NORMAL EDGE-TRANSITIVE CAYLEY GRAPHS WHOSE ORDER ARE A PRODUCT OF THREE PRIMES

Modjtaba Ghorbani*
Mahin Songhori
Mina Rajabi Parsa
Department of Mathematics
Faculty of Science
Shahid Rajaee
Teacher Training University
Tehran, 16785-136
I. R. Iran
mghorbani@srttu.edu

Abstract. The Cayley graph $X = Cay(G, S)$ on group G with respect to connection set S is normal edge-transitive, if $N_{Aut(X)}(R(G))$ acts transitively on edge set. In this paper we determine the structure of automorphism group of all tetravalent normal edge-transitive Cayley graphs of order pqr.

Keywords: wreath product, normal edge-transitive Cayley graph, automorphism group.

1. Introduction

All graphs considered here are finite, simple, undirected and connected. The vertex set and the edge set of graph Γ are denoted by $V(\Gamma)$ and $E(\Gamma)$, respectively. An automorphism of graph Γ is a mapping from the vertices of Γ back to vertices of Γ such that the resulting graph is isomorphic with Γ. The set of all automorphisms of Γ under the composition of mappings forms a group known as the automorphism group denoted by $Aut(\Gamma)$. Among all graphs, determining the automorphism group of Cayley graphs is a very difficult task. Thus, the majority of this research has been focused on normal edge-transitive Cayley graphs. Due to the complexity, this article only briefly examines properties of Cayley graphs of order pqr, where p, q, r are prime numbers.

Here, in the next section, we give the necessary definitions and some preliminary results. Section three contains main results of this paper and the automorphism group of Cayley graphs of order pqr is verified in this section.

*. Corresponding author

2. Definitions and preliminaries

The non-empty set $S \subseteq G$ is symmetric connecting set if S generates the group, $1 \notin S$ and $S = S^{-1}$. The Cayley graph $X = Cay(G, S)$ has the vertex set $V(X) = G$ and edge set $E(X) = \{(g, sg) | g \in G, s \in S\}$. Let S be a generating set of G, it is a well-known fact that X is connected.

Let $R(G) = \{\rho_g : G \to G, \rho_g(x) = xg, \forall x \in G\}$ and $Aut(G, S) = \{\alpha \in Aut(G), \alpha(S) = S\}$, then $Aut(G, S)$ is a subgroup of $Aut(G)$ which fixes the subset S. Let $A = Aut(X)$, the normalizer of $R(G)$ in A is equal to

$$N_A(R(G)) = R(G) \rtimes Aut(G, S),$$

where $\rtimes$ denotes the semidirect product of two groups. The Cayley graph X is called normal edge-transitive or normal arc-transitive if $N_A(R(G))$ acts transitively on the set of edges or arcs of X, respectively. The Cayley graph $X = Cay(G, S)$ is said to be normal if $R(G)$ is normal in $Aut(X)$, for more details on normal Cayley graphs see [2, 4, 6, 12, 14, 15, 16].

In [2, 3] authors studied respectively normal edge-transitivity of Cayley graphs on abelian groups of valency at most five and edge-transitivity of Cayley graphs of valency four on non-abelian simple groups. In [5, 13] authors obtained all normal edge-transitive Cayley graphs of valency four and in [12] the authors classified all connected tetravalent non-normal arc-transitive Cayley graphs on dihedral groups. Darafsheh et al. in [4] studied the normal edge-transitive Cayley graphs on non-abelian groups of order $4p$, where p is a prime number and Ghorbani et al. in [8] studied hexavelant normal edge-transitive Cayley graphs on groups of order pqr. The main results of this paper are based on three following fundamental results:

Proposition 2.1 ([2, 14])**.** *Let $\Gamma = Cay(G, S)$ be a connected Cayley graph. Then Γ is normal edge-transitive if and only if $Aut(G, S)$ is either transitive on S or has two orbits in S in the form of T and T^{-1}, where T is a non-empty subset of S and $S = T \cup T^{-1}$.*

Corollary 2.2. *Let $\Gamma = Cay(G, S)$ and H be the subset of all involutions of the group G. If $\langle H \rangle \neq G$ and Γ is connected normal edge-transitive, then its valency is even.*

Corollary 2.3. *Let Γ be connected normal edge-transitive Cayley graph, then all elements of S have the same order.*

Let $p > q > r$ be three primes greater than 2. Due to Corollary 2.2, we consider the tetravalent normal edge-transitive Cayley graphs of order pqr.

3. Main results

Here, for easily a normal edge-transitive Cayley graph is denoted by NET Cayley graph. Let $p > q$ be two prime numbers such that $q|p - 1$. A non-abelian group of order pq with the following presentation is called a Frobenius group:

$$F_{p,q} = \langle x, y : \ x^p = y^q = 1, y^{-1}xy = x^u \rangle,$$

where $u^q \equiv 1 \ (mod\ p)$, see [10]. Let $p > q > r$ be prime numbers, in [9] all groups of order pqr are determined and in [7] the following structures are given

$$\begin{aligned} G_1 &= \mathbb{Z}_{pqr}, G_2 = F_{p,qr}(qr|p-1), G_3 = \mathbb{Z}_r \times F_{p,q}(q|p-1), \\ G_4 &= \mathbb{Z}_p \times F_{q,r}(r|q-1), G_5 = \mathbb{Z}_q \times F_{p,r}(r|p-1), \\ G_{d+5} &= \langle x, y, z : x^p = y^q = z^r = 1, xy = yx, c^{-1}yz = y^u, z^{-1}xz = x^v \rangle, \end{aligned}$$

where $r|p-1, q-1, u^r \equiv 1 \ (mod\ q)$ and $v^r \equiv 1 \ (mod\ p)(1 \leq d \leq r-1)$.

For graph Γ, let X be a subgroup of $Aut(\Gamma)$, Γ is called X-vertex-transitive or X-edge-transitive, if X is transitive on the set of vertices or the set of edges, respectively.

Theorem 3.1. *Let $\Gamma = Cay(G, S)$ be a tetravalent X- normal edge-transitive Cayley graph, where G is a cyclic group and $|G| = pqr$ (p, q, r are odd prime numbers). Let 1 be the vertex corresponding to the identity element. Then*

i) Γ is $(X, 1)$- transitive and $S = \{g, g^\rho, g^{\rho^2}, g^{\rho^3}\}$,

ii) $X_1 \leq \mathbb{Z}_2 \times \mathbb{Z}_2$ and $S = \{g, g^\tau, g^{-1}, (g^{-1})^\tau\}$, where $o(\tau) = 2$.

Proof. i) Let $l = pqr$ and k be an integer, where $k^2 \equiv -1(mod\ pqr)$. Then G has an automorphism ρ with $\rho(g) = g^k$ and $\rho^2(g) = g^{k^2}$. Suppose $S = \{g, g^k, g^{k^2}, g^{k^3}\}$, thus $S = \{g, g^k, g^{-1}, g^{-k}\}$ and so $X = G \rtimes \mathbb{Z}_4$. This means that Γ is $(X, 1)$-transitive.

ii) Let l has two odd factors. Let $\tau \in Aut(G)$ and $\tau(g) = g^{-1}$. Then $Aut(G)$ contains an automorphism $\sigma \in Aut(G) \setminus \{\tau\}$ such that $\sigma\tau = \tau\sigma$. Let $\sigma(g) = g^k$ where $k^2 \equiv 1 \ (mod\ l)$. Let $S = \{g, g^{-1}, g^k, g^{-k}\}$, then $X = G \rtimes \langle \sigma, \tau \rangle \cong \mathbb{Z}_l \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$ and so Γ is X-NET Cayley graph. □

Proposition 3.2 ([8])**.** *Let G is a finite group and S be a subset of G where $G = \langle S \rangle$ and $|S| = 4$. Then $Aut(G, S)$ is a subgroup of D_8.*

Let H_n be the hyper-cube graph of dimension n. The graph $\Gamma = H_n^+$ can be defined as follows:

$$H_n^+ = H_n + E', \text{ where } E' = \{\{x, y\}; x; y \in V(H_n), d(x, y) = n\}.$$

For given group G and positive integer d, by G^d we mean the direct product group $G \times \cdots \times G(d\ times)$. Let n, m, k and t be positive integers with $m|n, n = mk, n \geq 3, m > 1, (t, k) = 1$ and $0 \leq t \leq k-1$. Let $G = \mathbb{Z}_n \times \mathbb{Z}_m = \langle a \rangle \times \langle b \rangle$ and $S_t = \{a; a^{-1}; a^t b; a^{-t} b^{-1}\}$. We denote the Cayley graph $Cay(G; S_t)$ by $AC(n, m, t)$ which is a tetravalent graph. The wreath product of two groups G and H is also denoted by $G \wr H$.

Theorem 3.3 ([1]). *Let $\Gamma = Cay(G, S)$ be a Cayley graph on non-cyclic abelian group G with regarding to connecting set S of valency at most 5. Then Γ is normal edge-transitive if one of the following happens:*

1. $G = \langle a_1, a_2, \cdots, a_d\rangle \cong Z_2^d, S = \{a_1, a_2, \cdots, a_d\}, \Gamma = H_d, Aut(\Gamma) \cong S_2 \wr S_d = S_2^d : S_d$, *for* $d = 2, 3,$ *and* 4.

2. $G = \langle a_1, a_2, \cdots, a_d\rangle \cong Z_2^d, S = \{a_1, a_2, \cdots, a_d, a_1, a_2, \cdots, a_d\}, \Gamma = H_d^+$, $Aut(\Gamma) \cong S_2 \wr S_{d+1}$, *for* $d = 2, 3,$ *and* 4.

3. $G = \langle a_1, a_2, \cdots, a_d\rangle \cong Z_n^d$, *where* $n \geq 3$, $S = \{a_1, a_1^{-1}, a_2, a_2^{-1}, ..., a_d, a_d^{-1}\}$, $\Gamma = AC(n, n, 0)$, *for* $d = 2$. *Also if* $n = 4$, *then* Γ *is non-normal and for* $n \geq 3$, $n \neq 4$, Γ *is normal.*

4. $\Gamma = AC(2m, m, 1)$ *for* $m \geq 3$.

5. $\Gamma = AC(n, m, w)$ *for* $k \geq 3$ *and* $w^2 \equiv \pm 1 \ (mod\ k)$.

6. $G = \mathbb{Z}_m \times \mathbb{Z}_2 = \langle a\rangle \times \langle b\rangle$, $m \geq 3$, $m \neq 4$, *m is even,* $S = \{a, ab, a^{-1}, a^{-1}b\}$, $\Gamma = AC(m, 2, \pm 1) = C_m[2K_1]$, $Aut(\Gamma) = \mathbb{Z}_2 \wr D_{2m}$.

Due to Theorems 3.1 and 3.3, we focous only on non-abelian groups.

Lemma 3.4 ([8]). *Let p is a prime number and $q \mid p - 1$, then $Aut(F_{p,q}) \cong F_{p,p-1}$.*

Theorem 3.5 ([8]). *Let $G = F_{p,q}$ and $S = \{b^ia^m, b^ia^n, (b^ia^m)^{-1}, (b^ia^n)^{-1}\}$. If the Cayley graph $\Gamma = Cay(G, S)$ is NET, then $Aut(G, S) \cong \mathbb{Z}_2$.*

Theorem 3.6 ([8]). *Let $S = \{cba^m, (cba^m)^{-1}, cba^n, (cba^n)^{-1}\}$. If $Cay(G_3, S)$ is NET Cayley graph, then $Aut(G_3, S) \cong \mathbb{Z}_2$.*

A similar discussion shows that $Aut(G_4, S) \cong Aut(G_5, S) \cong \mathbb{Z}_2$. Consider the group G_6 and put $S = \{c, c^{-1}, cb^la^s, (cb^la^s)^{-1}\}$, if $\beta \in Aut(G_6)$ is an automorphism such that $\beta(c) = cb^la^s$ and $\beta(cb^la^s) = c$, then necessarily $\beta(a) = a^{-s}, \beta(b) = b^{-l}$ and $\beta(c) = cb^la^s$ $(1 \leq l \leq q, 1 \leq s \leq p)$. This means that $o(\beta) = 2$. On the other hand, an automorphism of G_6 maps c to c^{-1} or $(cb^la^s)^{-1}$. Hence $Aut(G_6, S) \cong \langle\beta\rangle$. Therefore, we proved the following theorem:

Theorem 3.7. *Let $\Gamma = Cay(G_6, S)$ be NET Cayley graph with the connecting set $S = \{c, c^{-1}, cb^la^s, (cb^la^s)^{-1}\}$, then $Aut(G_6, S) \cong \mathbb{Z}_2$.*

Corollary 3.8. *Let G be a non-cyclic group of order pqr and $\Gamma = Cay(G, S)$ be tetravalent NET Cayley graph, then $Aut(G, S) \cong \mathbb{Z}_2$.*

3.1 Groups of order pq^2

In [9] it is proved that there are five groups of order pq^2 $(p > q)$ with following structures:

$$\begin{aligned} H_1 &= \mathbb{Z}_p \times \mathbb{Z}_{q^2}, H_2 = \mathbb{Z}_p \times (\mathbb{Z}_q \times \mathbb{Z}_q), H_3 = \mathbb{Z}_q \times F_{p,q}\ (q|p-1), \\ H_4 &= F_{p,q^2}\ (q^2|p-1), H_5 = \langle a, b : a^p = b^{q^2} = 1, b^{-1}ab = a^r,\ r^q \equiv 1\ (mod\ p)\rangle. \end{aligned}$$

By using Theorems 3.1 and 3.2, one can compute easily $Aut(H, S)$, where $H \cong H_1$ or $H \cong H_2$. For non-abelian group H_3, we have the following theorem.

Theorem 3.9. *Let $S = \{c, c^{-1}, abc, (abc)^{-1}\}$ be a connecting set of group H_3, then Cayley graph $\Gamma = Cay(H_3, S)$ is NET and $Aut(H_3, S) \cong \mathbb{Z}_2$.*

Proof. Let $S = \{c, c^{-1}, abc, (abc)^{-1}\}$, we show that $Aut(H_3, S)$ has two orbits on S. Let $\alpha \in Aut(H_3, S)$ and $\alpha(a) = a^i, \alpha(b) = b^j$ and $\alpha(c) = a^k b^l c$ $(1 \le i \le p-1, 1 \le j \le q-1, 0 \leqslant k \leqslant p-1,\ 0 \leqslant l \leqslant q-1)$. It is not difficult to see that if $\alpha(c^{-1}ac) = \alpha(a^t)$, where $o(t) = q$ in $\mathbb{Z}_p^*$, then $a^{it^k} = a^{ti}$ and so $k = 1$. If $l = 1$ then $\alpha(c) = abc$. On the other hand, suppose $\alpha(abc) = c^{-1}$ then $ca^{kt+i}b^{j+l} = c^{-1}$, a contradiction. Also, it is obvious that $\alpha(c) \neq (abc)^{-1}$ and so $Aut(H_3, S)$ on S has two orbits. This completes the proof. □

Theorem 3.10. *Let $\Gamma = Cay(H_3, S)$ be tetravalent NET Cayley graph, then $S = \{a^m b^n c^h, (a^m b^n c^h)^{-1}, a^r b^n c^h, (a^r b^n c^h)^{-1}\}$.*

Proof. Let $S = \{a^m b^n c^h, (a^m b^n c^h)^{-1}, a^r b^s c^f, (a^r b^s c^f)^{-1}\}$ and $\alpha \in Aut(H_3, S)$, then α is as given in Theorem 3.9.

Clearly, $\alpha(a^m b^n c^h) = a^r b^s c^f$ and thus $a^{mi+k-r} c^h a^{k(t^{h-1}+\ldots+t)} b^{nj+lh} = c^f b^s$.

By a similar method, if $\alpha(a^r b^s c^f) = a^m b^n c^h$, then we have $nj+lh \equiv s\ (mod\ q)$ and $sj + lf \equiv n\ (mod\ q)$. So $(n-s)(j+1) + l(h-f) \equiv 0\ (mod\ q)$. On the other hand, $q - h \equiv -f\ (mod\ q)$ and $q - f \equiv -h\ (mod\ q)$. Then $h = f$ and $q|(n-s)(j+1)$. It yields that $n \equiv s\ (mod\ q)$ or $j \equiv -1\ (mod\ q)$. This completes the proof. □

Theorem 3.11. *The Cayley graph $\Gamma = Cay(H_4, S)$ is not NET, where $S = \{a^i b^{qk_1+1}, (a^i b^{qk_1+1})^{-1}, a^j b^{qk_2+1}, (a^j b^{qk_2+1})^{-1}\},\ k_1 \neq k_2$.*

Proof. Let $\alpha(a) = a^l, \alpha(a) = a^k b^s$, where $1 \leqslant l, k \leqslant p-1$ and $0 \leqslant s \leqslant q^2 - 1$. Similar to the proof of Theorem 3.10, one can see that $s = 1$. We show that there is no $\alpha \in Aut(H_4, S)$ such that $o(\alpha) \neq 1$. Let $\alpha(a^i b^{qk_1+1}) = a^j b^{qk_2+1}$. Hence, $a^{il+k(r^{-1}+\ldots r^{-qk_1-1})} b^{qk_1+1} = a^j b^{qk_2+1}$, where $o(r) = q^2$. It can be verified that $il + k(r^{-1} + \ldots r^{-qk_1-1}) b^{qk_1+1} \equiv j\ (mod\ p)$ and $qk_1 + 1 \equiv qk_2 + 1\ (mod\ q^2)$ which is a contradiction. Now, assume that $\alpha(a^i b^{qk_1+1}) = (a^i b^{qk_1+1})^{-1}$, thus $a^{il+k(r^{-1}+\ldots r^{-qk_1-1})} b^{qk_1+1} = a^{-ir^{qk_1+1}} b^{-qk_1-1}$, a contradiction. By a similar method, if $\alpha(a^i b^{qk_1+1}) = (a^j b^{qk_2+1})^{-1}$, then

$$a^{il+k(r^{-1}+\ldots r^{-qk_1-1})} b^{qk_1+1} = a^{-jr^{qk_2+1}} b^{-qk_2-1}$$

and so $q^2 | q(k_1 + k_2) + 2$, a contradiction. This means that $Aut(H_4, S) \cong id$. □

Theorem 3.12. *$\Gamma = Cay(H_4, S)$ is tetravalent NET if and only if $S = \{a^m b^n, (a^m b^n)^{-1}, a^r b^n, (a^r b^n)^{-1}\}$.*

Proof. If $\alpha \in Aut(H_4)$ then necessarily $\alpha(a) = a^i$ and $\alpha(b) = a^j b$, where $1 \leqslant i \leqslant p-1$ and $1 \leqslant j \leqslant p$. In particular, there is no automorphism of H_4 where $\alpha(b) = b^{-1}$. Now assume that $\alpha(a^m b^n) = b^{-n} a^{-m}$, then $b^n a^{(mi+j)s^n + j(s^{n-1}+\ldots+s)} = b^{-n} a^{-m}$, where $o(s) = q^2$ in $\mathbb{Z}_p^*$. It yields $2n \equiv 0$ $(mod\ q^2)$, contradiction. Finally, if $\alpha(a^m b^n) = a^r b^t$, then $b^{n-t} a^{(mi+j)s^n + j(s^{n-1}+\ldots+s)} = b^{-t} a^r b^t$. Hence, $n - t \equiv 0$ $(mod\ q^2)$. By a similar method, if $\alpha(a^r b^t) = b^m a^n$, then

$$\begin{cases} (mi+j)s^n + j(s^n - 1) - rs^t \equiv 0\ (mod\ p) \\ (ri+j)s^n + j(s^n - 1) - ms^n \equiv 0\ (mod\ p) \end{cases}.$$

So $s^n(m-r)(i+1) \equiv 0$ $(mod\ p)$. Hence $i \equiv -1$ $(mod\ p)$. In this case, $Aut(H_4, S)$ has two orbits on S and the proof is completed. □

Theorem 3.13. *The Cayley graph $\Gamma = Cay(H_5, S)$ is NET, if $S = \{a^i b^{kq+1}, a^{-ir} b^{-kq-1}, a^j b^{kq+1}, a^{-jr} b^{-kq-1}\}$.*

Proof. Let $S = \{a^i b^{qk+1}, a^{-ir} b^{-qk-1}, a^j b^{qk+1}, a^{-jr} b^{-qk-1}\}$, it is not difficult to show that $b^q \in Z(H_5)$ and $a^i b^j (0 \leq i \leq p-1, 1 \leq j \leq q^2 - 1)$ is of order q^2. Let $\alpha \in Aut(H_5, S)$ and $\alpha(a) = a^l, \alpha(b) = a^s b^{qt+1}$ where $1 \leq l, s \leq p-1$ and $1 \leq t \leq q-1$. We claim that there is no $\alpha \in Aut(H_5, S)$ where $\alpha(a^i b^{qk+1}) = a^{-ir} b^{-qk-1}$, otherwise $\alpha(a^i b^{qk+1}) = a^{il+s} b^{q(k+t)+1}$, then $a^{il+s} b^{q(k+t)+1} = a^{-ir} b^{-qk-1}$. So $il + s \equiv -ir$ $(mod\, p)$ *and* $q(k+t)+1 \equiv -qk-1$ $(mod\ q^2)$. Similarly, $\alpha(a^{-ir} b^{-qk-1}) = a^{-irl-sr} b^{-qt-qk+1}$ and so $-irl - sr \equiv i$ $(mod\, p)$ and $-qt-qk+1 \equiv qk+1$ $(mod\ q^2)$. But in this case, $2 \equiv 0$ $(mod\ q^2)$, a contradiction. It can be shown there is $\alpha \in Aut(H_5, S)$ where $\alpha(a^i b^{qk+1}) = a^{-jr} b^{-qk-1}$. Hence $Aut(H_5, S)$ is not transitive on S. Let $\alpha(a^i b^{qk+1}) = a^j b^{qk+1}$, then $a^{il+s} b^{q(k+t)+1} = a^j b^{qk+1}$ and so $t = 0$. On the other hand, $\alpha(a^{il+s} b^{qk+1}) = a^{il^2+sl} (a^s b)^{qk+1}$ which means $a^{il^2+sl+s} b^{qk+1} = a^i b^{qk+1}$ and by a similar method, if $\alpha(a^j b^{qk+1}) = a^i b^{qk+1}$, then we have $il + s \equiv j$ $(mod\ p), il^2 + sl + s = i$ $(mod\ p)$, and $jl + s \equiv i$ $(mod\ p)$. It follows that $l(i-j) + (i-j) \equiv 0$ $(mod\, p)$ and thus $l \equiv (p-1)$ $(mod\, p)$, $i + j = s$. This completes the proof. □

Corollary 3.14. *Let G be a non-abelian group of order pq^2 and $\Gamma = Cay(G, S)$ is NET tetravalent Cayley graph, then $Aut(G, S) \cong \mathbb{Z}_2$.*

3.2 Groups of order p^3

Let $X = Cay(G, S)$ be a connected tetravalent NET Cayley graph on non-abelian group G of order p^3, where $G = \langle S \rangle$, $S^{-1} = S$ and $|S| = 4$. By [5, Corollary 3.2], X is normal, hence $Aut(X)_1 = Aut(G, S)$ and by Proposition 3.2. $Aut(G, S) \leq D_8$. Consequently 2 or 4 divides $|Aut(G, S)|$. By the elementary group theory, there are two non-abelian groups of order p^3

$$K_1(p^3) = \langle a, b | a^{p^2} = b^p = 1, b^{-1}ab = a^{p+1} \rangle,$$
$$K_2(p^3) = \langle a, b, c | a^p = b^p = c^p = 1, [a, b] = c, [a, c] = [b, c] = 1 \rangle.$$

Theorem 3.15. *Let $\Gamma = Cay(K_1, S)$ and $S = \{a^f b^l, (a^f b^l)^{-1}, a^r b^s, (a^r b^s)^{-1}\}$ then $Aut(K_1, S) \cong id$.*

Proof. Let $Aut(K_1, S) \ncong id$, it is not difficult to see that the following map is an automorphism of $Aut(K_1)$: $\alpha(a) = a^i b^j$ $0 \leq j \leq p-1, (i, p^2) = 1$ and $\alpha(b) = a^{pk} b$, $0 \leq k \leq p-1$. Since α is an automorphism, $\alpha(b^{-1}ab) = \alpha(a^{p+1}) = (a^i b^j)^{p+1} = a^{i(p+1)} b^j$. On the other hand, $\alpha(b^{-1}ab) = \alpha(b^{-1})\alpha(a)\alpha(b)$. It is easy to see that $\alpha(b^{-1}ab) = a^{i(p+1)} b^j$. If $p \equiv 3$ $(mod\ 4)$, then $4 \nmid |Aut(K_1)|$, because $|Aut(K_1)| = p^3(p-1)$. If $p \equiv 1$ $(mod\ 4)$, then we show there is no an element of order four in $Aut(K_1)$, too. To do this assume $S = \{a^f b^l, (a^f b^l)^{-1}, a^r b^s, (a^r b^s)^{-1}\}$.

Let $\alpha(a^r b^s) = \alpha((a^r b^s)^{-1})$, then $b^{rj+s} a^{r(p+1)i+ip+r} = a^{-r}$, a contradiction. Also, if $\alpha(a^f b^l) = \alpha(a^r b^s)$ then $(a^i b^j)^f a^{pkl} b^{l-s} = a^r$ and so $b^{fj} a^{k(p+1)i+ip+pkl} b^l = a^r b^s$. Hence we have $b^{fj+l-s} a^{k(p+1)i+ip+pkl} = a^r$. Similarly, let $\alpha(a^r b^s) = \alpha(a^f b^l)$, then $i = 0$, a contradiction, since $o(a) \neq o(b^j)$, ($o(x)$ denotes the order of x in K_1). This means that $Aut(K_1, S)$ has no element of order two. Consequently it has no element of order four which yields that $Aut(K_1, S) \cong id$. $\square$

Let $G = K_2 = \langle x, y \rangle$, where $o(x) = o(y) = p$. Since $[x, y] \in Z(K_2) = \langle c \rangle$, we have $\beta(a) = x, \beta(b) = y$ and $\beta(c) = [x, y]$, where $\beta \in Aut(K_2)$. Two following maps are elements of $Aut(K_2)$: $\alpha_1(a) = b, \alpha_1(b) = a$ and $\alpha_1(c) = c$ and $\alpha_2(a) = b, \alpha_2(b) = a^{-1}, \alpha_2(c) = c$. Clearly, α_1, $\alpha_2 \in Aut(K_2, S)$ and $\langle \alpha_1, \alpha_2 \rangle \cong D_8$. On the other hand, $Aut(K_2, S)$ is a subgroup of dihedral group D_8 and so $Aut(K_2, S) \cong D_8$. So we proved the following theorem.

Theorem 3.16. *Let $X = Cay(K_2, S)$ is NET Cayley graph. Then $Aut(K_2, S)$ is isomorphic with group D_8, where $S = \{a^f b^l, (a^f b^l)^{-1}, a^r b^s, (a^r b^s)^{-1}\}$.*

Theorem 3.17. *Let G be a group of order pqr and $X = Cay(G, S)$ is a tetravalent NET Cayley graph. Then*

i) if $p > q > r$ are three primes or $p = q > r$, then $Aut(G, S) \cong \mathbb{Z}_2$,

ii) if $p = q = r$, then $Aut(G, S) \cong id$ or D_8.

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Accepted: 30.10.2017