A BAYESIAN METHOD TO FIT AN ARMA MODEL

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Abstract. The method of time series analysis is widely used in many fields of science, engineering, finance and economics etc, and fitting a time series model accurately is the important basis of time series analysis. Based on the Bayesian statistical theory, this paper presents a Bayesian method which can identify an ARMA (autoregressive moving-average) model and estimate the model parameters simultaneously. Firstly, in order to determine the orders of the ARMA model, an identification model with the recognition variables is constructed. Moreover, the problem of determining the orders of the ARMA model is transformed into two sets of hypothesis tests. By the principle of Bayesian hypothesis testing, it is suggested to solve the above hypothesis test problems by calculating the posterior probabilities of the hypotheses. However, due to the large number of the unknown parameters in the identification model, this paper proposes to obtain the samples by Gibbs sampling and then calculate the posterior probabilities of the hypotheses, the AR coefficients, the MA coefficients and the variance of the random errors to fit the ARMA model. Finally, in order to illustrate the good performance of

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the method proposed in this article, we design three simulation examples and compare the results of our method with two existing methods: RJMCMC method and EACF method. It can be found clearly that the method proposed in this paper has more accurate results for fitting an ARMA model.

Keywords: time series, ARMA model, model fitting, Bayesian statistic, Gibbs sampling.

1. Introduction

Time series analysis is a kind of method of dynamic data analysis which is widely used in various fields, like science, engineering, finance and economics etc [1-3]. Using the time series analysis method, we must know the internal mechanism of generating the time series data, establish the accurate time series model to forecast the future values. Therefore, the establishment of an accurate time series model on the basis of the time series data is the most fundamental and critical part of the time series analysis. For the identification of the time series ARMA model, many domestic and foreign scholars have done a lot of works and got a wealth of research results. The identification methods of the time series ARMA model can be divided into two categories. One is the non-Bayesian method, for example, Tsay and Tiao (1984) determined the orders of the ARMA model by using the EACF (extended autocorrelation function) method [4]; In 1985, Tsay and Tiao proposed a canonical correlation approach for determining the orders of the ARMA model [5], but these two methods cannot determine the orders of the ARMA model accurately and do not think about the estimation of the model parameters. The other is the Bayesian method. Such as Schwarz (1978) putted forward a BIC criteria to identify the ARMA model [6], however this method is cumbersome; Ong et al. (2005) suggested to use the genetic algorithm to identify the ARIMA model [7], and this method includes the BIC theory and it is also cumbersome; Ehlers and Brooks (2004) illustrated a RJ (reversible jump) MCMC approach to select the orders of the ARIMA model and estimate the parameters simultaneously [8], but the method has an imprecise result. Therefore, it is necessary to establish a more accurate method of identifying the ARMA model. Based on the Bayesian statistical theory, this paper presents a Bayesian method which can identify the ARMA model and estimate the model parameters simultaneously for fitting the ARMA model exactly. The rest of the paper is organized as follows. In section 2, an identification model based on the recognition variables is established for fitting the ARMA model, and the problem of the ARMA model identification is transformed into two sets of hypothesis tests. What's more, we proposed to solve the problems of hypothesis tests by the Bayesian statistical theory. Section 3 provides the conditional posterior distributions of the unknown parameters to calculate the posterior probabilities of the hypotheses and estimate the AR coefficients, the MA coefficients and the variance of the random errors based on the Gibbs sampling. In section 4, a Bayesian method of fitting an ARMA model on the basis of the Gibbs sampling is presented. Section 5 shows the better performances of the method proposed in this article comparing with the other existing approaches by some simulating examples. Finally, some conclusions are given in section 6.

2. The criterion of ARMA model identification

In general, the ARMA (p, q) model [1-3] is:

$$\begin{cases} \phi(B)z_t = \theta(B)\varepsilon_t \\ \varepsilon_t \;\; i.i.d \;\; N(0,\sigma^2) \end{cases} \tag{2.1}$$

where, $\{x_t\}$ is the time series data, which can be recorded as $X{*}{=}(x_1, x_2, \cdots, x_n)^T$, $\phi(B) = I - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p$, $\theta(B) = I - \theta_1 B - \theta_2 B^2 - \cdots - \theta_q B^q$, B is a backshift operator such that $B^k x_t = x_{t-k}$, p and q are the autoregressive order and the moving-average order of the model respectively, $\Phi = (\phi_1, \phi_2, \cdots, \phi_p)^T$ and $\Theta = (\theta_1, \theta_2, \cdots, \theta_q)^T$ are the autoregressive coefficients and the moving-average coefficients of the model, respectively. $\{\varepsilon_t\}$ is a sequence of the independent random errors identically distributed $N(0, \sigma^2)$. To ensure the ARMA (p,q) model being stationary and invertible, assume that all of the zeros of $\phi(B) = I - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p$ and $\theta(B) = I - \theta_1 B - \theta_2 B^2 - \cdots - \theta_q B^q$ are on or outside the unite circle.

When we use the ARMA model to fit the time series data $\{x_t\}$, the model orders p, q, the unknown parameters $\Phi = (\phi_1, \phi_2, \cdots, \phi_p)^T$, $\Theta = (\theta_1, \theta_2, \cdots, \theta_q)^T$ and the variance σ^2 of the random errors need to be determined.

Firstly, assume a larger family M of the ARMA models, that is $M = \{ARMA(a, b), a = 0, 1, \cdots, e; b = 0, 1, \cdots f\}$. By the recognition variables [12], an identification model is proposed as follows:

$$\begin{cases} x_t = k_1\phi_1 x_{t-1} + k_2\phi_2 x_{t-1} + \cdots k_e\phi_e x_{t-e} \\ +\varepsilon_t - l_1\theta_1\varepsilon_{t-1} - l_2\theta_2\varepsilon_{t-2} - \cdots - l_f\theta_f\varepsilon_{t-f} \\ \varepsilon_t \;\; i.i.d \;\; N(0,\sigma^2) \end{cases} \tag{2.2}$$

to determine the orders of the ARMA model, where $K = (k_1, k_2, \cdots k_e)$ are the recognition variables for the autoregressive items, $L = (l_1, l_2, \cdots l_f)$ are the recognition variables for the moving-average items, and the values of $k_i, i = 1, 2, \cdots e$ and $l_j, j = 1, 2, \cdots f$ only can be 0 or 1. If $k_i{=}\,1(i = 1, 2, \cdots e)$, the ARMA model includes the i-th autoregressive item $\phi_i x_{t-i}$, otherwise, the ARMA model does not include the i-th autoregressive item $\phi_i x_{t-i}$; if $l_j{=}\,1(j = 1, 2, \cdots e)$, the ARMA model includes the j-th moving-average item $\phi_j \varepsilon_{t-j}$, otherwise, the ARMA model does not include the j-th moving-average item $\phi_j \varepsilon_{t-j}$.

To determine the orders of the ARMA model, we construct the following two sets of hypothesis test questions:

$$H^A_{1,i} : k_i = 1, \qquad H^A_{2,i} : k_i = 0 \qquad (i = 1, 2, \cdots, e) \tag{2.3}$$

(2.4) $H_{1,j}^M : l_j = 1, \qquad H_{2,j}^M : l_j = 0 \qquad (j = 1, 2, \cdots, f)$

For each of the above hypothesis test questions, based on the Bayesian statistical theory [13-17], after $X = (x_{m+1}, x_{m+2}, \cdots, x_n)^T$ $(m = \max(e, f), m << n)$ is got, we can calculate the posterior probabilities of the hypotheses:

$$P(H_{1,i}^A|X), P(H_{2,i}^A|X), P(H_{1,j}^M|X) \text{ and } P(H_{2,j}^M|X).$$

If $P(H_{1,i}^A|X) \geq P(H_{2,i}^A|X)$, the hypothesis $H_{1,i}^A$ is accepted, otherwise the hypothesis $H_{2,i}^A$ is accepted; Similarly, if $P(H_{1,j}^M|X) \geq P(H_{2,j}^M|X)$, the hypothesis $H_{1,j}^M$ is accepted, otherwise the hypothesis $H_{2,j}^M$ is accepted.

3. Caculate the conditional posterior distributions of the unknown parameters

Because the autoregressive coefficients, the moving-average coefficients and the variance of the random errors in the identification model (2.2) are unknown, it is difficult to calculate directly the posterior probabilities of the hypotheses. Therefore we need to use the Gibbs sampling method [18-20] to get the samples based on the condition posterior distributions of the unknown parameters in the identification model, and then calculate the posterior probabilities of the hypotheses so as to determine the orders of the ARMA model and estimate the unknown parameters.

3.1 Determine the prior distributions of the unknown parameters

When calculating the conditional posterior distributions of the unknown parameters, the priori distributions of the unknown parameters are required. By the selection methods of the prior distribution [12], the prior distributions of the unknown parameters are given as follows.

Because the values of $k_i, i = 1, 2, \cdots e$ and $l_j, j = 1, 2, \cdots f$ only can be 0 or 1, we take the Bernoulli distribution as the prior distributions of them, $k_i \sim b(1, \alpha_1), i = 1, 2, \cdots e$, $l_j \sim b(1, \alpha_2), j = 1, 2, \cdots f$, where α_1 and α_2 are the hyper parameters that are determined in advance.

To the autoregressive coefficients $\Phi = (\phi_1, \phi_2, \cdots, \phi_e)^T$ and the moving-average coefficients $\Theta = (\theta_1, \theta_2, \cdots, \theta_f)^T$, we take the multiple normal distribution as the prior distributions of them, $\Phi \sim N_e(\Phi_0, \Sigma_1)$, $\Theta \sim N_f(\Theta_0, \Sigma_2)$, where $\Phi_0, \Sigma_1, \Theta_0$ and Σ_2 are the hyper parameters that are determined in advance, and Φ_0 is an e-dimensional column vector, Θ_0 is an f-dimensional column vector, Σ_1 is an $e \times e$ nonnegative definite matrix, Σ_2 is an $f \times f$ nonnegative definite matrix.

To the variance σ^2 of the random errors, we take the inverted gamma distribution as the prior distribution of it, $\sigma^2 \sim IG(v, \lambda)$, where v and λ are the hyper parameters that are determined in advance.

3.2 Calculate the conditional posterior distributions of the unknown parameters

The joint probability density function of $X = (x_{m+1}, x_{m+2}, \cdots, x_n)^T (m = \max(e, f))$ has to be calculated before we calculate the conditional posterior distribution of every unknown parameter, and it is

$$p(X|\Phi, \Theta, K, L) \propto \exp\{-\frac{1}{2\sigma^2} \sum_{t=m+1}^{n} (x_t - k_1\phi_1 x_{t-1} - \cdots - k_e\phi_e x_{t-e} + l_1\theta_1\varepsilon_{t-1} + \cdots l_f\theta_f\varepsilon_{t-f})^2\}$$

By the Bayesian formula proposed earlier by J. O. Berger (1985) and P. E. Rossi and G. M. Allenby (2005), the conditional posterior distribution of every unknown parameter is calculated as follows:

(1) The conditional posterior distribution of $k_i, i = 1, 2, \cdots e$ is

$$k_i|X, K_{(-i)}, L, \Phi, \Theta, \sigma^2 \sim b(1, \beta_i^A) \tag{3.1}$$

where, $K_{(-i)} = (k_1, k_2, \cdots, k_{i-1}, k_{i+1}, \cdots, k_e)$, $\beta_i^A = \frac{p_{2,i}^A}{p_{1,i}^A + p_{2,i}^A}$,

$$\begin{aligned} p_{1,i}^A &= p(k_i = 0|X, K_{(-i)}, L, \Phi, \Theta) \\ &= (1 - \alpha_1) \exp\{-\frac{1}{2\sigma^2} \sum_{t=m+1}^{n} (x_t - k_1\phi_1 x_{t-1} - \cdots - k_e\phi_e x_{t-e} \\ &+ l_1\theta_1\varepsilon_{t-1} + \cdots l_f\theta_f\varepsilon_{t-f})^2\} \\ p_{2,i}^A &= p(k_i = 1|X, K_{(-i)}, L, \Phi, \Theta) \\ &= \alpha_1 \exp\{-\frac{1}{2\sigma^2} \sum_{t=m+1}^{n} (x_t - k_1\phi_1 x_{t-1} - \cdots - k_e\phi_e x_{t-e} \\ &+ l_1\theta_1\varepsilon_{t-1} + \cdots l_f\theta_f\varepsilon_{t-f})^2\} \end{aligned}$$

(2) The conditional posterior distribution of $l_j, j = 1, 2, \cdots f$ is

$$l_j|X, K, L_{(-j)}, \Phi, \Theta, \sigma^2 \sim b(1, \beta_j^M) \tag{3.2}$$

where, $L_{(-j)} = (l_1, l_2, \cdots, l_{j-1}, l_{j+1}, \cdots, l_f)$, $\beta_j^M = \frac{p_{2,j}^M}{p_{1,j}^M + p_{2,j}^M}$,

$$\begin{aligned} p_{1,j}^M &= p(l_j = 0|X, K, L_{(-j)}, \Phi, \Theta) \\ &= (1 - \alpha_2) \exp\{-\frac{1}{2\sigma^2} \sum_{t=m+1}^{n} (x_t - k_1\phi_1 x_{t-1} - \cdots - k_e\phi_e x_{t-e} \\ &+ l_1\theta_1\varepsilon_{t-1} + \cdots l_f\theta_f\varepsilon_{t-f})^2\}, \end{aligned}$$

$$
\begin{aligned}
p_{2,j}^M &= p(l_j = 1|X, K, L_{(-j)}, \Phi, \Theta) \\
&= \alpha_2 \exp\{-\frac{1}{2\sigma^2} \sum_{t=m+1}^{n} (x_t - k_1\phi_1 x_{t-1} - \cdots - k_e\phi_e x_{t-e} \\
&+ l_1\theta_1\varepsilon_{t-1} + \cdots l_f\theta_f\varepsilon_{t-f})^2\}
\end{aligned}
$$

(3) After $K = (k_1, k_2, \cdots k_e)$ and $L = (l_1, l_2, \cdots l_f)$ are determined, the orders p and q of the ARMA model also can be determined. And then we can calculate the conditional posterior distribution of Φ_p which consists of the first p components in Φ,

$$
\Phi_p|X, K, L, \Theta, \sigma^2 \sim N_p(\hat{\Phi}_0, \hat{\Sigma}_1) \tag{3.3}
$$

where,

$$
\begin{aligned}
\hat{\Sigma}_1 &= [\frac{1}{\sigma^2} \sum_{t=m+1}^{n} X_{(-t)} X_{(-t)}^T + \Sigma_1^*]^{-1}, \\
\hat{\Phi}_0 &= \hat{\Sigma}_1[\frac{1}{\sigma^2} \sum_{t=m+1}^{n} X_{(-t)}(x_t + \Theta^T\varepsilon_{(-t)}) + (\Sigma_1^*)^{-1}\Phi_0^*], \\
X_{(-t)} &= (x_{(t-1)}, x_{(t-2)}, \cdots, x_{(t-p)})^T
\end{aligned}
$$

and $\varepsilon_{(-t)} = (\varepsilon_{(t-1)}, \varepsilon_{(t-2)}, \cdots, \varepsilon_{(t-q)})^T$. Φ_0^* is a p-dimensional column vector which consists of the first p components in Φ_0, Φ_0 is an e-dimensional column vector as described previously, and $\Phi_0 = (\varphi_1^{(0)}, \varphi_2^{(0)}, \cdots, \varphi_p^{(0)}, \varphi_{p+1}^{(0)}, \cdots, \varphi_e^{(0)})^T$, $\Phi_0^* = (\varphi_1^{(0)}, \varphi_2^{(0)}, \cdots, \varphi_p^{(0)})^T$. What's more, Σ_1^* is a $p \times p$ nonnegative definite matrix which consists of the top p rows and top p columns components in Σ_1, Σ_1 is an $e \times e$ nonnegative definite matrix as described previously, and

$$
\Sigma_1 = \begin{pmatrix}
\sigma_{1,1}^1 & \sigma_{1,2}^1 & \cdots & \sigma_{1,p}^1 & \sigma_{1,p+1}^1 & \cdots & \sigma_{1,e}^1 \\
\sigma_{2,1}^1 & \sigma_{2,2}^1 & \cdots & \sigma_{2,p}^1 & \sigma_{2,p+1}^1 & \cdots & \sigma_{2,e}^1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\sigma_{p,1}^1 & \sigma_{p,2}^1 & \cdots & \sigma_{p,p}^1 & \sigma_{p,p+1}^1 & \cdots & \sigma_{p,e}^1 \\
\sigma_{p+1,1}^1 & \sigma_{p+1,2}^1 & \cdots & \sigma_{p+1,p}^1 & \sigma_{p+1,p+1}^1 & \cdots & \sigma_{p+1,e}^{1)} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\sigma_{e,1}^{(1)} & \sigma_{e,2}^{(1)} & \cdots & \sigma_{e,p}^{(1)} & \sigma_{e,p+1}^{(1)} & \cdots & \sigma_{e,e}^{(1)}
\end{pmatrix},
$$

$$
\Sigma_1^* = \begin{pmatrix}
\sigma_{1,1}^{(1)} & \sigma_{1,2}^{(1)} & \cdots & \sigma_{1,p}^{(1)} \\
\sigma_{2,1}^{(1)} & \sigma_{2,2}^{(1)} & \cdots & \sigma_{2,p}^{(1)} \\
\vdots & \vdots & \ddots & \vdots \\
\sigma_{p,1}^{(1)} & \sigma_{p,2}^{(1)} & \cdots & \sigma_{p,p}^{(1)}
\end{pmatrix}
$$

(4) After the orders p and q of the ARMA model be determined, we can calculate the conditional posterior distribution of Θ_q which consists of the first q components in Θ,

$$\Theta_q|X,K,L,\Phi,\sigma^2 \sim N_q(\hat{\Theta}_0,\hat{\Sigma}_2) \tag{3.4}$$

where, $\hat{\Sigma}_2=[\frac{1}{\sigma^2}\sum_{t=m+1}^{n}\varepsilon_{(-t)}\varepsilon_{(-t)}^T+\Sigma_2^*]^{-1}$, $\hat{\Phi}_0=\hat{\Sigma}_2[(\Sigma_2^*)^{-1}\Theta_0^*-\frac{1}{\sigma^2}\sum_{t=m+1}^{n}\varepsilon_{(-t)}(x_t-\Phi^T x_{(-t)})]$. Θ_0^* is a q-dimensional column vector which consists of the first q components in Θ_0, Θ_0 is an f-dimensional column vector as described previously, and $\Theta_0=(\theta_1^{(0)},\theta_2^{(0)},\cdots,\theta_q^{(0)},\theta_{q+1}^{(0)},\cdots,\theta_f^{(0)})^T$, $\Theta_0^*=(\theta_1^{(0)},\theta_2^{(0)},\cdots,\theta_q^{(0)})^T$. What is more, Σ_2^* is a $q\times q$ nonnegative definite matrix which consists of the top q rows and top q columns components in Σ_2, Σ_2 is an $f\times f$ nonnegative definite matrix as described previously, and

$$\Sigma_2=\begin{pmatrix} \sigma_{1,1}^{(2)} & \sigma_{1,2}^{(2)} & \cdots & \sigma_{1,q}^{(2)} & \sigma_{1,q+1}^{(2)} & \cdots & \sigma_{1,f}^{(2)} \\ \sigma_{2,1}^{(2)} & \sigma_{2,2}^{(2)} & \cdots & \sigma_{2,q}^{(2)} & \sigma_{2,q+1}^{(2)} & \cdots & \sigma_{2,f}^{(2)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{q,1}^{(2)} & \sigma_{q,2}^{(2)} & \cdots & \sigma_{q,q}^{(2)} & \sigma_{q,q+1}^{(2)} & \cdots & \sigma_{q,f}^{(2)} \\ \sigma_{q+1,1}^{(2)} & \sigma_{q+1,2}^{(2)} & \cdots & \sigma_{q+1,q}^{(2)} & \sigma_{q+1,q+1}^{2} & \cdots & \sigma_{q+1,f}^{2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{f,1}^{2} & \sigma_{f,2}^{2} & \cdots & \sigma_{f,q}^{2} & \sigma_{f,q+1}^{2} & \cdots & \sigma_{f,f}^{2} \end{pmatrix},$$

$$\Sigma_2^*=\begin{pmatrix} \sigma_{1,1}^{(2)} & \sigma_{1,2}^{(2)} & \cdots & \sigma_{1,q}^{(2)} \\ \sigma_{2,1}^{(2)} & \sigma_{2,2}^{(2)} & \cdots & \sigma_{2,q}^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{q,1}^{(2)} & \sigma_{q,2}^{(2)} & \cdots & \sigma_{q,q}^{(2)} \end{pmatrix}$$

(5) The conditional posterior distribution of σ^2 can be calculated as follows:

$$\sigma^2|X,K,L,\Phi_p,\Theta_q \sim IG(\hat{\nu},\hat{\lambda}) \tag{3.5}$$

where, $\hat{v}=v-\frac{n-m}{2}$, $\hat{\lambda}=\lambda+\frac{1}{2}\sum_{t=m+1}^{n}(x_t-\Phi_p^T X_{(-t)}+\Theta_q^T\varepsilon_{(-t)})^2$. The orders of $\Phi_p,\Theta_q,X_{(-t)}$ and $\varepsilon_{(-t)}$ change with p and q.

4. The implementation of the Bayesian method of fitting an ARMA model

The implementation of the Bayesian method of fitting an ARMA model is given as follows:

Step 1: Choose the hyper parameters $\alpha_1,\alpha_2,\Phi_0,\Sigma_1,\Theta_0,\Sigma_2,v,\lambda$.

Step 2: Choose the initial values $K^{(0)},L^{(0)},\Phi^{(0)},\Theta^{(0)},(\sigma^2)^{(0)}$.

Step 3: Implement the Gibbs sampling as follows and get the Gibbs samples. Suppose that the (r-1)-th sample $(K^{(r-1)}, L^{(r-1)}, \Phi^{(r-1)}, \Theta^{(r-1)}, (\sigma^2)^{(r-1)})$ has been acquired. Then, the r-th sample can be obtained by the following procedure:

Step 3.1: Determine p and q based on $K^{(r-1)}$ and $L^{(r-1)}$.

Step 3.2: Obtain $\Phi_p^{(r)}$ from $\Phi_p|X, K^{(r-1)}, L^{(r-1)}, \Theta^{(r-1)}, (\sigma^2)^{(r-1)}$, and make the previous p components of $\Phi^{(r)}$ equal to the corresponding components in $\Phi_p^{(r)}$, and make the last e-p components of $\Phi^{(r)}$ equal to the corresponding components in $\Phi^{(r-1)}$.

Step 3.3: Obtain $\Theta_q^{(r)}$ from $\Theta_q|X, K^{(r-1)}, L^{(r-1)}, \Phi^{(r)}, (\sigma^2)^{(r-1)}$, and make the previous q components of $\Theta^{(r)}$ equal to the corresponding components in $\Theta_q^{(r)}$, and make the last f-q components of $\Theta^{(r)}$ equal to the corresponding components in $\Theta^{(r-1)}$.

Step 3.4: Obtain $(\sigma^2)^{(r)}$ from $\sigma^2|X, K^{(r-1)}, L^{(r-1)}, \Phi^{(r)}, \Theta^{(r)}$.

Step 3.5: Calculate $(\beta_i^A)^{(r)}$ and obtain $k_i^{(r)}$ from $k_i|X, K_{(-i)}^{(r,r-1)}, L^{(r-1)}$, $\Phi^{(r)}, \Theta^{(r)}, (\sigma^2)^{(r)}$, where $K_{(-i)}^{(r,r-1)} = (k_1^{(r)}, k_2^{(r)}, \cdots k_{i-1}^{(r)}, k_{i+1}^{(r-1)}, \cdots k_e^{(r-1)})$.

Step 3.6: Calculate $(\beta_j^M)^{(r)}$ and obtain $l_j^{(r)}$ from $l_j|X, L_{(-j)}^{(r,r-1)}, K^{(r)}$, $\Phi^{(r)}$, $\Theta^{(r)}$, $(\sigma^2)^{(r)}$, where $L_{(-j)}^{(r,r-1)} = (l_1^{(r)}, l_2^{(r)}, \cdots l_{j-1}^{(r)}, l_{j+1}^{(r-1)}, \cdots l_f^{(r-1)})$.

Implement and end the iterative procedure after the Gibbs sampling is convergent.

Step 4: Make the Bayesian inference and identify the ARMA model. Supposing that N samples are acquired and the Gibbs sampling is convergent after acquiring the M-th sample, we use the last N-M samples to make the following Bayesian inference.

Step 4.1. Determine the model orders p and q. By the above Gibbs sampling, we use the following formulas to calculate the posterior probabilities of the hypotheses

$$P(H_{1,i}^A) = p(k_i = 1|X) = \frac{1}{N-M} \sum_{r=M+1}^{N} (\beta_i^A)^{(r)}; \tag{4.1}$$

$$P(H_{2,i}^A) = p(k_i = 0|X) = \frac{1}{N-M} \sum_{r=M+1}^{N} [1 - (\beta_i^A)^{(r)}]; \tag{4.2}$$

$$P(H_{1,j}^M) = p(l_j = 1|X) = \frac{1}{N-M} \sum_{r=M+1}^{N} (\beta_j^M)^{(r)}; \tag{4.3}$$

$$P(H_{2,j}^M) = p(l_j = 0|X) = \frac{1}{N-M} \sum_{r=M+1}^{N} [1 - (\beta_j^M)^{(r)}]. \tag{4.4}$$

If $P(H_{1,i}^A) \geq P(H_{2,i}^A)$, the hypothesis $H_{1,i}^A$ will be accepted, otherwise the hypothesis $H_{2,i}^A$ is accepted; If $P(H_{1,j}^M) \geq P(H_{2,j}^M)$, the hypothesis $H_{1,j}^M$ will be accepted, otherwise the hypothesis $H_{2,j}^{M}$ is accepted. And we can determine the model orders by $p = \max\{i, k_i = 1\}$ and $q = \max\{j, l_j = 1\}$.

Step 4.2. Estimate the unknown parameters Φ, Θ and σ^2. After the orders p and q of the ARMA model are determined, we use the following formulas to estimate the unknown parameters Φ, Θ and σ^2.

$$\Phi = \frac{1}{N-M} \sum_{r=M+1}^{N} \Phi^{(r)}, \tag{4.5}$$

$$\Theta = \frac{1}{N-M} \sum_{r=M+1}^{N} \Theta^{(r)}, \tag{4.6}$$

$$\sigma^2 = \frac{1}{N-M} \sum_{r=M+1}^{N} (\sigma^2)^{(r)}. \tag{4.7}$$

5. Examples and analysis

In order to illustrate the performance of the Bayesian approach for fitting the ARMA model, three examples are designed as follows.

Example 1. Get 100 observations from the model ARMA(2,2):

$$\begin{cases} x_t = 0.5x_{t-1} + 0.4x_{t-2} + \varepsilon_t - 0.3\varepsilon_{t-1} \\ \qquad -0.2\varepsilon_{t-2} \\ \varepsilon_t i.i.d N(0,1), \end{cases}$$

use the Bayesian method proposed in this article to fit the model and compare with the existing RJMCMC method [8] and EACF method [4]. The results are shown in Table 1 and Table 2.

It can be found clearly that the model is determined as ARMA(2,2) by the Bayesian method proposed in this article, and the AR coefficients are estimated as $\hat{\phi}_1 = 0.5183$ and $\hat{\phi}_2 = 0.4307$, the MA coefficients are estimated as $\hat{\theta}_1 = 0.3135$ and $\hat{\theta}_2 = 0.2515$, and the variance σ^2 of the random errors is estimated as $\hat{\sigma}^2 = 0.8926$. So, by the method proposed in this article, the model can be identified correctly and the model parameters can be estimated accurately.

However, when we use the RJMCMC method to identify the model, it is identified as ARMA (3,2), so the RJMCMC method has an imprecise result of model identification. Similarly, the EACF method also has an imprecise result of model identification because the model is identified as ARMA (1,2) by it.

Example 2. Get three groups of data from ARMA(2,1) and ARMA(3,2), respectively, and the sizes of three groups of data are 25, 50 and 100, and then we use the Bayesian method proposed in this article, the RJMCMC method and the EACF method to identify the model. Repeat 100 times of the above experiments and the results are shown in Table 3.

From the experimental results, we can find that the Bayesian method proposed in this article has a higher correct recognition rate to model than two existing methods in most cases. Whats more, the correct recognition rate to model is increase with the size of data.

Example 3. To show the effect of estimating parameters, we design some experiments as follows: Get two groups of data from ARMA(1,1), ARMA(2,2) and ARMA(3,2) respectively, and the sizes of two groups of data are 50 and 100, and then we use the Bayesian method proposed in this paper and the RJMCMC method to identify the model and estimate the model parameters. The comparative results of estimating parameters are shown in Table 4. From the experimental results, we can find that the Bayesian method proposed in this article has a higher accuracy for estimating parameters than the RJMCMC method. In particularly, the model parameters could be estimated precisely by the Bayesian method based on the data whose size is 100.

6. Conclusions

Firstly, an identification model for fitting the ARMA model is established based on the method of recognition variable, and the ARMA model identification problem is reduced to two sets of hypothesis test questions for recognition variables.

TABLE 1: Comparison of two methods

The method proposed in this article		RJMCMC method	
Lag items	The posterior probabilities	Lag items	The posterior probabilities
x_{t-1}	0.9354	x_{t-1}	0.8470
x_{t-2}	0.8610	x_{t-2}	0.7885
x_{t-3}	0.1588	x_{t-3}	0.5799
x_{t-4}	0.0942	x_{t-4}	0.2518
ε_{t-1}	0.8668	ε_{t-1}	0.3799
ε_{t-2}	0.7780	ε_{t-2}	0.6856
ε_{t-3}	0.1200	ε_{t-3}	0.0925
ε_{t-4}	0.0002	ε_{t-4}	0.1021

Then, based on the Bayesian statistical theory, this paper proposes to solve the above hypothesis test by calculating the posterior probabilities of the hy-

TABLE 2: The results of model identification by the EACF method

AR \ MA	0	1	2	3	4	5	6
0	x	x	x	x	x	x	x
1	x	x	O	O	O	O	O
2	x	x	O	O	O	O	O
3	x	O	O	O	O	O	O
4	x	O	O	O	O	O	O
5	x	x	O	O	O	O	O
6	x	O	O	O	O	O	O

TABLE 3: The correct recognition rate of three methods

Model	n	Bayesian method	RJMCMC method	EACF method
ARMA (2,1)	25	68%	46%	52%
	50	55%	51%	56%
	100	76%	67%	72%
ARMA (3,2)	25	30%	18%	26%
	50	35%	32%	33%
	100	43%	38%	40%

TABLE 4: Comparison of estimating parameters

Model and parameters	n	Bayesian method	RJMCMC method
ARMA (1,1): $\phi_1 = 0.5$, $\theta_1 = 0.3$	50	$\phi_1 = 0.436$, $\theta_1 = 0.386$	$\phi_1 = 0.602$, $\theta_1 = 0.212$
	100	$\phi_1 = 0.503$, $\theta_1 = 0.291$	$\phi_1 = 0.552$, $\theta_1 = 0.366$
ARMA (2,2): $\phi_1 = 0.5$, $\phi_2 = 0.3$, $\theta_1 = 0.3$, $\theta_2 = 0.2$	50	$\phi_1 = 0.565, \phi_2 = 0.352$, $\theta_1 = 0.336, \theta_2 = 0.326$	$\phi_1 = 0.683, \phi_2 = 0.366$, $\theta_1 = 0.236, \theta_2 = 0.469$
	100	$\phi_1 = 0.523, \phi_2 = 0.313$, $\theta_1 = 0.306, \theta_2 = 0.236$	$\phi_1 = 0.556, \phi_2 = 0.353$ $\theta_1 = 0.312, \theta_2 = 0.252$
ARMA (3,2): $\phi_1 = 0.5$, $\phi_2 = 0.3$, $\phi_3 = 0.1$, $\theta_1 = 0.3$, $\theta_2 = 0.2$	50	$\phi_1 = 0.566, \phi_2 = 0.386$, $\phi_3 = 0.165$ $\theta_1 = 0.396, \theta_2 = 0.232$	$\phi_1 = 0.623, \phi_2 = 0.529$, $\phi_3 = 0.003$ $\theta_1 = 0.231, \theta_2 = 0.168$
	100	$\phi_1 = 0.523, \phi_2 = 0.333$, $\phi_3 = 0.113$ $\theta_1 = 0.328, \theta_2 = 0.201$	$\phi_1 = 0.596, \phi_2 = 0.389$, $\phi_3 = 0.018$ $\theta_1 = 0.263, \theta_2 = 0.187$

potheses. However, since the autoregressive coefficients, the moving-average coefficients and the variance of the random errors in the identification model are unknown, the posterior probabilities of the hypotheses cannot be calculated directly. Therefore, this paper proposes to use the Gibbs sampling method to get the samples from the conditional posterior distribution of the unknown parameters, and then calculate the posterior probabilities of the hypotheses to determine the model orders and estimate the unknown parameters in the model. Finally, in order to show the good performance of Bayesian method proposed in this article, we design three simulation examples and compare the Bayesian method proposed in this paper with the two existing methods: RJMCMC method and EACF method. And some advantages of the Bayesian method proposed in this article can be found as follows: (i) The Bayesian method proposed in this article can identify the ARMA model more accurately than the existing methods. And it is obvious that our method has a larger correct recognition rate of the ARMA model than these existing methods. (ii) The Bayesian method proposed in this article can estimate the autoregressive coefficients, the moving-average coefficients and the variance of the random errors in the model accurately at the same time as identifying the ARMA model.

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References

[1] R. H. Shumway and D. S. Stoffer, *Time Series Analysis and Its Applications with R Examples,* Springer, 2006.

[2] R. S. Tsay, *Analysis of Financial Time Series,* Wiley, 2002.

[3] M. Memmedli and O. Ozdemir, An application of fuzzy time series to improve ISE forecasting, *Wseas Transactions on Mathematics,* Vol.9, No.1, 2010, pp. 12-21.

[4] Y. Z. Cai and N. Davies, A Simple Diagnostic Method of Outlier Detection for Stationary Gaussian Time Series, *Journal of Applied Statistics,* Vol.30, No.2, 2003, pp. 205-223.

[5] V. Karioti, *Theory and Practice of Risk Assessment,* Springer Proceedings in Mathematics & Statistics, 2015.

[6] R. S. Tsay, Outliers, Level Shifts, and Variance Changes in Time Series, *Journal of Forecasting,* Vol.7, No.1, 1988, pp. 1-20.

[7] K. Choy, Outlier Detection for Stationary Time Series, *Journal of Statistical Planning and Inference,* Vol.99, No.2, 2001, pp. 111-127.

[8] B. Abraham and G. E. P. Box, Bayesian Analysis of Some Outlier Problems in Time Series, *Biometrika,* Vol.66, No.2, 1979, pp. 229-236.

[9] R. E. Mcculloch and R. S. Tsay, Bayesian Analysis of Autoregressive Time Series via the Gibbs Sampler, *Journal of Time Series Analysis,* Vol.15, No.2, 1994, pp. 235-250.

[10] D. Pena and P. Galeano, *Economic Time Series: Modeling and Seasonality,* Chapman & Hall, 2012.

[11] R. Baragona, F. Battaglia and D. Cucina, Empirical Likelihood for Outlier Detection and Estimation in Autoregressive Time Series, *Journal of Time Series Analysis,* Vol.37, No.3, 2016, pp. 315-336.

[12] J. Lu, L. Shi and F. Chen, Outlier Detection in Time Series Models Using Local Influence Method, *Communication in Statistics- Theory and Methods,* Vol.41, No.12, 2012, pp. 2202-2220.

[13] H. Louni, Outlier Detection in ARMA Models, *Journal of time series analysis,* Vol.29, No.6, 2008, pp. 1057-1065.

[14] S. James, *Subjective and Objective Bayesian Statistics: Principles, Models, and Applications,* John Wiley & Sons, 2002.

[15] J. Lund and M. Rudemo, Generalised Gibbs Sampler and Multigrid Monte Carlo for Bayesian Computation, *Biometrika,* Vol.87, No.2, 2000, pp. 353-369.

[16] J. O. Berger, *Statistical Decision Theory and Bayesian Analysis,* Wiley, 1985.

[17] A. E. Gelfand, Gibbs Sampling, *Journal of the American Statistical Association,* Vol.95, No.452, 2000, pp. 1300-1304.

[18] N. Chopin and S. S. Singh, On particle Gibbs sampling, *Bernoulli,* Vol.21, No.3, 2015, pp. 1855-1883.

[19] B. C. Sutradhar and A. J. Oyet, Maximum studentized score tests for the detection of outliers in time series regression models, *Journal of Statistical Computation & Simulation,* Vol.80, No.12, 2010, pp. 1355-1372.

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