SOME GENERALIZED SEQUENCE SPACES OF INVARIANT MEANS DEFINED BY IDEAL AND MODULUS FUNCTIONS IN N-NORMED SPACES

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Abstract. In the present paper we introduce and study some generalized difference sequence spaces of invariant means defined by ideal and a sequence of modulus functions over n-normed space. We study some topological properties and prove some inclusion results between these spaces. Further, we also study some results on statistical convergence.
Keywords: ideal, Difference sequence space, modulus function, Lacunary sequence, Invariant mean, Statistical convergence.

1. Introduction and preliminaries

Let σ be the mapping of the set of positive integers into itself. A continuous linear functional φ on l_∞ is said to be an invariant mean or σ-mean (c.f. [5], [34], [35]) if and only if

1. $\varphi(x) \geq 0$ when the sequence $x = (x_k)$ has $x_k \geq 0$, for all k,

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2. $\varphi(e) = 1$, where $e = (1, 1, 1, \cdots)$ and
3. $\varphi(x_{\sigma(k)}) = \varphi(x)$, for all $x \in l_\infty$.

If $x = (x_n)$, write $Tx = Tx_n = (x_{\sigma(n)})$. It can be shown in [41] that

$$V_\sigma = \Big\{x \in l_\infty : \lim_k t_{k_n}(x) = l, \quad \text{uniformly in } \ n, \ l = \sigma - \lim x\Big\},$$

where

$$t_{k_n}(x) = \frac{x_n + x_{\sigma^1 n} + ... + x_{\sigma^k n}}{k+1}.$$

In the case σ is the translation mapping $n \to n+1$, σ-mean is often called a Banach limit and V_σ, the set of bounded sequences all of whose invariant means are equal, is the set of almost convergent sequences [24] (also see [27]). By using the concept of invariant means Mursaleen et al. [36] introduced the following sequence spaces:

$$w_\sigma = \Big\{x : \lim_{n\to\infty} \frac{1}{n+1}\sum_{k=0}^{n} t_{k_m}(x_k - l) \to 0, \ \ \text{uniformally in} \ \ m\Big\},$$

$$[w]_\sigma = \Big\{x : \lim_{n\to\infty} \frac{1}{n+1}\sum_{k=0}^{n} |t_{k_m}(x_k - l)| \to 0, \ \ \text{uniformally in} \ \ m\Big\},$$

$$[w_\sigma] = \Big\{x : \lim_{n\to\infty} \frac{1}{n+1}\sum_{k=0}^{n} t_{k_m}(|x_k - l|) \to 0, \ \ \text{uniformally in} \ \ m\Big\},$$

and investigate some of its properties.

The notion of statistical convergence has been introduced by Fast [11] in 1951 and later developed by Fridy [12], Šalát [40], Mohiuddine and Belen [31] and many others. Furthermore, Kostyrko et al. [21] presented a very interesting generalization of statistical convergence called as I-convergence. The detailed history and development in this regard can be found by Connor [6], Maddox [25] and many others.

By a lacunary sequence $\theta = (k_r)$ where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers with $k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$. We write $h_r = k_r - k_{r-1}$. The ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r. The space of lacunary strongly convergent sequence was defined by Freedman et al. [15] as follows:

$$N_\theta = \Big\{x = (x_k) : \lim_{r\to\infty} \frac{1}{h_r}\sum_{k\in I_r} |x_k - L| = 0 \ \text{ for some } \ L\Big\}.$$

Fridy and Orhan [13] generalized the concept of statistical convergence by using lacunary sequence which is called lacunary statistical convergence. Further, lacunary sequences have been studied by Fridy and Orhan [14]. Quite recently, Karakaya [22] combined the approach of lacunary sequence with invariant means and introduced the notion of strong $\sigma-$lacunary statistically convergence as follows:

Definition 1.1. [22] Let $\theta = (k_r)$ be a lacunary sequence. A sequence $x = (x_k)$ is said to be lacunary strong σ lacunary statistically convergent if for every $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r}\{k \in I_r : |t_{k_m}(x - L)| \geq \varepsilon\} = 0 \text{ uniformaly in } m$$

where St_{θ_σ} denotes the set of all lacunary strong $\sigma-$lacunary statistically convergent sequences.

Let $\mathbb{N}$ be a non empty set. Then a family of sets $I \subseteq 2^{\mathbb{N}}$ (Power set of $\mathbb{N}$) is said to be an ideal if I is additive i.e $A, B \in I \Rightarrow A \cup B \in I$ and $A \in I, B \subseteq A \Rightarrow B \in I$. A non empty family of sets $\pounds(I) \subseteq 2^{\mathbb{N}}$ is said to be filter on $\mathbb{N}$ if and only if $\Phi \notin \pounds(I)$ for $A, B \in \pounds(I)$ we have $A \cap B \in \pounds(I)$ and for each $A \in \pounds(I)$ and $A \subseteq B$ implies $B \in \pounds(I)$. An ideal $I \subseteq 2^{\mathbb{N}}$ is called non trivial if $I \neq 2^{\mathbb{N}}$. A non trivial ideal $I \subseteq 2^{\mathbb{N}}$ is called admissible if $\{\{x\} : x \in \mathbb{N}\} \subseteq I$. A non-trivial ideal is maximal if there cannot exist any non trivial ideal $J \neq I$ containing I as a subset. For each ideal I, there exist a filter $\pounds(I)$ corresponding to I i.e $\pounds(I) = \{K \subseteq \mathbb{N} : K^c \in I\}$, where $K^c = \mathbb{N} \setminus K$. Recently, Das et al. [7] unified the idea of lacunary statistical convergence with ideal convergence and presented the following interesting generalization of statistical convergence.

Definition 1.2. [7] Let $\theta = (k_r)$ be a lacunary sequence. A sequence $x = (x_k)$ is said to be $I-$lacunary statistical convergent or $S_\theta(I)-$convergent to L, if for every $\varepsilon > 0$ and $\delta > 0$,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}|\{k \in I_r : |x_k - L| \geq \varepsilon\}| \geq \delta\right\} \in I.$$

In this case $x_k \to L(S_\theta(I))$ or $S_\theta(I) - \lim_{k\to\infty} x_k = L$. The set of all I-lacunary statistically convergent sequences will be denoted by $S_\theta(I)$.

Definition 1.3. [7] Let $\theta = (k_r)$ be a lacunary sequence. A sequence $x = (x_k)$ is said to be $N_\theta(I)-$ convergent to L, if for every $\varepsilon > 0$ we have

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r} |x_k - L| \geq \varepsilon\}\right\} \in I.$$

In this case $x_k \to L(N_\theta(I))$.

The concept of 2-normed spaces was initially developed by Gähler [16] in the mid of 1960's, while that of n-normed spaces one can see in Misiak [26]. Since then, many others have studied this concept and obtained various results, see Gunawan ([17],[18]) and Gunawan and Mashadi [19]. Let $n \in \mathbb{N}$ and X be a linear space over the field $\mathbb{R}$ of reals of dimension d, where $d \geq n \geq 2$. A real valued function $\|\cdot, \cdots, \cdot\|$ on X^n satisfying the following four conditions:

1. $\|x_1, x_2, \cdots, x_n\| = 0$ if and only if $x_1, x_2, \cdots, x_n$ are linearly dependent in X;

2. $\|x_1, x_2, \cdots, x_n\|$ is invariant under permutation;

3. $\|\alpha x_1, x_2, \cdots, x_n\| = |\alpha| \ \|x_1, x_2, \cdots, x_n\|$ for any $\alpha \in \mathbb{R}$, and

4. $\|x + x', x_2, \cdots, x_n\| \leq \|x, x_2, \cdots, x_n\| + \|x', x_2, \cdots, x_n\|$

is called a n-norm on X, and the pair $(X, \|\cdot, \cdots, \cdot\|)$ is called a n-normed space over the field $\mathbb{R}$.

For example, we may take $X = \mathbb{R}^n$ being equipped with the Euclidean n-norm $\|x_1, x_2, \cdots, x_n\|_E$ = the volume of the n-dimensional parallelopiped spanned by the vectors $x_1, x_2, \cdots, x_n$ which may be given explicitly by the formula

$$\|x_1, x_2, \cdots, x_n\|_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots, n$. Let $(X, \|\cdot, \cdots, \cdot\|)$ be a n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \cdots, a_n\}$ be linearly independent set in X. Then the following function $\|\cdot, \cdots, \cdot\|_\infty$ on X^{n-1} defined by

$$\|x_1, x_2, \cdots, x_{n-1}\|_\infty = \max\{\|x_1, x_2, \cdots, x_{n-1}, a_i\| : i = 1, 2, \cdots, n\}$$

defines an $(n-1)$-norm on X with respect to $\{a_1, a_2, \cdots, a_n\}$.

A sequence (x_k) in a n-normed space $(X, \|\cdot, \cdots, \cdot\|)$ is said to converge to some $L \in X$ if

$$\lim_{k\to\infty} \|x_k - L, z_1, \cdots, z_{n-1}\| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$

A sequence (x_k) in a n-normed space $(X, \|\cdot, \cdots, \cdot\|)$ is said to be Cauchy if

$$\lim_{k,p\to\infty} \|x_k - x_p, z_1, \cdots, z_{n-1}\| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be n-Banach space.

Definition 1.4. Let $I \subseteq 2^{\mathbb{N}}$. A sequence $x = (x_k)$ in a n-normed space $(X, \|\cdot, \cdots, \cdot\|)$ is said to be I-convergent to a number L if for every $\epsilon > 0$, the set $A(\varepsilon) = \{k \in \mathbb{N} : \|x_k - L, z_1, \cdots, z_{n-1}\| \geq \epsilon\} \in I$. In this case we write $I - \lim_{k\to\infty} \|x_k, z_1, \cdots, z_{n-1}\| = \|L, \ z_1, \cdots, z_{n-1}\|$.

Definition 1.5. A sequence $x = (x_k)$ in a n-normed space $(X, \|\cdot, \cdots, \cdot\|)$ is said to be statistical convergent to some $L \in X$ if for each $\varepsilon > 0$, the set $A(\varepsilon) = \{k \in \mathbb{N} : \|x_k - L, z_1, \cdots, z_{n-1}\| \geq \varepsilon\}$ having its natural density zero.

The notion of difference sequence spaces was introduced by Kızmaz [20] who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et. and Çolak [9] by introducing the spaces $l_\infty(\Delta^n)$,

$c(\Delta^n)$ and $c_0(\Delta^n)$. Let w be the space of all complex or real sequences $x = (x_k)$ and let m, v be non-negative integers, then for $Z = l_\infty,\ c,\ c_0$ we have sequence spaces $Z(\Delta_v^m) = \{x = (x_k) \in w : (\Delta_v^m x_k) \in Z\}$, where $\Delta_v^m x = (\Delta_v^m x_k) = (\Delta_v^{m-1} x_k - \Delta_v^{m-1} x_{k+1})$ and $\Delta^0 x_k = x_k$, for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta_v^m x_k = \sum_{s=0}^{m} (-1)^s \begin{pmatrix} m \\ s \end{pmatrix} x_{k+vs}.$$

Taking $v = 1$, we get the spaces which were introduced and studied by Et. and Çolak [9]. Taking $m = v = 1$, we get the spaces which were studied by Kızmaz [20]. For more details about sequence spaces (see [23, 32, 33, 36, 37, 39]) and reference therein.

A modulus function is a function $f : [0, \infty) \to [0, \infty)$ such that

1. $f(x) = 0$ if and only if $x = 0$,
2. $f(x + y) \leq f(x) + f(y)$, for all $x, y \geq 0$,
3. f is increasing,
4. f is continuous from the right at 0.

It follows that f must be continuous everywhere on $[0, \infty)$. The modulus function may be bounded or unbounded. For example, if we take $f(x) = \frac{x}{x+1}$, then $f(x)$ is bounded. If $f(x) = x^p, 0 < p < 1$ then the modulus function $f(x)$ is unbounded. Subsequently, modulus function has been discussed in ([2], [38]) and references therein.

Lemma 1.6. *Let $\mathcal{F} = (f_k)$ be a sequence of modulus functions and $0 < \delta < 1$. Then for each $x > \delta$ we have $f_k(x) \leq \frac{2 f_k(1) x}{\delta}$.*

Let $\mathcal{F} = (f_k)$ be a sequence of modulus functions, $(X, \|\cdot, \cdots, \cdot\|)$ be a n-normed space, $p = (p_k)$ be a bounded sequence of strictly positive real numbers and $u = (u_k)$ be any sequence of positive real numbers. By $S(n - X)$ we denote the space of all sequences defined over $(X, \|\cdot, \cdots, \cdot\|)$. In this paper we define the following sequence spaces:

$$w_\sigma^0\Big[\mathcal{F}, I, u, p, \|\cdot, \cdots, \cdot\|\Big]_\theta(\Delta_v^m)$$
$$= \Big\{x \in S(n - X) : \Big\{r \in \mathbb{N}: \frac{1}{h_r} \sum_{k \in I_r} u_k \Big[f_k\Big(\|\frac{t_{k_n}(\Delta_v^m x_k)}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_k} \geq \varepsilon\Big\} \in I,$$
$$\text{for some } \ \rho > 0\Big\},$$

$$w_\sigma\Big[\mathcal{F}, I, u, p, \|\cdot, \cdots, \cdot\|\Big]_\theta(\Delta_v^m)$$
$$= \Big\{x \in S(n - X) : \Big\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} u_k \Big[f_k\Big(\|\frac{t_{k_n}(\Delta_v^m x_k - l)}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_k} \geq \varepsilon\Big\} \in I,$$
$$\text{for some } \ l \ \text{ and } \ \rho > 0, \Big\}$$

and

$$w_\sigma^\infty\Big[\mathcal{F}, I, u, p, \|\cdot, \cdots, \cdot\|\Big]_\theta(\Delta_v^m)$$
$$= \Big\{x \in S(n-X): \ \exists K > 0,$$
$$\Big\{r\in\mathbb{N}: \frac{1}{h_r}\sum_{k\in I_r} u_k\Big[f_k\Big(\|\frac{t_{k_n}(\Delta_v^m x_k)}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_k} \geq K\Big\} \in I, \text{for some } \ \rho > 0\Big\},$$

uniformly in n.

If we take $\mathcal{F}(x) = x$, we get the spaces

$$w_\sigma^0\Big[I, u, p, \|\cdot, \cdots, \cdot\|\Big]_\theta(\Delta_v^m)$$
$$= \Big\{x\in S(n-X): \Big\{r\in\mathbb{N}: \frac{1}{h_r}\sum_{k\in I_r} u_k\Big[\Big(\|\frac{t_{k_n}(\Delta_v^m x_k)}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_k} \geq \varepsilon\Big\} \in I,$$
$$\text{for some } \ \rho > 0\Big\},$$

$$w_\sigma\Big[I, u, p, \|\cdot, \cdots, \cdot\|\Big]_\theta(\Delta_v^m)$$
$$= \Big\{x\in S(n-X): \Big\{r\in\mathbb{N}: \frac{1}{h_r}\sum_{k\in I_r} u_k\Big[\Big(\|\frac{t_{k_n}(\Delta_v^m x_k - l)}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_k} \geq \varepsilon\Big\} \in I,$$
$$\text{for some } \ l \ \text{ and } \ \rho > 0,\Big\}$$

and

$$w_\sigma^\infty\Big[I, u, p, \|\cdot, \cdots, \cdot\|\Big]_\theta(\Delta_v^m)$$
$$= \Big\{x \in S(n-X): \ \exists K > 0, \Big\{r \in \mathbb{N}: \frac{1}{h_r}\sum_{k\in I_r} u_k \cdot$$
$$\cdot \Big[\Big(\|\frac{t_{k_n}(\Delta_v^m x_k)}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_k} \geq K\Big\} \in I, \text{for some } \ \rho > 0\Big\}.$$

If we take $p = (p_k) = 1$, we get the spaces

$$w_\sigma^0\Big[\mathcal{F}, I, u, \|\cdot, \cdots, \cdot\|\Big]_\theta(\Delta_v^m)$$
$$= \Big\{x \in S(n-X): \Big\{r \in \mathbb{N}: \frac{1}{h_r}\sum_{k\in I_r} u_k \cdot$$
$$\cdot \Big[f_k\Big(\|\frac{t_{k_n}(\Delta_v^m x_k)}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big] \geq \varepsilon\Big\} \in I, \ \text{ for some } \ \rho > 0\Big\},$$

$$w_\sigma\Big[\mathcal{F}, I, u, \|\cdot, \cdots, \cdot\|\Big]_\theta(\Delta_v^m)$$
$$= \Big\{x \in S(n-X): \Big\{r \in \mathbb{N}: \frac{1}{h_r}\sum_{k\in I_r} u_k \cdot$$
$$\cdot \Big[f_k\Big(\|\frac{t_{k_n}(\Delta_v^m x_k - l)}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big] \geq \varepsilon\Big\} \in I, \text{for some } \ l \ \text{ and } \ \rho > 0,\Big\}$$

and

$$w_\sigma^\infty\Big[\mathcal{F}, I, u, \|\cdot, \cdots, \cdot\|\Big]_\theta(\Delta_v^m)$$
$$= \Big\{x \in S(n-X) : \exists\, K > 0, \Big\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r} u_k \cdot$$
$$\cdot \Big[f_k\Big(\|\frac{t_{k_n}(\Delta_v^m x_k)}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big] \geq K\Big\} \in I, \text{for some } \rho > 0\Big\}.$$

The following inequality will be used throughout the paper. If $0 \leq p_k \leq \sup p_k = H$, $K = \max(1, 2^{H-1})$ then

$$|a_k + b_k|^{p_k} \leq D\{|a_k|^{p_k} + |b_k|^{p_k}\} \tag{1.1}$$

for all k and $a_k, b_k \in \mathbb{R}$. Also $|a|^{p_k} \leq \max(1, |a|^H)$, for all $a \in \mathbb{R}$.

The main aim of the present paper is to study some topological properties and prove some inclusion relations between above defined sequence spaces.

2. Main results

Theorem 2.1. *Let $\mathcal{F}$=(f_k) be a sequence of modulus functions, p=(p_k) be a bounded sequence of strictly positive real numbers and u=(u_k) be any sequence of positive real numbers. Then the classes of sequences $w_\sigma^0[\mathcal{F}, I, u, p, \|\cdot, \cdots, \cdot\|]_\theta(\Delta_v^m)$, $w_\sigma[\mathcal{F}, I, u, p, \|\cdot, \cdots, \cdot\|]_\theta(\Delta_v^m)$ and $w_\sigma^\infty[\mathcal{F}, I, u, p, \|\cdot, \cdots, \cdot\|]_\theta(\Delta_v^m)$ are linear spaces over the real field $\mathbb{R}$.*

Proof. We shall prove the assertion for $w_\sigma^0[\mathcal{F}, I, u, p, \|\cdot, \cdots, \cdot\|]_\theta(\Delta_v^m)$ only and the others can be proved similarly.

Let $x = (x_k)$, $y = (y_k) \in w_\sigma^0[\mathcal{F}, I, u, p, \|\cdot, \cdots, \cdot\|]_\theta(\Delta_v^m)$ and $\alpha, \beta \in \mathbb{R}$. Then there exist positive real numbers ρ_1 and ρ_2 such that for every $\varepsilon > 0$ and $z_1, \cdots, z_{n-1} \in X$, we have

$$A_\theta\Big(\frac{\varepsilon}{2}\Big) = \Big\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r} u_k\Big[f_k\Big(\|\frac{t_{k_n}(\Delta_v^m x_k)}{\rho_1}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_k} \geq \frac{\varepsilon}{2}\Big\}$$

and

$$B_\theta\Big(\frac{\varepsilon}{2}\Big) = \Big\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r} u_k\Big[f_k\Big(\|\frac{t_{k_n}(\Delta_v^m y_k)}{\rho_2}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_k} \geq \frac{\varepsilon}{2}\Big\}$$

belongs to I. Let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $\|\cdot,\cdots,\cdot\|$ is a n-norm on X and f_k's are modulus functions so by using inequality (1.1), we have

$$\begin{aligned}
&\frac{1}{h_r}\sum_{k\in I_r} u_k\Big[f_k\Big(\|\frac{t_{k_n}(\alpha\Delta_v^m x_k+\beta\Delta_v^m y_k)}{\rho_3}, z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_k}\\
&\le \frac{1}{h_r}\sum_{k\in I_r} u_k\Big[f_k\Big(|\alpha|\|\frac{t_{k_n}(\Delta_v^m x_k)}{\rho_3}, z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_k}\\
&+ \frac{1}{h_r}\sum_{k\in I_r} u_k\Big[f_k\Big(|\beta|\|\frac{t_{k_n}(\Delta_v^m y_k)}{\rho_3}, z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_k}\\
&\le D.\frac{1}{h_r}\sum_{k\in I_r} \frac{1}{2^{p_k}}u_k\Big[f_k\Big(\|\frac{t_{k_n}(\Delta_v^m x_k)}{\rho_1}, z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_k}\\
&+ D.\frac{1}{h_r}\sum_{k\in I_r} \frac{1}{2^{p_k}}u_k\Big[f_k\Big(\|\frac{t_{k_n}(\Delta_v^m y_k)}{\rho_2}, z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_k}.
\end{aligned}$$

For given $\varepsilon > 0$ and for all $z_1,\cdots,z_{n-1}\in X$, we have the following containment

$$\begin{aligned}
&\Big\{r\in\mathbb{N}: \frac{1}{h_r}\sum_{k\in I_r} u_k\Big[f_k\Big(\|\frac{t_{k_n}(\alpha\Delta_v^m x_k+\beta\Delta_v^m y_k)}{\rho_3}, z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_k}\ge\varepsilon\Big\}\\
&\subseteq\Big\{r\in\mathbb{N}: \frac{1}{h_r}\sum_{k\in I_r} u_k\Big[f_k\Big(\|\frac{t_{k_n}(\Delta_v^m x_k)}{\rho_1}, z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_k}\ge\frac{\varepsilon}{2D}\Big\}\\
&\cup\Big\{r\in\mathbb{N}: \frac{1}{h_r}\sum_{k\in I_r} u_k\Big[f_k\Big(\|\frac{t_{k_n}(\Delta_v^m y_k)}{\rho_2}, z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_k}\ge\frac{\varepsilon}{2D}\Big\}.
\end{aligned}$$

By using the property of an ideal the set on the left hand side in the above expression belongs to I. Thus, $\alpha x+\beta y\in w_\sigma^0[\mathcal{F}, I, u, p, \|\cdot,\cdots,\cdot\|]_\theta(\Delta_v^m)$. This completes the proof. □

Theorem 2.2. *Let $p = (p_k)$ be a bounded sequence of strictly positive real numbers and $u = (u_k)$ be a sequence of positive real numbers. Then for $m \ge 1$, we have:*
(i) $w_\sigma^0[\mathcal{F}, I, u, p, \|\cdot,\cdots,\cdot\|]_\theta(\Delta_v^{m-1}) \subset w_\sigma^0[\mathcal{F}, u, p, \|\cdot,\cdots,\cdot\|]_\theta(\Delta_v^m)$ is strict.
(ii) $w_\sigma[\mathcal{F}, I, u, p, \|\cdot,\cdots,\cdot\|]_\theta(\Delta_v^{m-1}) \subset w_\sigma[\mathcal{F}, u, p, \|\cdot,\cdots,\cdot\|]_\theta(\Delta_v^m)$ is strict.
(iii) $w_\sigma^\infty[\mathcal{F}, I, u, p, \|\cdot,\cdots,\cdot\|]_\theta(\Delta_v^{m-1}) \subset w_\sigma^\infty[\mathcal{F}, u, p, \|\cdot,\cdots,\cdot\|]_\theta(\Delta_v^m)$ is strict.

Proof. We shall prove the result for $w_\sigma^0[\mathcal{F}, I, u, p, \|\cdot,\cdots,\cdot\|]_\theta(\Delta_v^{m-1})$ only. The others can be proved similarly. Suppose $x\in w_\sigma^0[\mathcal{F}, I, u, p, \|\cdot,\cdots,\cdot\|]_\theta(\Delta_v^{m-1})$, by definition for every $\varepsilon > 0$ and $z_1,\cdots,z_{n-1}\in X$, we have

$$\Big\{r\in\mathbb{N}: \frac{1}{h_r}\sum_{k\in I_r} u_k\Big[f_k\Big(\|\frac{t_{k_n}(\Delta_v^{m-1} x_k)}{\rho}, z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_k}\ge\varepsilon\Big\}\in I, \tag{2.1}$$

uniformly in n. Since $\mathcal{F}=(f_k)$ is a sequence of modulus functions, we have the following inequality:

$$\begin{aligned}
&\frac{1}{h_r}\sum_{k\in I_r} u_k\Big[f_k\Big(\|\frac{t_{k_n}(\Delta_v^m x_k)}{\rho}, z_1,\cdots, z_{n-1}\|\Big)\Big]^{p_k}\\
&\leq \frac{1}{h_r}\sum_{k\in I_r}\Bigg[u_k\Big[f_k\Big(\|\frac{t_{k_n}(\Delta_v^{m-1} x_k)}{\rho}, z_1,\cdots, z_{n-1}\|\Big)\Big]\\
&+ u_k\Big[f_k\Big(\|\frac{t_{k_n}(\Delta_v^{m-1} x_{k+1})}{\rho}, z_1,\cdots, z_{n-1}\|\Big)\Big]\Bigg]^{p_k}\\
&\leq D\frac{1}{h_r}\sum_{k\in I_r} u_k\Big[f_k\Big(\|\frac{t_{k_n}(\Delta_v^{m-1} x_k)}{\rho}, z_1,\cdots, z_{n-1}\|\Big)\Big]^{p_k}\\
&+ D\frac{1}{h_r}\sum_{k\in I_r} u_k\Big[f_k\Big(\|\frac{t_{k_n}(\Delta_v^{m-1} x_{k+1})}{\rho}, z_1,\cdots, z_{n-1}\|\Big)\Big]^{p_k}
\end{aligned}$$

uniformly in n.

Now for given $\varepsilon>0$, we have

$$\begin{aligned}
&\Bigg\{r\in\mathbb{N}: \frac{1}{h_r}\sum_{k\in I_r} u_k\Big[f_k\Big(\|\frac{t_{k_n}(\Delta_v^m x_k)}{\rho}, z_1,\cdots, z_{n-1}\|\Big)\Big]^{p_k}\geq\varepsilon\Bigg\}\\
&\subseteq\Bigg\{r\in\mathbb{N}: \frac{1}{h_r}\sum_{k\in I_r} u_k\Big[f_k\Big(\|\frac{t_{k_n}(\Delta_v^{m-1} x_k)}{\rho}, z_1,\cdots, z_{n-1}\|\Big)\Big]^{p_k}\geq\frac{\varepsilon}{2D}\Bigg\}\\
&\cup\Bigg\{r\in\mathbb{N}: \frac{1}{h_r}\sum_{k\in I_r} u_k\Big[f_k\Big(\|\frac{t_{k_n}(\Delta_v^{m-1} x_{k+1})}{\rho}, z_1,\cdots, z_{n-1}\|\Big)\Big]^{p_k}\geq\frac{\varepsilon}{2D}\Bigg\},
\end{aligned}$$

uniformly in n.

Both the sets on the right hand side in the above containment belong to I by (2.1). It follows that $x\in w_\sigma^0[\mathcal{F},I,u,p,\|\cdot,\cdots,\cdot\|]_\theta(\Delta_v^m)$. Clearly the inclusion is strict because if we take $x=(x_k)=k^{m-1}$, $f_k(x)=x$, $p_k=1$, $u_k=1$, $t_{0_n}(x)=(x_n)$ and $\theta=(2^r)$ for all $k\in\mathbb{N}$, then $x_k\in w_\sigma^0[\mathcal{F},u,p,\|\cdot,\cdots,\cdot\|]_\theta(\Delta_v^m)$ but $x_k\notin w_\sigma^0[\mathcal{F},I,u,p,\|\cdot,\cdots,\cdot\|]_\theta$ (Δ_v^{m-1}). □

Theorem 2.3. *Let $\mathcal{F}'=(f_k')$ and $\mathcal{F}''=(f_k'')$ be two sequences of modulus functions. If* $\limsup_{t\to\infty}\frac{f_k'(t)}{f_k''(t)}=P>0$, *then* $w_\sigma^0[\mathcal{F}',u,p,\|\cdot,\cdots,\cdot\|]_\theta(\Delta_v^m)\subset w_\sigma^0[\mathcal{F}'',I,u,p,\|\cdot,\cdots,\cdot\|]_\theta(\Delta_v^m)$.

Proof. Let $\limsup_{t\to\infty}\frac{f_k'(t)}{f_k''(t)}=P$, then there exists a positive number $K>0$ such that $f_k'(t)\geq Kf_k''(t)$, for all $t\geq 0$. Therefore, for each $z_1,\cdots,z_{n-1}\in X$, we have

$$\frac{1}{h_r}\sum_{k\in I_r} u_k\Big[f'_k\Big(\|\frac{t_{k_n}(\Delta_v^m x_k)}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_k}$$
$$\geq (K)^H \frac{1}{h_r}\sum_{k\in I_r} u_k\Big[f''_k\Big(\|\frac{t_{k_n}(\Delta_v^m x_k)}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_k},$$

uniformly in n. Then for every $\varepsilon > 0$ and $z_1, \cdots, z_{n-1} \in X$, we have following relationship

$$\Big\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r} u_k\Big[f''_k\Big(\|\frac{t_{k_n}(\Delta_v^m x_k)}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_k} \geq \varepsilon\Big\}$$
$$\subseteq \Big\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r} u_k\Big[f'_k\Big(\|\frac{t_{k_n}(\Delta_v^m x_k)}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_k} \geq \varepsilon(K)^H\Big\},$$

uniformly in n. Therefore, the above containment gives the result. □

Theorem 2.4. *Suppose $\mathcal{F} = (f_k)$, $\mathcal{F}' = (f'_k)$ and $\mathcal{F}'' = (f''_k)$ are sequences of modulus functions, then*
(i) $w_\sigma[\mathcal{F}', I, u, p, \|\cdot, \cdots, \cdot\|]_\theta(\Delta_v^m) \subset w_\sigma[\mathcal{F} \circ \mathcal{F}', I, u, p, \|\cdot, \cdots, \cdot\|]_\theta(\Delta_v^m)$.
(ii) $w_\sigma[\mathcal{F}', I, u, p, \|\cdot, \cdots, \cdot\|]_\theta(\Delta_v^m) \cap w_\sigma[\mathcal{F}'', I, u, p, \|\cdot, \cdots, \cdot\|]_\theta(\Delta_v^m) \subset w_\sigma[\mathcal{F}' + \mathcal{F}'', I, u, p, \|\cdot, \cdots, \cdot\|]_\theta(\Delta_v^m)$.

Proof. (i) Let $x = (x_k) \in w_\sigma[\mathcal{F}', I, u, p, \|\cdot, \cdots, \cdot\|]_\theta(\Delta_v^m)$, then for every $\varepsilon > 0$ choose $\delta \in (0, 1)$ such that $f_k(t) < \varepsilon$, for all $0 < t < \delta$, we have

$$(2.2)\quad A_\delta = \Big\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r} u_k\Big[f'_k\Big(\|\frac{t_{k_n}(\Delta_v^m x_k - l)}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_k} \geq \delta\Big\} \in I,$$

uniformly in n. On the other hand, we have

$$\frac{1}{h_r}\sum_{k\in I_r} u_k\Big[(f_k \circ f'_k)\Big(\|\frac{t_{k_n}(\Delta_v^m x_k - l)}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_k}$$
$$= \frac{1}{h_r} \sum_{k\in I_r \& u_k[f'_k(\|\frac{t_{k_n}(\Delta_v^m x_k - l)}{\rho}, z_1, \cdots, z_{n-1}\|)]^{p_k} < \delta} .$$
$$\cdot u_k\Big[(f_k \circ f'_k)\Big(\|\frac{t_{k_n}(\Delta_v^m x_k - l)}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_k}$$
$$+ \frac{1}{h_r} \sum_{k\in I_r \& u_k[f'_k(\|\frac{t_{k_n}(\Delta_v^m x_k - l)}{\rho}, z_1, \cdots, z_{n-1}\|)]^{p_k} \geq \delta} .$$
$$\cdot u_k\Big[(f_k \circ f'_k)\Big(\|\frac{t_{k_n}(\Delta_v^m x_k - l)}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_k}$$
$$\leq (\varepsilon)^H + \max(1, (2.\frac{f_k(1)}{\delta})^H)\frac{1}{h_r}\sum_{k\in I_r} u_k\Big[f'_k\Big(\|\frac{t_{k_n}(\Delta_v^m x_k - l)}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_k}$$

uniformly in n by lemma (1.6). Then for any $\eta > 0$,

$$\Big\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} u_k \Big[(f_k \circ f'_k)\Big(\|\frac{t_{k_n}(\Delta_v^m x_k - l)}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_k} \geq \eta\Big\}$$
$$\subseteq \Big\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} u_k \Big[f'_k\Big(\|\frac{t_{k_n}(\Delta_v^m x_k - l)}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_k} \geq \frac{\eta - \varepsilon}{K}\Big\},$$

where $K = \max(1, (2.\frac{f_k(1)}{\delta})^H)$.

By using (2.2), we obtain $x \in w_\sigma[\mathcal{F} \circ \mathcal{F}', I, u, p, \|\cdot, \cdots, \cdot\|]_\theta(\Delta_v^m)$.

(ii) This part of the theorem proved by using the following inequality

$$\frac{1}{h_r} \sum_{k \in I_r} u_k \Big[(f'_k + f''_k)\Big(\|\frac{t_{k_n}(\Delta_v^m x_k - l)}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_k}$$
$$\leq \frac{D}{h_r} \sum_{k \in I_r} u_k \Big[f'_k\Big(\|\frac{t_{k_n}(\Delta_v^m x_k - l)}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_k}$$
$$+ \frac{D}{h_r} \sum_{k \in I_r} u_k \Big[f''_k\Big(\|\frac{t_{k_n}(\Delta_v^m x_k - l)}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_k},$$

where $\sup_k p_k = H$ and $D = \max(1, 2^{H-1})$. □

Theorem 2.5. *Let $\mathcal{F} = (f_k)$ be a sequence of modulus functions and $p = (p_k)$ be a bounded sequence of strictly positive real numbers, then*

$$w_\sigma[I, u, p, \|\cdot, \cdots, \cdot\|]_\theta(\Delta_v^m) \subseteq w_\sigma[\mathcal{F}, I, u, p, \|\cdot, \cdots, \cdot\|]_\theta(\Delta_v^m).$$

Proof. This can be proved by using the techniques similar to those used in Theorem 2.4 (i). □

Theorem 2.6. *Let $\mathcal{F} = (f_k)$ be a sequence of modulus functions and $p = (p_k)$ be a bounded sequence of strictly positive real numbers, if $\limsup_{t \to \infty} \frac{f_k(t)}{t} = Q > 0$, then*

$$w_\sigma\Big[\mathcal{F}, I, u, p, \|\cdot, \cdots, \cdot\|\Big]_\theta(\Delta_v^m) \subseteq w_\sigma\Big[u, p, \|\cdot, \cdots, \cdot\|\Big]_\theta(\Delta_v^m).$$

Proof. Suppose $x = (x_k) \in w_\sigma[\mathcal{F}, I, u, p, \|\cdot, \cdots, \cdot\|]_\theta(\Delta_v^m)$ and $\limsup_{t \to \infty} \frac{f_k(t)}{t} = Q > 0$, then there exists a constant $R > 0$ such that $f_k(t) \geq Rt$, for all $t \geq 0$. Thus we have

$$\frac{1}{h_r} \sum_{k \in I_r} u_k \Big[f_k\Big(\|\frac{t_{k_n}(\Delta_v^m x_k - l)}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_k}$$
$$\geq (R)^H \frac{1}{h_r} \sum_{k \in I_r} u_k \Big[\Big(\|\frac{t_{k_n}(\Delta_v^m x_k - l)}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_k},$$

uniformly in n and for each $z_1, \cdots, z_{n-1} \in X$. Which gives the result. □

Theorem 2.7. *If* $0 < p_k \leq q_k$ *and* $(\frac{q_k}{p_k})$ *be bounded, then*

$$w_\sigma[\mathcal{F}, I, u, q, \|\cdot, \cdots, \cdot\|]_\theta(\Delta_v^m) \subset w_\sigma[\mathcal{F}, I, u, p, \|\cdot, \cdots, \cdot\|]_\theta(\Delta_v^m)$$

Proof. The proof is easy so omitted. □

3. Statistical convergence

The concept of convergence of a sequence of real numbers had been extended to statistical convergence by Fast [11] and later studied by many authors. We refer to the recent work in [1, 3, 4, 8, 28, 29, 30] for some applications of statistical summability to approximation theorems. Here, we define the notion of $S_{\theta_\sigma}^{\Delta_v^m}[I, u]$-convergence with the help of an ideal and invariant means. We also made an effort to establish a strong connection between the spaces $S_{\theta_\sigma}^{\Delta_v^m}[I, u, \|\cdot, \cdots, \cdot\|]$ and $w_\sigma[\mathcal{F}, I, u, p, \|\cdot, \cdots, \cdot\|]_\theta(\Delta_v^m)$.

Definition 3.1. Let $I \subseteq P(\mathbb{N})$ be a non-trivial ideal. A sequence $x = (x_k) \in X$ is said to be $S_{\theta_\sigma}^{\Delta_v^m}[I, u]$-convergent to a number l provided that for every $\varepsilon > 0$, $\delta > 0$ and $z_1, \cdots, z_{n-1} \in X$, the set

$$\Big\{r \in \mathbb{N} : \frac{1}{h_r}\Big|\Big\{k \in I_r : u_k\|\frac{t_{k_n}(\Delta_v^m x_k - l)}{\rho}, z_1, \cdots, z_{n-1}\| \geq \varepsilon\Big\}\Big| \geq \delta\Big\} \in I.$$

uniformly in n. In this case, we write $S_{\theta_\sigma}^{\Delta_v^m}[I, u] - \lim_{k\to\infty} x_k = l$.

Let $S_{\theta_\sigma}^{\Delta_v^m}[I, u, \|\cdot, \cdots, \cdot\|]$, denotes the set of all $S_{\theta_\sigma}^{\Delta_v^m}[I, u]$-convergent sequences in X.

Theorem 3.2. *Let* $\mathcal{F} = (f_k)$ *be a sequence of modulus functions and* $0 < \inf_k p_k = h \leq p_k \leq \sup_k p_k = H < \infty$. *Then* $w_\sigma[\mathcal{F}, I, u, p, \|\cdot, \cdots, \cdot\|]_\theta(\Delta_v^m) \subset S_{\theta_\sigma}^{\Delta_v^m}[I, u\|\cdot, \cdots, \cdot\|]$.

Proof. Suppose $x = (x_k) \in w_\sigma[\mathcal{F}, I, u, p, \|\cdot, \cdots, \cdot\|]_\theta(\Delta_v^m)$ and $\varepsilon > 0$ be given. Then for each $z_1, \cdots, z_{n-1} \in X$, we obtain

$$\begin{aligned}
&\frac{1}{h_r}\sum_{k\in I_r} u_k[f_k(\|\frac{t_{k_n}(\Delta_v^m x_k - l)}{\rho}, z_1, \cdots, z_{n-1}\|)]^{p_k} \\
&= \frac{1}{h_r}\sum_{k\in I_r \& f_k(\|\frac{t_{k_n}(\Delta_v^m x_k - l)}{\rho}, z_1, \cdots, z_{n-1}\|)\geq\varepsilon} u_k\Big[f_k\Big(\|\frac{t_{k_n}(\Delta_v^m x_k - l)}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_k} \\
&+ \frac{1}{h_r}\sum_{k\in I_r \& f_k(\|\frac{t_{k_n}(\Delta_v^m x_k - l)}{\rho}, z_1, \cdots, z_{n-1}\|)<\varepsilon} u_k\Big[f_k\Big(\|\frac{t_{k_n}(\Delta_v^m x_k - l)}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_k} \\
&\geq \frac{1}{h_r}\sum_{k\in I_r \& f_k(\|\frac{t_{k_n}(\Delta_v^m x_k - l)}{\rho}, z_1, \cdots, z_{n-1}\|)\geq\varepsilon} u_k\Big[f_k\Big(\|\frac{t_{k_n}(\Delta_v^m x_k - l)}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_k}
\end{aligned}$$

$$\geq \frac{1}{h_r}\sum_{k\in I_r}[f_k(\varepsilon)]^{p_k} \geq \sum_{k\in I_r}\min([f_k(\varepsilon)]^h,[f_k(\varepsilon)]^H)$$
$$\geq K\frac{1}{h_r}\Big|\Big\{k\in I_r : \|\frac{t_{k_n}(\Delta_v^m x_k - l)}{\rho}, z_1,\cdots,z_{n-1}\| \geq \varepsilon\Big\}\Big|,$$

where $K = \min([f_k(\varepsilon)]^h,[f_k(\varepsilon)]^H)$. Then for every $\delta > 0$ and $z_1,\cdots,z_{n-1} \in X$, we have

$$\Big\{r\in\mathbb{N} : \frac{1}{h_r}\Big|\Big\{k\in I_r : u_k\|\frac{t_{k_n}(\Delta_v^m x_k - l)}{\rho}, z_1,\cdots,z_{n-1}\| \geq \varepsilon\Big\}\Big| \geq \delta\Big\}$$
$$\subseteq \left\{r\in\mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r}u_k\Big[f_k\Big(\|\frac{t_{k_n}(\Delta_v^m x_k - l)}{\rho}, z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_k} \geq K\delta\right\}.$$

Since $x_k \in w_\sigma[\mathcal{F}, I, u, p, \|\cdot,\cdots,\cdot\|]_\theta(\Delta_v^m)$ so that $x \in S_{\theta_\sigma}^{\Delta_v^m}[I, u, \|\cdot,\cdots,\cdot\|]$. □

Theorem 3.3. *Let $\mathcal{F} = (f_k)$ be a sequence of modulus functions and $p = (p_k)$ be a bounded sequence of strictly positive real numbers. If $0 < \inf_k p_k = h \leq p_k \leq \sup_k p_k = H < \infty$ then $S_{\theta_\sigma}^{\Delta_v^m}[I, u, \|\cdot,\cdots,\cdot\|] \subset w_\sigma[\mathcal{F}, I, u, p, \|\cdot,\cdots,\cdot\|]_\theta(\Delta_v^m)$.*

Proof. Using the same technique of ([10], Theorem 3.5), it is easy to prove. □

Theorem 3.4. *Let $\mathcal{F} = (f_k)$ be a bounded sequence of modulus functions and $p = (p_k)$ be a bounded sequence of strictly positive real numbers. If $0 < \inf_k p_k = h \leq p_k \leq \sup_k p_k = H < \infty$ then*

$$S_{\theta_\sigma}^{\Delta_v^m}[I, u, \|\cdot,\cdots,\cdot\|] = w_\sigma[\mathcal{F}, I, u, p, \|\cdot,\cdots,\cdot\|]_\theta(\Delta_v^m)$$

if and only if $\mathcal{F} = (f_k)$ is bounded.

Proof. Direct part can be obtained by combining Theorems 3.2. and 3.3. Conversely suppose $\mathcal{F} = (f_k)$ be unbounded defined by $f_k(k) = k$, for all $k \in \mathbb{N}$. Let $\theta = (2^r)$ be a lacunary sequence. We take a fixed set $B \in I$, where I be an admissible ideal and define $x = (x_k)$ as follows

$$x_k = \begin{cases} k^{m+1}, & \text{if } r \notin B,\ 2^{r-1}+1 \leq k \leq 2^{r-1} + [\sqrt{h_r}], \\ k^{m+1}, & \text{if } r \in B,\ 2^{r-1} < k \leq 2^{r-1} + h_r, \\ 0, & \text{otherwise.} \end{cases}$$

where $I_r = (2^{r-1},\ 2^r]$ and $h_r = 2^r - 2^{r-1}$. For given $\varepsilon > 0$ and for each $z_1,\cdots,z_{n-1} \in X$ we have,

$$\lim_{r\to\infty}\frac{1}{h_r}\big|\big\{k\in I_r : u_k\|\frac{t_{k_n}(\Delta_v^m x_k - 0)}{\rho}, z_1,\cdots,z_{n-1}\| \geq \varepsilon\big\}\big| < \frac{[\sqrt{h_r}]}{h_r} \to 0,$$

for all $r \notin B$, uniformly in n. Hence for $\delta > 0$, there exists a positive integer r_0 such that $\frac{1}{h_r}\Big|\Big\{k \in I_r : u_k\|\frac{t_{k_n}(\Delta_v^m x_k - 0)}{\rho}, z_1, \cdots, z_{n-1}\| \geq \varepsilon\Big\}\Big| < \delta$ for $r \notin B$ and $r \geq r_0$. Now we have

$$\Big\{r \in \mathbb{N} : \frac{1}{h_r}|\{k \in I_r : u_k\|\frac{t_{k_n}(\Delta_v^m x_k - 0)}{\rho}, z_1, \cdots, z_{n-1}\| \geq \varepsilon\}| \geq \delta\Big\}$$
$$\subset \{B \cup (1, 2, ..., r_0 - 1)\}.$$

Since I be an admissible ideal.

It follows that $S_{\theta_\sigma}^{\Delta_v^m}[I, u]{-}\lim_{k\to\infty} \|\frac{x_k}{\rho}, z_1, \cdots, z_{n-1}\| \to 0$ for each $z_1, \cdots, z_{n-1} \in X$. On the other hand, if we take $p = (p_k) = 1$, for all $k \in \mathbb{N}$ then $x_k \notin w_\sigma[\mathcal{F}, I, u, p, \|\cdot, \cdots, \cdot\|]_\theta(\Delta_v^m)$. This contradicts the fact $S_{\theta_\sigma}^{\Delta_v^m}[I, u, \|\cdot, \cdots, \cdot\|] = w_\sigma[\mathcal{F}, I, u, p, \|\cdot, \cdots, \cdot\|]_\theta(\Delta_v^m)$, so our supposition is wrong. □

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