-1}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\frac{|f''(\varphi(a))|^q+|f''(\varphi(b))|^q}{1-s}\right)^{\frac{1}{q}}. \tag{3.10}$$

Theorem 3.9. *Let $\varphi : I \longrightarrow A$ be a continuous function. Suppose $A \subseteq \mathbb{R}$ be an open nonempty m-invex subset with respect to $\eta : A\times A\times(0,1] \longrightarrow \mathbb{R}$ for any fixed $s \in [0,1)$, some fixed $m \in (0,1]$ and let $a < b$ with $\eta(\varphi(a),\varphi(b),m) > 0$. Assume that $f : A \longrightarrow \mathbb{R}$ be a twice differentiable function on A°. If $|f''|^q$ is a generalized (s,m,φ)-preinvex Godunova-Levin function of second kind on $[m\varphi(b), m\varphi(b)+\eta(\varphi(a),\varphi(b),m)]$, $q \geq 1$, then for each $x \in [m\varphi(b), m\varphi(b)+\eta(\varphi(a),\varphi(b),m)]$ and $\alpha \in (0,1]$, we have*

$$\left|\frac{f(m\varphi(b)+\eta(\varphi(a),\varphi(b),m))+f(m\varphi(b))}{2}-\frac{\Gamma(\alpha+1)}{2\eta(\varphi(a),\varphi(b),m)^\alpha}\right.$$
$$\left.\times\left[J^\alpha_{(m\varphi(b))+}f(m\varphi(b)+\eta(\varphi(a),\varphi(b),m))+J^\alpha_{(m\varphi(b)+\eta(\varphi(a),\varphi(b),m))-}f(m\varphi(b))\right]\right|$$

$$\leq \frac{\eta^2(\varphi(a),\varphi(b),m)}{2(\alpha+1)}\left(\frac{\alpha}{\alpha+2}\right)^{1-\frac{1}{q}} \tag{3.11}$$
$$\times\left(\frac{\alpha+1}{(1-s)(\alpha-s+2)}-\beta(\alpha+2,1-s)\right)^{\frac{1}{q}}\left[|f''(\varphi(a))|^q+m|f''(\varphi(b))|^q\right]^{\frac{1}{q}}.$$

Proof. Suppose that $q \geq 1$. Using Lemma 3.2, relation (3.9), the fact that $|f''|^q$ is a generalized (s, m, φ)-preinvex Godunova-Levin function of second kind, the well-known power mean inequality and properties of the modulus, we have

$$|R_{f,\eta,\varphi}(\alpha,m,a,b)| \leq \frac{\eta^2(\varphi(a),\varphi(b),m)}{2(\alpha+1)}$$
$$\times\int_0^1\left(1-(1-t)^{\alpha+1}-t^{\alpha+1}\right)|f''(m\varphi(b)+t\eta(\varphi(a),\varphi(b),m))|dt$$
$$\leq \frac{\eta^2(\varphi(a),\varphi(b),m)}{2(\alpha+1)}\left(\int_0^1\left(1-(1-t)^{\alpha+1}-t^{\alpha+1}\right)dt\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\left(1-(1-t)^{\alpha+1}-t^{\alpha+1}\right)|f''(m\varphi(b)+t\eta(\varphi(a),\varphi(b),m))|^q dt\right)^{\frac{1}{q}}$$
$$\leq \frac{\eta^2(\varphi(a),\varphi(b),m)}{2(\alpha+1)}\left(\frac{\alpha}{\alpha+2}\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\left(1-(1-t)^{\alpha+1}-t^{\alpha+1}\right)\left(\frac{|f''(\varphi(a))|^q}{t^s}+\frac{m|f''(\varphi(b))|^q}{(1-t)^s}\right)dt\right)^{\frac{1}{q}}$$
$$= \frac{\eta^2(\varphi(a),\varphi(b),m)}{2(\alpha+1)}\left(\frac{\alpha}{\alpha+2}\right)^{1-\frac{1}{q}}$$
$$\times\left(\frac{\alpha+1}{(1-s)(\alpha-s+2)}-\beta(\alpha+2,1-s)\right)^{\frac{1}{q}}\left[|f''(\varphi(a))|^q+m|f''(\varphi(b))|^q\right]^{\frac{1}{q}}.$$

□

Corollary 3.10. *Under the conditions of Theorem 3.9, if we choose $m = 1$ and $\eta(\varphi(a), \varphi(b), 1) = \varphi(a) - \varphi(b)$, we get the following generalized Hermite-Hadamard type inequality for fractional integrals:*

$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{2}-\frac{\Gamma(\alpha+1)}{2(\varphi(a)-\varphi(b))^\alpha}\left[J^\alpha_{\varphi(b)+}f(\varphi(a))+J^\alpha_{\varphi(a)-}f(\varphi(b))\right]\right|$$
$$\leq \frac{(\varphi(b)-\varphi(a))^2}{2(\alpha+1)}\left(\frac{\alpha}{\alpha+2}\right)^{1-\frac{1}{q}} \tag{3.12}$$
$$\times\left(\frac{\alpha+1}{(1-s)(\alpha-s+2)}-\beta(\alpha+2,1-s)\right)^{\frac{1}{q}}\left[|f''(\varphi(a))|^q+|f''(\varphi(b))|^q\right]^{\frac{1}{q}}.$$

4. Applications to special means

Consider the following special means for arbitrary real numbers α, β ($\alpha \neq \beta$) as follows:

1. The arithmetic mean:

$$A := A(\alpha, \beta) = \frac{\alpha + \beta}{2},$$

2. The harmonic mean:

$$H := H(\alpha, \beta) = \frac{2}{\frac{1}{\alpha} + \frac{1}{\beta}}, \ \alpha, \beta \in \mathbb{R}\backslash\{0\},$$

3. The logarithmic mean:

$$L := L(\alpha, \beta) = \frac{\beta - \alpha}{\ln|\beta| - \ln|\alpha|}; \ \ |\alpha| \neq |\beta|, \ \ \alpha\beta \neq 0,$$

4. The generalized log-mean:

$$L_n := L_n(\alpha, \beta) = \left[\frac{\beta^{n+1} - \alpha^{n+1}}{(n+1)(\beta - \alpha)}\right]^{\frac{1}{n}}; \ \ n \in \mathbb{Z} \setminus \{-1, 0\}, \ \ \alpha \neq \beta.$$

It is well known that L_n is monotonic nondecreasing over $n \in \mathbb{R}$ with $L_{-1} := L$. In particular, we have the following inequality $H \leq L \leq A$. Now, using the theory results in Section 3, we give some applications to special means of real numbers.

Theorem 4.1. *Let $\varphi : I \longrightarrow A$ be a continuous function. Suppose $A \subseteq \mathbb{R}$ be an open nonempty 1-invex subset with respect to $\eta : A \times A \times (0, 1] \longrightarrow \mathbb{R}$ for any fixed $s \in [0, 1)$. Then the following inequality for fractional integrals holds:*

$$\begin{aligned} &\left|\frac{1}{H(\varphi(a), \varphi(b))} - \frac{1}{L(\varphi(a), \varphi(b))}\right| \\ &\leq (\varphi(b) - \varphi(a)) \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{\varphi^{2q}(a) + \varphi^{2q}(b)}{\varphi^{2q}(a)\varphi^{2q}(b)(1-s)}\right)^{\frac{1}{q}}, \end{aligned} \tag{4.1}$$

where $q > 1$, $p^{-1} + q^{-1} = 1$.

Proof. Applying Theorem 3.3 for $f(x) = \frac{1}{x}$, $\eta(\varphi(a), \varphi(b), 1) = \varphi(a) - \varphi(b), \alpha = \lambda = 1$ one can obtain the result immediately. □

Theorem 4.2. *Let $\varphi : I \longrightarrow A$ be a continuous function. Suppose $A \subseteq \mathbb{R}$ be an open nonempty 1-invex subset with respect to $\eta : A \times A \times (0,1] \longrightarrow \mathbb{R}$ for any fixed $s \in [0,1)$ and $n \in \mathbb{Z}\backslash\{-1,0\}$. Then the following inequality for fractional integrals holds:*

$$\left|A(\varphi^n(a),\varphi^n(b)) - L_n^n(\varphi(a),\varphi(b))\right| \leq 2^{\frac{1}{q}-2} n(\varphi(b)-\varphi(a))$$

$$\times\left[\left(\frac{\varphi^{(n-1)q}(a)}{2-s}+\frac{\varphi^{(n-1)q}(b)}{(s-1)(s-2)}\right)^{\frac{1}{q}}+\left(\frac{\varphi^{(n-1)q}(a)}{(s-1)(s-2)}+\frac{\varphi^{(n-1)q}(b)}{2-s}\right)^{\frac{1}{q}}\right], \tag{4.2}$$

where $q \geq 1$.

Proof. Applying Theorem 3.5 for $f(x) = x^n$, $\eta(\varphi(a),\varphi(b),1) = \varphi(a)-\varphi(b)$, $\alpha = \lambda = 1$ one can obtain the result immediately. □

Theorem 4.3. *Let $\varphi : I \longrightarrow A$ be a continuous function. Suppose $A \subseteq \mathbb{R}$ be an open nonempty 1-invex subset with respect to $\eta : A \times A \times (0,1] \longrightarrow \mathbb{R}$ for any fixed $s \in [0,1)$. Then the following inequality for fractional integrals holds:*

$$\left|\frac{1}{H(\varphi(a),\varphi(b))} - \frac{1}{L(\varphi(a),\varphi(b))}\right|$$

$$\leq 2^{\frac{1}{q}-2}(\varphi(b)-\varphi(a))^2\left(\frac{2p-1}{2p+1}\right)^{\frac{1}{p}}\left(\frac{\varphi^{3q}(a)+\varphi^{3q}(b)}{\varphi^{3q}(a)\varphi^{3q}(b)(1-s)}\right)^{\frac{1}{q}}, \tag{4.3}$$

where $q > 1$, $p^{-1}+q^{-1} = 1$.

Proof. Applying Theorem 3.7 for $f(x) = \frac{1}{x}$, $\eta(\varphi(a),\varphi(b),1) = \varphi(a)-\varphi(b)$, $\alpha = 1$ one can obtain the result immediately. □

Theorem 4.4. *Let $\varphi : I \longrightarrow A$ be a continuous function. Suppose $A \subseteq \mathbb{R}$ be an open nonempty 1-invex subset with respect to $\eta : A \times A \times (0,1] \longrightarrow \mathbb{R}$ for any fixed $s \in [0,1)$ and $n \in \mathbb{Z}\backslash\{-1,0\}$. Then the following inequality for fractional integrals holds:*

$$\left|A(\varphi^n(a),\varphi^n(b)) - L_n^n(\varphi(a),\varphi(b))\right| \leq \frac{2^{\frac{1}{q}-2}}{3^{1-\frac{1}{q}}} n(n-1)(\varphi(b)-\varphi(a))^2$$

$$\times\left(\frac{1}{(s-1)(s-3)}+\frac{1}{(s-1)(s-2)(s-3)}\right)^{\frac{1}{q}}\left[\varphi^{(n-2)q}(a)+\varphi^{(n-2)q}(b)\right]^{\frac{1}{q}}, \tag{4.4}$$

where $q \geq 1$.

Proof. Applying Theorem 3.9 for $f(x) = x^n$, $\eta(\varphi(a),\varphi(b),1) = \varphi(a)-\varphi(b)$, $\alpha = 1$ one can obtain the result immediately. □

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