A HIGH-ORDER ACCURACY EXPLICIT DIFFERENCE SCHEME WITH BRANCHING STABILITY FOR SOLVING FOUR-DIMENSIONAL PARABOLIC EQUATIONS

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Abstract. A high-order explicit difference scheme for solving four-dimensional parabolic equations is given. The scheme is constructed by the method of undetermined coefficients, and appropriate parameter is chosen to endow the truncation error of schemes is $O\left(\Delta t^4 + \Delta x^4\right)$. And the new difference scheme is proved to be stable if $r \leq \frac{1}{12}$ with the Fourier analysis method. Finally, the numerical experiment shows the numerical solutions of difference scheme and the exact solutions are matched and the difference scheme is effective.

Keywords: four-dimensional parabolic equations, explicit difference scheme, truncation error.

1. Introduction

Many mathematical models of modern science, technology and engineering can be described by partial differential equation and lots of basic equations themselves in natural science are partial differential equations. People have been always using differential equation to describe, explain and predict all kinds of natural phenomena, which turns out to be successful. However, many partial differential equations dont have analytical solutions that people can only figure out its numerical solution through all kinds of methods. Up till now, differential equations main numerical computation methods are finite difference method and finite element method, meanwhile, there are boundary element, mixed finite element, spectral method and finite volume method etc, among which, finite difference method is still the more effective method to solve the numerical solution of partial differential equation.

The difference method of parabolic equation is a classical problem in numerical solution of partial differential equation. Equations described the motion law of underground flow are mostly parabolic equations, in particular, the motion of underground petroleum and natural gas are the typical examples of using parabolic equation to describe. We can also encounter the following multidimensional parabolic equations in the fields of seepage, diffusion, heat conduction and so on

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial w^2}, 0 < x, y, z, w < 1, t > 0, \tag{1}$$

$$u(x, y, z, w, 0) = \varphi(x, y, z, w), 0 \leq x, y, z, w \leq 1, \tag{2}$$

$$u(0, y, z, w, t) = f_1(y, z, w, t), u(1, y, z, w, t) = f_2(y, z, w, t), 0 \leq y, z, w \leq 1, 0 \leq t \leq T, \tag{3}$$

$$u(x, 0, z, w, t) = g_1(x, z, w, t), u(x, 1, z, w, t) = g_2(x, z, w, t), 0 \leq x, z, w \leq 1, 0 \leq t \leq T, \tag{4}$$

$$u(x, y, 0, w, t) = h_1(x, y, w, t), u(x, y, 1, w, t) = h_2(x, y, w, t), 0 \leq x, y, w \leq 1, 0 \leq t \leq T, \tag{5}$$

$$u(x, y, z, 0, t) = k_1(x, y, z, t), u(x, y, z, 1, t) = k_2(x, y, z, t), 0 \leq x, y, z \leq 1, 0 \leq t \leq T. \tag{6}$$

Where $\varphi, f_1, f_2, g_1, g_2, h_1, h_2, k_1$ and k_2 are sufficiently smooth functions.

Various numerical finite difference schemes have been proposed to solve parabolic problems approximately. For multidimensional problems, the explicit difference scheme and implicit difference scheme are the common finite difference schemes. The implicit difference scheme has the advantage of good stability, but it is needed to solve different linear equations on each time layer which will cost to big computation. The alternating-direction implicit (ADI) difference scheme can overcome these disadvantages.

As we known, the ADI scheme is unconditional stable and only need to solve a sequence of tridiagonal linear systems [1–2]. In recent years, there are many new methods which use ADI scheme to solve the multidimensional parabolic equations [3–9], some of them have the accuracy of $O\left(\Delta t^2 + \Delta x^4\right)$ [3,6,7,8,9].

The explicit difference scheme has worse stability than the implicit difference scheme, but it has the advantage of smaller amount of calculation. The general explicit scheme is the classical explicit scheme with the stability condition of $r \leq \frac{1}{8}$, where $r = \frac{\Delta t}{\Delta x^2}$ is the mesh spacing ratio. Its deficiency is that the accuracy is not high, and its truncation error is $O\left(\Delta t + \Delta x^2\right)$. Recently, there has been a interest in the development and application of explicit difference schemes for the numerical solution of multidimensional parabolic equations [10−12], but the schemes only have the truncation error of $O\left(\Delta t^2 + \Delta x^4\right)$. For four-dimensional situation, in reference [12], Ma constructed an explicit scheme with the truncation error of $O(\Delta t^2 + \Delta x^4)$ and the stability condition is $r < \frac{3}{8}$, the scheme's accuracy is not high enough. This paper presents an explicit scheme for solving Eq.(1), the stability condition is $r \leq \frac{1}{12}$, and the truncation error is $O(\Delta t^4 + \Delta x^4)$, the scheme has higher accuracy than the above schemes.

The remainder of this paper is organized as follows. In Section 2, we construct a three-layer explicit difference scheme with the accuracy of $O(\Delta t^4 + \Delta x^4)$;

In Section 3, by using the Fourier analysis method, it is proved that the difference scheme is stable when $r \leq \frac{1}{12}$. In Section 4, though choosing the proper parameter θ, we obtain a three-level explicit scheme with branching stability. In Section 5, we compare the difference of exact solution and the scheme constructed in this paper with that in the reference [12], and compare the computational efficiency of the two difference schemes with the classical explicit scheme. The results shows that the difference scheme in this paper is effective.

2. Construction of the difference scheme

Let Δt denote the step length of time and $\Delta x = \Delta y = \Delta z = \Delta w$ be the step length of space in the direction of x, y, z, w respectively. We approximate the Eq.(1) using the following difference equation with parameters

$$\begin{aligned} &\Delta_t u_{ijkl}^n + \theta_1 \Delta_t u_{ijkl}^{n-1} + \theta_2 \Delta_t \left(\frac{1}{3}\Diamond\right) u_{ijkl}^{n-1} + \theta_3 \Delta_t \left(\frac{1}{12}\Box\right) u_{ijkl}^{n-1} \\ &= \frac{1}{\Delta x^2}\left[\left(\frac{\theta_4}{12}\Box + \frac{\theta_5}{3}\Diamond\right) u_{ijkl}^n + \left(\frac{\theta_6}{12}\Box + \frac{\theta_7}{3}\Diamond\right) u_{ijkl}^{n-1}\right], \end{aligned} \tag{7}$$

where u_{ijkl}^n denotes the value of u at node $(i\Delta x, j\Delta y, k\Delta z, l\Delta w, n\Delta t)$, $\Delta_t u_{ijkl}^n = \frac{u_{ijkl}^{n+1} - u_{ijkl}^n}{\Delta t}$, and

$$\begin{aligned} \Box u_{ijkl}^n &= \left({}_x\Box + {}_y\Box + {}_z\Box + {}_w\Box\right) u_{ijkl}^n, \\ \Diamond u_{ijkl}^n &= \left({}_x\Diamond + {}_y\Diamond + {}_z\Diamond + {}_w\Diamond\right) u_{ijkl}^n, \\ {}_x\Box u_{jkl}^n &= u_{i,j+1,k+1,l+1}^n + u_{i,j-1,k+1,l+1}^n \\ &\quad + u_{i,j+1,k-1,l+1}^n + u_{i,j-1,k-1,l+1}^n + u_{i,j+1,k+1,l-1}^n \\ &\quad + u_{i,j-1,k+1,l-1}^n + u_{i,j+1,k-1,l-1}^n + u_{i,j-1,k-1,l-1}^n - 8u_{ijkl}^n \\ {}_x\Diamond u_{ijkl}^n &= u_{i,j+1,k,l}^n + u_{i,j-1,k,l}^n + u_{i,j,k+1,l}^n \\ &\quad + u_{i,j,k-1,l}^n + u_{i,j,k,l+1}^n + u_{i,j,k,l-1}^n - 6u_{ijkl}^n, \end{aligned}$$

the rest can be inferred by analogy. $\theta_1 - \theta_7$ are parameters to be determined. A proper choice of undetermined parameters $\theta_1 - \theta_7$ can make difference equation (7) approach Eq.(1), and not only has truncation error with order as high as possible, but also has higher stability.

When the solution of Eq.(1) is smooth enough, we can get the following relation:

$$\frac{\partial^n}{\partial t^n}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial w^2}\right)^m u = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial w^2}\right)^{m+2n} u. \tag{8}$$

Using Taylor's expansion of u at node $(i\Delta x, j\Delta y, k\Delta z, l\Delta w, n\Delta t)$, we have

$$
\begin{aligned}
\Delta_t u_{ijkl}^n &= \frac{\partial u}{\partial x} + \frac{\Delta t}{2}\frac{\partial^2 u}{\partial x^2} + \frac{\Delta t^2}{6}\frac{\partial^3 u}{\partial x^3} + O\left(\Delta t^3\right) \\
\Delta_t u_{ijkl}^{n-1} &= \frac{\partial u}{\partial x} - \frac{\Delta t}{2}\frac{\partial^2 u}{\partial x^2} + \frac{\Delta t^2}{6}\frac{\partial^3 u}{\partial x^3} + O\left(\Delta t^3\right) \\
\Diamond u_{ijkl}^n &= 3\Delta x^2 \frac{\partial u}{\partial t} + \frac{\Delta x^4}{4}\frac{\partial^2 u}{\partial t^2} - \frac{\Delta x^4}{2}\left(\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial x^2 \partial z^2}\right. \\
&\quad \left. + \frac{\partial^4 u}{\partial x^2 \partial w^2} + \frac{\partial^4 u}{\partial y^2 \partial z^2} + \frac{\partial^4 u}{\partial y^2 \partial w^2} + \frac{\partial^4 u}{\partial z^2 \partial w^2}\right) + O\left(\Delta x^6\right) \\
\Box u_{ijkl}^n &= 12\Delta x^2 \frac{\partial u}{\partial t} + \Delta x^4 \frac{\partial^2 u}{\partial t^2} + 2\Delta x^4\left(\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial x^2 \partial z^2}\right. \\
&\quad \left. + \frac{\partial^4 u}{\partial x^2 \partial w^2} + \frac{\partial^4 u}{\partial y^2 \partial z^2} + \frac{\partial^4 u}{\partial y^2 \partial w^2} + \frac{\partial^4 u}{\partial z^2 \partial w^2}\right) + O\left(\Delta x^6\right).
\end{aligned}
$$

Substituting the above Taylor expansions into (7) and using relation (8), we can obtain

$$
\begin{aligned}
&(1+\theta_1)\frac{\partial u}{\partial t} + \frac{\Delta t}{2}(1-\theta_1)\frac{\partial^2 u}{\partial t^2} + \frac{\Delta t^2}{6}(1+\theta_1)\frac{\partial^3 u}{\partial t^3} + \frac{\Delta t^3}{24}(1-\theta_1)\frac{\partial^4 u}{\partial t^4} \\
&+ \Delta x^2(\theta_2+\theta_3)\frac{\partial^2 u}{\partial t^2} - \frac{\Delta t \Delta x^2}{2}(\theta_2+\theta_3)\frac{\partial^3 u}{\partial t^3} + \frac{\Delta t^2 \Delta x^2}{6}(\theta_2+\theta_3)\frac{\partial^4 u}{\partial t^4} \\
&= (\theta_4+\theta_5+\theta_6+\theta_7)\frac{\partial u}{\partial t} + \frac{\Delta x^2}{12}(\theta_4+\theta_5+\theta_6+\theta_7)\frac{\partial^2 u}{\partial t^2} - \Delta t(\theta_6+\theta_7)\frac{\partial^2 u}{\partial t^2} \\
&+ \frac{\Delta x^2}{6}\left[(\theta_4+\theta_6)-(\theta_5+\theta_7)\right]\left(\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial x^2 \partial z^2} + \frac{\partial^4 u}{\partial x^2 \partial w^2}\right. \\
&\left. + \frac{\partial^4 u}{\partial y^2 \partial z^2} + \frac{\partial^4 u}{\partial y^2 \partial w^2} + \frac{\partial^4 u}{\partial z^2 \partial w^2}\right) \\
&+ \frac{\Delta t^2}{2}(\theta_6+\theta_7)\frac{\partial^3 u}{\partial t^3} - \frac{\Delta t \Delta x^2}{12}(\theta_6+\theta_7)\frac{\partial^3 u}{\partial t^3} + \frac{\Delta t^2 \Delta x^2}{24}(\theta_6+\theta_7)\frac{\partial^4 u}{\partial t^4} \\
&- \frac{\Delta t \Delta x^2}{6}(\theta_6-\theta_7)\left(\frac{\partial^5 u}{\partial x^2 \partial y^2 \partial t} + \frac{\partial^5 u}{\partial x^2 \partial z^2 \partial t} + \frac{\partial^5 u}{\partial x^2 \partial w^2 \partial t}\right. \\
&\left. + \frac{\partial^5 u}{\partial y^2 \partial z^2 \partial t} + \frac{\partial^5 u}{\partial y^2 \partial w^2 \partial t} + \frac{\partial^5 u}{\partial z^2 \partial w^2 \partial t}\right) \\
&+ \frac{\Delta t^2 \Delta x^2}{12}(\theta_6-\theta_7)\left(\frac{\partial^6 u}{\partial x^2 \partial y^2 \partial t^2} + \frac{\partial^6 u}{\partial x^2 \partial z^2 \partial t^2} + \frac{\partial^6 u}{\partial x^2 \partial w^2 \partial t^2}\right. \\
&\left. + \frac{\partial^6 u}{\partial y^2 \partial z^2 \partial t^2} + \frac{\partial^6 u}{\partial y^2 \partial w^2 \partial t^2} + \frac{\partial^6 u}{\partial z^2 \partial w^2 \partial t^2}\right) \\
&- \frac{\Delta t^3}{6}(\theta_6+\theta_7)\frac{\partial^4 u}{\partial t^4} + O\left(\Delta t^4 + \Delta x^4\right).
\end{aligned}
$$

In order to make the truncation error of scheme (7) getting to $O\left(\Delta t^4+\Delta x^4\right)$, the following equation system should be available.

$$
\left\{\begin{array}{l}
1+\theta_1=\theta_4+\theta_5+\theta_6+\theta_7 \\
\frac{r}{2}\left(1-\theta_1\right)+\theta_2+\theta_3=\frac{1}{12}\left(\theta_4+\theta_5+\theta_6+\theta_7\right)-r\left(\theta_6+\theta_7\right) \\
\theta_4+\theta_6-\theta_5-\theta_7=0 \\
\frac{r^2}{6}\left(1+\theta_1\right)-\frac{r}{2}\left(\theta_2+\theta_3\right)=\frac{r^2}{2}\left(\theta_6+\theta_7\right)-\frac{r}{12}\left(\theta_6+\theta_7\right) \\
\frac{r^3}{24}\left(1-\theta_1\right)+\frac{r^2}{6}\left(\theta_2+\theta_3\right)=\frac{r^2}{24}\left(\theta_6+\theta_7\right)-\frac{r^3}{6}\left(\theta_6+\theta_7\right) \\
\theta_6-\theta_7=0.
\end{array}\right. \tag{9}
$$

Where $r=\frac{\Delta t}{\Delta x^2}$. Let $\theta_3=\theta$, the solution of the above equation system is:

$$\theta_1{=}24r-1;\theta_2{=}-24r^3+6r^2+r-\theta;\theta_4{=}\theta_5=9r-12r^2;\theta_6{=}\theta_7=12r^2+3r.$$

Substituting the above values into (7), we obtain the following single parameter three-level explicit difference scheme with its truncation error getting to $O\left(\Delta t^4+\Delta x^4\right)$.

$$
\begin{aligned}
12u_{ijkl}^{n+1} &= [24(1-2r)+(9r^2-12r^3-\theta)\Box+(48r^3+12r^2-4r+4\theta)\Diamond]u_{ijkl}^n \\
&+[12(24r-1)+(12r^3+3r^2+\theta)\Box+(-48r^3+36r^2+4r-4\theta)\Diamond]u_{ijkl}^{n-1}.
\end{aligned} \tag{10}
$$

3. Analysis of stability

According to Fourier method for analyzing stability. The two-level equation system equivalent to (10) is

$$
\left\{\begin{aligned}
u_{ijkl}^{n+1} &= \frac{24(1-2r)+(9r^2-12r^3-\theta)\Box+(48r^3+12r^2-4r+4\theta)\Diamond}{12}u_{ijkl}^n \\
&+\frac{12(24r-1)+(12r^3+3r^2+\theta)\Box+(-48r^3+36r^2+4r-4\theta)\Diamond}{12}v_{ijkl}^n \\
v_{ijkl}^{n+1} &= u_{ijkl}^n.
\end{aligned}\right. \tag{11}
$$

Let

$$u_{ijkl}^n=U^n e^{I(i\theta+j\varphi+k\psi+l\zeta)}, v_{ijkl}^n=V^n e^{I(i\theta+j\varphi+k\psi+l\zeta)}, \tag{12}$$

where $I=\sqrt{-1}$. And through simple calculation, we have

$$\Box u_{ijkl}^n=-4s_2u_{ijkl}^n, \Diamond u_{ijkl}^n=-12s_1u_{ijkl}^n, \tag{13}$$

where

$$
\begin{aligned}
s_1 &= \sin^2\frac{\theta}{2} + \sin^2\frac{\varphi}{2} + \sin^2\frac{\psi}{2} + \sin^2\frac{\zeta}{2} \in [0,4] \\
s_2 &= \sin^2\frac{\theta+\varphi+\psi}{2} + \sin^2\frac{\theta-\varphi+\psi}{2} + \sin^2\frac{\theta+\varphi-\psi}{2} \\
&+ \sin^2\frac{\theta-\varphi-\psi}{2} + \sin^2\frac{\theta+\varphi+\zeta}{2} + \sin^2\frac{\theta+\varphi-\zeta}{2} + \sin^2\frac{\theta-\varphi+\zeta}{2} \\
&+ \sin^2\frac{\theta-\varphi-\zeta}{2} + \sin^2\frac{\theta+\psi+\zeta}{2} + \sin^2\frac{\theta+\psi-\zeta}{2} + \sin^2\frac{\theta-\psi+\zeta}{2} + \sin^2\frac{\theta-\psi-\zeta}{2} \\
&+ \sin^2\frac{\varphi+\psi+\zeta}{2} + \sin^2\frac{\varphi+\psi-\zeta}{2} + \sin^2\frac{\varphi-\psi+\zeta}{2} + \sin^2\frac{\varphi-\psi-\zeta}{2} \in [0,16]\,.
\end{aligned}
$$

Substituting (12) into (11) and using (13) we obtain

$$
\begin{bmatrix} U^{k+1} \\ V^{k+1} \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} U^{k} \\ V^{k} \end{bmatrix} = \mathbf{G}\left(\mathbf{s_1},\mathbf{s_2}\right) \begin{bmatrix} U^{k} \\ V^{k} \end{bmatrix}
$$

where

$$
g_{11} = 2\left(1-12r\right) - \left(3r^2 - 4r^3 - \frac{\theta}{3}\right) s_2 - 4\left(12r^3 + 3r^2 - r + \theta\right) s_1,
$$

$$
g_{12} = 24r - 1 - \left(4r^3 + r^2 + \frac{\theta}{3}\right) s_2 - 4\left(-12r^3 + 9r^2 + r - \theta\right) s_1, g_{21} = 1, g_{22} = 0.
$$

The characteristic equation of propagation matrix $\mathbf{G}\left(\mathbf{s_1},\mathbf{s_2}\right)$ is

$$
\lambda^2 - g_{11}\lambda - g_{12} = 0. \tag{14}
$$

Lemma 1 ([13])**.** *The two roots of real coefficient quadratic Eq.(14) are less than or equal to 1 in norm if and only if*

$$
|g_{11}| \le 1 - g_{12} \le 2. \tag{15}
$$

Lemma 2 ([13])**.** *The difference scheme* (10) *is stable, i.e., the family of matrices* $\mathbf{G^n}\left(\mathbf{s_1},\mathbf{s_2}\right)\left((s_1,s_2) \in [0,4] \times [0,16]\,, n = 1,2,\cdots\right)$ *is uniformly bounded if and only if*

(1) $|\lambda_{1,2}| \le 1$ ($\lambda_{1,2}$ are roots of (14));

(2) (s_1, s_2) which assures $1 - g_{11}^2/4 = g_{11}^2 + 4g_{12} = 0$ is not existent or not in the region of $[0,4] \times [0,16]$.

Theorem 1. *A sufficient condition for scheme* (10) *being stable is*

$$
r \le \frac{1}{12}, \max\left\{-12r^3 + 3r^2, -12r^3 + 15r^2 - \frac{3}{4}r\right\} \le \theta \le -12r^3 + 3r^2 + r. \tag{16}
$$

Proof. If $g_{12} \neq -1, 1 - g_{11}^2/4 = g_{11}^2 + 4g_{12} = 0$ do not hold for any (s_1, s_2). By Lemma 1 and Lemma 2, the stability conditions of scheme (10) become

$$-1 + g_{12} \leq g_{11} \leq 1 - g_{12} < 2.$$

From $g_{11} \leq 1 - g_{12}$, we have

$$-4r^2 s_2 - 48r^2 s_1 \leq 0. \tag{17}$$

it is hold unconditionally.

Because $1 - g_{12} < 2$, we have

$$24r - \left(4r^3 + r^2 + \frac{\theta}{3}\right) s_2 - 4\left(-12r^3 + 9r^2 + r - \theta\right) s_1 > 0. \tag{18}$$

A sufficient condition which assures the above inequality hold is

$$4r^3 + r^2 + \frac{\theta}{3} \geq 0, \tag{19}$$

$$-12r^3 + 9r^2 + r - \theta \geq 0, \tag{20}$$

$$24r - 16\left(4r^3 + r^2 + \frac{\theta}{3}\right) - 16\left(-12r^3 + 9r^2 + r - \theta\right) > 0, \tag{21}$$

it is equivalent to

$$-12r^3 - 3r^2 \leq \theta \leq -12r^3 + 9r^2 + r, \tag{22}$$

$$\theta > -12r^3 + 15r^2 - \frac{3}{4}r. \tag{23}$$

Using $-1 + g_{12} \leq g_{11}$, we obtain

$$3(16r - 1) - \left(8r^3 - 2r^2 + \frac{2}{3}\theta\right) s_2 - 4\left(-24r^3 + 6r^2 + 2r - 2\theta\right) s_1 \leq 1. \tag{24}$$

A sufficient condition which assures the above inequality hold is

$$8r^3 - 2r^2 + \frac{2}{3}\theta \geq 0, \tag{25}$$

$$-24r^3 + 6r^2 + 2r - 2\theta \geq 0, \tag{26}$$

$$3(16r - 1) \leq 1, \tag{27}$$

it is equivalent to

$$-12r^3 + 3r^2 \leq \theta \leq -12r^3 + 3r^2 + r, \tag{28}$$

$$r \leq \frac{1}{12}. \tag{29}$$

By combining the inequalities (22),(23),(28) and (29), we complete the proof.

4. Choice of parameter and determination of difference scheme

We should choose parameters θ such that (16) is satisfied. Provide methods as follows:

If $-12r^3+3r^2 \geq -12r^3+15r^2-\frac{3}{4}r$, we have $r \leq \frac{1}{16}$, now $-12r^3+3r^2 \leq \theta \leq -12r^3+3r^2+r$. In particular, take $\theta = -12r^3+3r^2$, we obtain a three-level explicit scheme as follow:

$$\begin{aligned} 6u_{ijkl}^{n+1} &= \left[12(1-2r)+3r^2\Box+\left(12r^2-2r\right)\Diamond\right]u_{ijkl}^{n} \\ &+\left[6(24r-1)+3r^2\Box+\left(12r^2+2r\right)\Diamond\right]u_{ijkl}^{n-1}. \end{aligned} \tag{30}$$

The above scheme is stable if $r \leq \frac{1}{16}$, and the truncation error is $O(\Delta t^4+\Delta x^4)$.

If $-12r^3+3r^2 \leq -12r^3+15r^2-\frac{3}{4}r$, we have $r \geq \frac{1}{16}$. Consider the inequality (29), we obtain $\frac{1}{16} \leq r \leq \frac{1}{12}$. And when $\frac{1}{16} \leq r \leq \frac{1}{12}$, the inequality $-12r^3+15r^2-\frac{3}{4}r \leq -12r^3+3r^2+r$ holds. Now $-12r^3+15r^2-\frac{3}{4}r \leq \theta \leq -12r^3+3r^2+r$. In particular, take $\theta = -12r^3+15r^2-\frac{3}{4}r$, we obtain a three-level explicit scheme as follow:

$$\begin{aligned} 12u_{ijkl}^{n+1} &= \left[24(1-2r)+\left(-6r^2+\frac{3}{4}r\right)\Box+\left(72r^2-7r\right)\Diamond\right]u_{ijkl}^{n} \\ &+\left[12(24r-1)+\left(18r^2-\frac{3}{4}r\right)\Box+\left(-24r^2+7r\right)\Diamond\right]u_{ijkl}^{n-1}. \end{aligned} \tag{31}$$

The above scheme is stable if $\frac{1}{16} \leq r \leq \frac{1}{12}$, and the truncation is $O(\Delta t^4+\Delta x^4)$. If we use schemes (30) and (31) simultaneously, a three-level explicit scheme is constructed which is stable for arbitrary $0 < r \leq \frac{1}{12}$ and its truncation error is $O\left(\Delta t^4+\Delta x^4\right)$. Since the above two schemes are the same when $r = \frac{1}{16}$, we call them an explicit difference scheme with branching stability.

5. Numerical experiment

Consider initial and boundary value problem as follows:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}+\dfrac{\partial^2 u}{\partial w^2}, (0<x,y,z,w<1, t>0) \\ u(x,y,z,w,0) = \sin(x+y+z+w), (0 \leq x,y,z,w \leq 1) \\ u(0,y,z,w,t) = e^{-4t}\sin(y+z+w), u(1,y,z,w,t) \\ = e^{-4t}\sin(1+y+z+w), (0 \leq y,z,w \leq 1, t \geq 0) \\ u(x,0,z,w,t) = e^{-4t}\sin(x+z+w), u(x,1,z,w,t) \\ = e^{-4t}\sin(x+1+z+w), (0 \leq x,z,w \leq 1, t \geq 0) \\ u(x,y,0,w,t) = e^{-4t}\sin(x+y+w), u(x,y,1,w,t) \\ = e^{-4t}\sin(x+y+1+w), (0 \leq x,y,w \leq 1, t \geq 0) \\ u(x,y,z,0,t) = e^{-4t}\sin(x+y+z), u(x,y,z,1,t) \\ = e^{-4t}\sin(x+y+z+1), (0 \leq x,y,z \leq 1, t \geq 0). \end{cases} \tag{32}$$

Taking $\Delta x = \Delta y = \Delta z = \Delta w = 0.1, \Delta t = r\Delta x^2 = r/100, r = 1/18, 1/16, 1/15, 1/12$. For convenience, we use the exact solution of (32) $u(x, y, z, w, t) = e^{-4t} \sin(x + y + z + w)$ to calculate the value of the first level u^1_{ijkl}.

Table 1 shows the comparison of the exact solutions and the scheme constructed in this paper and that in the reference [12] at the time strata n=100. From Table 1, one can easily see that numerical results of schemes (30) and (31) are completely identical with theoretical analysis, it has higher accuracy than the scheme in reference [12]. Table 2 shows the comparison of the efficiency of the scheme constructed in this paper and the classical scheme one and that in the reference [12]. The result is tested by the unit of second. From table 2, we can see that the computational efficiency of three schemes are similar. Among these three schemes, the classical explicit scheme has the highest computational efficiency, the second one is the scheme in this paper, and the lowest is the one constructed in the reference [12]. From the result of these two tables, we can see that the scheme constructed in this paper is a high accuracy and efficiency difference scheme.

Table 1 Comparison of calculating results among difference schemes with exact solution

r	result	(x, y, z, w)			
		$(0.1, 0.1, 0.1, 0.1)$	$(0.3, 0.3, 0.3, 0.3)$	$(0.5, 0.5, 0.5, 0.5)$	$(0.7, 0.7, 0.7, 0.7)$
$\frac{1}{18}$	exact solution	0.311 821 832	0.746 318 557	0.728 108 460	0.268 237 541
	scheme (30)	0.311 821 790	0.746 318 060	0.728 107 769	0.268 237 293
	reference [12]	0.311 821 775	0.746 317 889	0.728 107 532	0.268 237 208
$\frac{1}{16}$	exact solution	0.303 279 309	0.725 872 770	0.708 161 548	0.260 889 033
	scheme (30)	0.303 279 269	0.725 872 281	0.708 160 857	0.260 888 784
	reference [12]	0.303 279 256	0.725 872 117	0.708 160 626	0.260 888 701
$\frac{1}{15}$	exact solution	0.298 266 543	0.713 875 148	0.696 456 667	0.256 576 917
	scheme (31)	0.298 266 507	0.713 874 702	0.696 456 031	0.256 576 687
	reference [12]	0.298 266 491	0.713 874 508	0.696 455 756	0.256 576 588
$\frac{1}{12}$	exact solution	0.279 030 435	0.667 835 187	0.651 540 076	0.240 029 498
	scheme (31)	0.279 030 413	0.667 834 911	0.651 539 673	0.240 029 351
	reference [12]	0.279 030 390	0.667 834 619	0.651 539 249	0.240 029 197

Table 2 Comparison of the calculation efficiency among three kinds of difference schemes (unit:s)

difference scheme	$r = \frac{1}{16}$,n=20	$r = \frac{1}{12}$,n=20	$r = \frac{1}{16}$,n=50	$r = \frac{1}{12}$,n=50
classical explicit scheme	74.382 486	75.544 137	184.658 025	181.476 046
schemes (30) and (31)	76.473 674	76.003 696	184.740 697	182.145 808
reference [12] scheme	77.470 928	76.269 907	185.713 634	183.290 512

6. Conclusions

In this paper, we proposed a high-order accuracy explicit difference scheme with branching stability for solving four-dimensional parabolic problems. The stable character of the scheme is which has been verified by a discrete Fourier analysis. The scheme which proposed in this paper is fourth-order accurate in space and fourth-order accurate in time and allows a considerable saving in computing time. Numerical examples are given to test its high accuracy and to show its superiority over some other schemes in terms of accuracy and computational costs.

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