ZARISKI TOPOLOGY FOR SECOND SUBHYPERMODULES

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Abstract. The purpose of this paper is introduce the concept of second subhypermodules over commutative hyperrings and topologize the collection of all second subhypermodules and also investigate the properties of this topological space.

Keywords: R-hypermodule, subhypermodule, second subhypermodule, top hypermodule.

1. Introduction

Hyperstructure theory was first initiated by Marty [8] in 1934 at the 8th congress of scandinavian mathematicians. He defined hypergroups as a generalization of groups and proved its unility in solving some problems of groups, algebraic functions and rational functions. Survey of the theory of hyperstructures can be found in the book of Corsini [4], Vougiouklis [12], Corsini and Leoreanu [5]. The Krasner hyperring [7] is a well known type of hyperring, with the property that the addition is a hyperoperation and the multiplication is a binary operation. This concept has been studied in depth by many authors, for example see [6]. The concept of hypermodule over a Krasner hyperring has been introduced and investigated by Massouros [9].

As a dual notion of prime submodules, Yassemi [13], introduced the notion of second submodules of a given nonzero module over a commutative ring.

An R-endomorphism r^* on M is called homothety if $r^*(x) = rx$, for all $x \in M$. The nonzero submodule K of M is said to be second if for each $r \in R$ the homothety $r^* : K \longrightarrow K$ is either surjective or zero. This implies that $Ann(K) = (0 : M) = p$ is a prime ideal of R, and K is said to be p-second. Also, we say that M is a second module if M is a second submodule of itself.

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The notion of second submodule can be generalized to second subhypermodule. In fact, in this paper we introduce the concept of second subhypermodules over a commutative hyperring and we topologize the spectrum of second subhypermodules and investigate the properties of the induced topology.

2. Preliminaries

The following definitions and preliminaries are required in the sequel of our work and hence presented in brief.

Let H be a nonempty set and $P^*(H)$ be the set of all nonempty subsets of H. A hyperoperation on H is a map $\circ : H \times H \longrightarrow P^*(H)$ and the couple $(H, \circ)$ is called a hypergroupoid.

For any two nonempty subsets A and B of H, we define

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b.$$

If $x \in H$, by $x \circ A$ and $A \circ x$, we mean $\{x\} \circ A$ and $A \circ \{x\}$, respectively.

A hypergroupid $(H, \circ)$ is called semihypergroup if $a \circ (b \circ c) = (a \circ b) \circ c$, for all $a, b, c \in H$.

A hypergroupoid $(H, \circ)$ is called quasihypergroup if $a \circ H = H = H \circ a$, for all $a \in H$. If $(H, \circ)$ is semihypergroup and quasihypergroup, then $(H, \circ)$ is called hypergroup.

A subhypergroup $(K, \circ)$ of $(H, \circ)$ is a nonempty subset K of H such that $k \circ K = K = K \circ k$, for all $k \in K$.

A hypergroup $(H, \circ)$ is called canonical [4], if:

(i) $(H, \circ)$ is commutative, which means $x \circ y = y \circ x$, for all $x, y \in H$;

(ii) there exists $e \in H$, such that $\{x\} = (x \circ e) \cap (e \circ x)$ for all $x \in H$;

(iii) for all $x \in H$ there exists a unique $x^{-1} \in H$, such that $e \in x \circ x^{-1}$;

(iv) $x \in y \circ z$ implies that $y \in x \circ z^{-1}$.

In (ii), an element e is called identity (scalar identity) of $(H, \circ)$.

There are several kinds of hyperrings and hypermodules that can be defined on a nonempty set. In what follows, we consider some of the most general types of hyperrings and hypermodules.

An algebraic system $(R, +, \cdot)$ is called Krasner hyperring, if

(i) $(R, +)$ is a canonical hypergroup;

(ii) $(R, \cdot)$ is a semigroup having zero as a bilaterally absorbing element;

(iii) $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(y + z) \cdot x = y \cdot x + z \cdot x$, for all $x, y, z \in R$.

The hyperring and R-hypermodule was studied in [10]. In this paper, a little change is done, which consists in using " $=$ " instead of " $\subseteq$ ", as in [10].

Definition 2.1 ([10]). *The triple $(R, +, \cdot)$ is a hyperring if:*

(*i*) *$(R, +)$ is a canonical hypergroup with scalar identity 0_R (we denote the opposite of $r \in R$, that is r^{-1}, by $-r$);*

(*ii*) *$(R, \cdot)$ is a semihypergroup;*

(*iii*) *$r \cdot (s + t) = r \cdot s + r \cdot t$ and $(s + t) \cdot r = s \cdot r + t \cdot r$, for all $r, s, t \in R$.*

Note that "$\cdot$" is a hyperoperation on R. The hyperring $(R, +, \cdot)$ satisfies the conditions:
for all $r, s \in R$, we have

$$r \cdot 0_R = \{0_R\} = 0_R \cdot r \text{ and } r \cdot (-s) = (-r) \cdot s = -(r \cdot s) = \{-t | t \in r \cdot s\}.$$

We abbreviate a hyperring $(R, +, \cdot)$ by R.

The hyperring R is said to be commutative, if R is commutative with respect to hyperoperation "$\cdot$". In the sequel, by a hyperring we mean a commutative hyperring. We say that a hyperring R is with identity element 1, if for all $r \in R$, $r \in r \cdot 1$.

Definition 2.2 ([10]). *Let R be a hyperring with identity element 1. An R-hypermodule is a structure $(M, +, \circ)$ such that $(M, +)$ is a canonical hypergroup with scalar identity 0_M and "$\circ$" is a multivalued scalar operation, i.e.,*

$$\circ : R \times M \longrightarrow P^*(M)$$

such that for all $a, b \in R$ and $x, y \in M$,

(*i*) *$a \circ (x + y) = a \circ x + a \circ y$;*

(*ii*) *$(a + b) \circ x = a \circ x + b \circ x$;*

(*iii*) *$(a \cdot b) \circ x = a \circ (b \circ x)$;*

(*iv*) *$x \in 1 \circ x$.*

The hypermodule $(M, +, \circ)$ also satisfies the conditions: for all $a \in R$ and $x \in M$, we have

$$a \circ 0_M = \{0_M\} = 0_R \circ x \text{ and}$$
$$a \circ (-x) = (-a) \circ x = -(a \circ x) = \{-y : y \in a \circ x\}$$

(we denote the opposite of $x \in M$, that is x^{-1}, by $-x$).

We abbreviate an R-hypermodule $(M, +, \circ)$ by M. This definition is a generalization of the concept of hypermodule over a Krasner hyperring (see [4]).

A nonempty subset N of an R-hypermodule M is subhypermodule, if N is an R-hypermodule under the same hyperoperations on M.

Proposition 2.3. *Let N be a nonempty subset of an R-hypermodule M. Then N is a subhypermodule of M if and only if for all $r \in R$ and $x, y \in N$, we have $r \circ x \subseteq N$ and $x - y \subseteq N$.*

Proof. Let for all $r \in R$ and $x, y \in M$, $r \circ x \subseteq N$ and $x - y \subseteq N$.

Since N is nonempty, then there exists $a \in N$. So $0 \in a - a \subseteq N$. If $y \in N$, then $-y = 0 - y \in N$. Also, if $x, y \in N$, then $x - (-y) = x + y \subseteq N$. All other conditions for N to be a hypermodule follows as hereditary proporties from M. Thus N is an R-hypermodule and so it is a subhypermodule of M. The converse is obvious. $\square$

For a family of subhypermodules $\{N_i\}_{i\in I}$ of M, we define:

$$\sum_{i\in I} N_i = \cup\{\sum_{i\in I} a_i | a_i \in N_i, \text{ for } i \in I \text{ and } \exists n \in \mathbb{N}; a_i = 0, \forall i \geq n\}.$$

By Proposition 2.3, we can see that $\sum_{i\in I} N_i$ is a subhypermodule of M.

Let R be a hyperring and M an R-hypermodule. For any nonempty subsets A, B of R and nonempty subset X of M, we set

$$A \circ X = \bigcup_{i\in I}\{a_i \circ x_i | a_i \in A \textit{ and } x_i \in X\};$$

$$[A \circ X] = \cup\Big\{\sum_{i=1}^{n} a_i \circ x_i | a_i \in A \textit{ and } x_i \in X, \textit{ for all } 1 \leq i \leq n\Big\};$$

$$[A \cdot B] = \cup\Big\{\sum_{i=1}^{n} a_i \cdot b_i | a_i \in A \textit{ and } b_i \in B, \textit{ for all } 1 \leq i \leq n\Big\}.$$

A nonempty subset I of a commutative hyperring R is called hyperideal, if $a - b \subseteq I$ and $r \cdot a \subseteq I$, for all $r \in R$ and $a, b \in I$. If I and J are hyperideals of R, then $I + J$ and $[I.J]$ are hyperideals of R.

A proper hyperideal P of R is called prime if $a \cdot b \subseteq P$ implies that either $a \in P$ or $b \in P$, for $a, b \in R$ (see [1]).

Let A be a nonempty subset of a hyperring R. Define $\langle A\rangle$ to be the smallest hyperidal of R containing A.

Lemma 2.4 ([10])**.** *Let A be a nonempty subset of a hyperring R with identity. Then $\langle A\rangle = [R \cdot A]$.*

Let I be a hyperideal of a hyperring R. Put $\frac{R}{I} = \{r + I | r \in R\}$. It is easy to see that $r + I = s + I$ if and only if $r - s \subseteq I$ for $r, s \in R$. So $\frac{R}{I}$ with the hyperoperations defined as follows, is a hyperring.

$$(r + I) \oplus (s + I) = \{t + I | t \in r + s\};$$
$$(r + I) \odot (s + I) = \{u + I | u \in r \cdot s\},$$

for all $r, s \in R$.

Denote the opposite of $r + I \in \frac{R}{I}$, by $(-r) + I$.

Proposition 2.5 ([10]). *Let M be an R-hypermodule, $r \in R$, I a hyperideal of R, N and K subhypermodules of M and X a nonempty subset of M. Then $r \circ N$, $[I \circ N]$ and $[R \circ X]$ are subhypermodules of M.*

Let X be a nonempty subset of an R-hypermodule M. Define $\langle X \rangle$ to be the smallest subhypermodule of M containing X. It is easy to see that, $\langle X \rangle = [R \circ X]$.

Let M be an R-hypermodule and X, Y be subsets of M. In general, $\langle X + Y \rangle \subseteq \langle X \rangle + \langle Y \rangle$. Now if $\{0_M\} \subseteq X \cap Y$, then $\langle X + Y \rangle \supseteq \langle X \rangle + \langle Y \rangle$. Because $[R \circ X] = [R \circ (X + 0_M)] \subseteq [R \circ (X + Y)]$. Similarly, $[R \circ Y] \subseteq [R \circ (X + Y)]$.

Corollary 2.6. *Let M be an R-hypermodule and X, Y be subsets of M such that $\{0_M\} \subseteq X \cap Y$. Then $\langle X + Y \rangle = \langle X \rangle + \langle Y \rangle$.*

Now it is natural to speak about homomorphism of hypermodules.

Definition 2.7. *Suppose that M and M' are R-hypermodules. A map $f : M \longrightarrow M'$ is called homomorphism if for all $x, y \in M$ and $r \in R$, we have*

$$f(x + y) \subseteq f(x) + f(y) \quad \textit{and} \quad f(r \circ x) = r \circ f(x).$$

Also, f is called strong homomorphism if for all $x, y \in M$ and $r \in R$,

$$f(x + y) = f(x) + f(y) \ \textit{and} \ \ f(r \circ x) = r \circ f(x).$$

Suppose that M and M' are R-hypermodules, f is a homomorphism from M into M' and N is a subhypermodule of M. Then for all $r \in R$,

$$f(r \circ N) = r \circ f(N).$$

Moreover, if f is a strong homomorphism, then $f(N)$ is a subhypermodule of M'.

Recall that two subhypermodules N, K of an R-hypermodule M are independent, if $K \cap N = \{0_M\}$. If K and N are independent, then $N + K$ is denoted by $N \oplus K$. Also, a subhypermodule N of M is called a direct summand of M, if $M = N \oplus L$, for some subhypermodule L of M (see [11]).

Note that if A, B, C are subhypermodules of M, then it need not be that

$$A \cap (B + C) = A \cap B + A \cap C.$$

For example, let $M = \{(x, y) | x, y \in \mathbb{Z}\}$ with trivial hyperoperations on the hyperring $\mathbb{Z}$. Also, let

$$A = \{(x, x) | x \in \mathbb{Z}\}, B = \{(x, 0) | x \in \mathbb{Z}\}, C = \{(0, x) | x \in \mathbb{Z}\}.$$

Then A, B, C are subhypermodules of M and we have $A \cap (B + C) = A$ and $A \cap B + A \cap C = \{(0, 0)\}$.

Lemma 2.8 ([11]). *Suppose that M is an R-hypermodule and A, B, C are subhypermodules of M such that $B \subseteq A$. Then $A \cap (B + C) = A \cap B + A \cap C$.*

Definition 2.9. *A subhypermodule N of an R-hypermodule M is said to be relatively divisible if $r \circ N = (r \circ M) \cap N$, for all $r \in R$.*

Lemma 2.10. *Let M be an R-hypermodule. Then every direct summand N of M is relatively divisible.*

Proof. There exists a subhypermodule K of M such that $M = N \oplus K$ (Since N is a direct summand of M). So

$$r \circ M = r \circ (N \oplus K) = (r \circ N) \oplus (r \circ K).$$

Since $r \circ N \subseteq N$, by Lemma 2.8, we have

$$\begin{aligned} N \cap (r \circ M) &= N \cap \big((r \circ N) \oplus (r \circ K)\big) \\ &= (r \circ N) \oplus \big(N \cap (r \circ K)\big) = (r \circ N) \oplus 0_M = r \circ N. \end{aligned}$$

□

Definition 2.11. *An R-hypermodule M is called divisible if $r \circ M = M$, for every $r \in R$.*

Example 2.12. Let M be a divisible R-module, N a submodule of M and I an ideal of R. Assume that $\overline{a} = a + I$ is the additive class for $a \in R$, and $\frac{R}{I}$ is the set of all these classes. For all $\overline{a}, \overline{b} \in \frac{R}{I}$, we define

$$\overline{a} \oplus \overline{b} = \{\overline{c} | c \in \overline{a} + \overline{b}\} \text{ and } \overline{a} \odot \overline{b} = \{\overline{ab}\}.$$

Then $(\frac{R}{I}, \oplus, \odot)$ is a hyperring.

Define the equivalence relation β on M by

$$x \beta y \text{ if and only if } x + N = y + N.$$

For all $\overline{x}, \overline{y} \in \frac{M}{N}, \overline{a} \in \frac{R}{I}$, we define

$$\overline{x} \oplus \overline{y} = \{\overline{z} | z \in \overline{x} + \overline{y}\} \text{ and } \overline{a} \circ \overline{x} = \{\overline{ax}\}.$$

Then $(\frac{M}{N}, \oplus, \circ)$ is an $\frac{R}{I}$-hypermodule. Also,

$$\overline{r} \circ \frac{M}{N} = r(\frac{M}{N}) = \frac{M}{N},$$

for all $r \in R$ (since M is divisible).

Moreover, if M is an R-hypermodule and $\{N_i\}_{i \in \Lambda}$ is a family of divisible subhypermodules of M, then $\sum_{i \in \Lambda} N_i$ is a divisible subhypermodule of M. Because

$$r \circ (\sum_{i \in \Lambda} N_i) = \sum_{i \in \Lambda} (r \circ N_i) = \sum_{i \in \Lambda} N_i$$

for all $r \in R$.

3. Second subhypermodules

In this section, we introduce the notion of second subhypermodules and investigate some properties of them.

Definition 3.1. *Let M be an R-hypermodule. A nonzero subhypermodule N of M is called second if for all $r \in R$, either $r \circ N = N$ or $r \circ N = \{0_M\}$. Also, we say that M is a second R-hypermodule if M is a second subhypermodule of itself.*

By $Spec^s(M)$, the spectrum of second subhypermodules of M, we mean the set of all second subhypermodules of M.

Example 3.2. Every divisible R-hypermodule is second.

Example 3.3. Every relatively divisible subhypermodule of a second R-hypermodule is second.

It is not necessary that every subhypermodule of a second R-hypermodule is second. For example, Suppose that $(M, +, \cdot)$ is a divisible hypermodule over a Krasner hyerring R. Let I be a hyperideal of R and N, K subhypermodules of M such that for all $r \in R$, $\{0_M\} \subset r \cdot K \subset K$. We define

$$\circ : R \times M \longrightarrow P^*(M) \text{ by } r \circ x = r \cdot x + N$$

for all $r \in R$ and $x \in M$. Then $(M, +, \circ)$ is a second R-hypermodule and K is not a second subhypermodule of M.

Lemma 3.4. *Let M and M' be R-hypermodules and f a strong homomorphism of M into M'. If $N \in Spec^s(M)$, then $f(N) \in Spec^s(M')$.*

Proof. Since f is a strong homomorphism of M into M' and N is second, then $r \circ f(N) = f(r \circ N) = f(0_M) = \{0_{M'}\}$ or $r \circ f(N) = f(r \circ N) = f(N)$, for all $r \in R$. $\square$

For an R-hypermodule M, we set

$$Ann(M) = \big\{r \in R \big| r \circ M = \{0_M\}\big\}.$$

Since $0_R \in Ann(M)$, then $Ann(M) \neq \emptyset$.

Lemma 3.5. *Let M be an R-hypermodule. Then $Ann(M)$ is a hyperideal of R.*

Proof. Let $a, b \in Ann(M)$ and $r \in R$. Then

$$\{0_M\} = 0_M + 0_M = a \circ M + b \circ M = (a + b) \circ M.$$

So $a + b \subseteq Ann(M)$. Also,

$$\{0_M\} = r \circ 0_M = r \circ (a \circ M) = (r \cdot a) \circ M.$$

Hence $r \cdot a \subseteq Ann(M)$. $\square$

Proposition 3.6. *Let N be a second subhypermodule of an R-hypermodule M. Then $p = Ann(N)$ is a prime hyperideal of R.*

Proof. Let $a, b \in R$ and $a \cdot b \subseteq Ann(N)$ and $b \notin Ann(N)$. Then $b \circ N = N$ and $(a \cdot b) \circ N = \{0_M\}$. So $\{0_M\} = a \circ (b \circ N) = a \circ N$. Hence $a \in Ann(N)$. □

In the meantime, we say that N is a p-second subhypermodule of an R-hypermodule M if N is a second subhypermodule of M with $p = Ann(N)$.

Lemma 3.7. *Let N be a subhypermodule of an R-hypermodule M and $W(N) = \{r \in R | r \circ N \neq N\}$. Then the following statements are equivalent:*

(i) *N is a p-second subhypermodule of M;*

(ii) *$Ann(N) = W(N) = p$.*

Proof. The proof is straightforward. □

Definition 3.8. *Let M be an R-hypermodule. A nonzero element $x \in M$ is called torsion-free if for $t \in R$, $0_M \in t \circ x$ implies that $t = 0_R$. Also, M is called torsion-free R-hypermodule, if all nonzero elements of M are torsion-free.*

Let M be a torsion-free R-hypermodule and I a hyperideal of R such that $I \subseteq Ann(M)$ and $\odot$ be a multivalued scalar operation define by

$$(r + I) \odot x := r \circ x,$$

for all $r \in R$. Then $(M, +, \odot)$ is an $\frac{R}{I}$-hypermodule.

Lemma 3.9. *Let N be a subhypermodule of a torsion-free R-hypermodule M and $Ann(N) = p$. Then the following statements are equivalent:*

(i) *N is a p-second subhypermodule of M;*

(ii) *N is a divisible $\frac{R}{p}$-hypermodule;*

(iii) *$r \circ N = N$ for all $r \in R - p$;*

(iv) *$W(N) = p$.*

Proof. The proof is straightforward. □

Definition 3.10. *Let M be an R-hypermodule and N a subhypermdule of M. N is called a minimal subhypermodule, if it is a minimal element in the set of all subhypermodules of M. On the other word, it is not strictly contains any other nonzero subhypermodules of M.*

Theorem 3.11. *Every minimal subhypermodule of a hypermodule is second.*

Proof. Let M be an R-hypermodule and N a minimal subhypermodule of M. Consider $r \in R$. Since $r \circ N \subseteq N$, then either $r \circ N = N$ or $r \circ N = \{0_M\}$ (by minimality of N). □

Let M be an R-hypermodule. We set

$$\sqrt{Ann(M)} = \big\{r \in R \big| r^n \circ M = \{0_M\}, \text{ for some } n \in \mathbb{N}\big\}.$$

Lemma 3.12. *Let M be an R-hypermodule. Then $\sqrt{Ann(M)}$ is a hyperideal of R.*

Proof. Ler $a, b \in \sqrt{Ann(M)}$ and $r \in R$. Then $-b \in \sqrt{Ann(M)}$ and there exist $m, n \in \mathbb{N}$ such that $a^m \circ M = \{0_M\} = (-b)^n \circ M$. But

$$(a-b)^{m+n} = \sum_{r=0}^{m+n} \tbinom{m+n}{r} (a^{m+n-r} \cdot (-b)^r).$$

So $(a-b)^{m+n} \circ M = \{0_M\}$ and hence $a - b \subseteq \sqrt{Ann(M)}$. Also,

$$(r \cdot a)^m \circ M = (r^m \cdot a^m) \circ M = r^m \circ (a^m \circ M) = r^m \circ \{0_M\} = \{0_M\}.$$

Thus $r \cdot a \subseteq \sqrt{Ann(M)}$. □

Definition 3.13. *An R-hypermodule M is called secondary if for all $r \in R$, either $r \circ M = M$ or there exists $n \in \mathbb{N}$ such that $r^n \circ M = \{0_M\}$.*

Obviously, every second R-hypermodule is secondary.

Lemma 3.14. *Let M be a secondary R-hypermodule. Then $\sqrt{Ann(M)} = p$ is a prime hyperideal of R.*

Proof. Let $a \cdot b \subseteq \sqrt{Ann(M)}$ and $b \notin \sqrt{Ann(M)}$. Then there exists $n \in \mathbb{N}$ such that $\{0_M\} = \big(a \cdot b)^n \circ M$. So $\{0_M\} = a^n \circ (b^n \circ M)$. Since M is secondary and $b \notin \sqrt{Ann(M)}$, then $a^n \circ M = \{0_M\}$. Hence $a \in \sqrt{Ann(M)}$. □

We say that N is a p-secondary subhypermodule of an R-hypermodule M if N is a secondary subhypermodule of M with $\sqrt{Ann(M)} = p$.

Proposition 3.15. *Let M be an R-hypermodule and N a subhypermodule of M. Then the following statements hold:*

(i) If N is secondary, then N is second if and only if $Ann(N)$ is a prime hyperideal of R.

(ii) If N is p-secondary contained in a p-second subhypermodule, then N is a p-second subhypermodule of M.

Proof. (i) This is obvious.

(ii) Let K be a p-second subhypermodule of M and $N \subseteq K$. Then

$$p = Ann(K) \subseteq Ann(N) \subseteq \sqrt{Ann(N)} = p.$$

So $Ann(N) = p$ and by (i), N is a p-second subhypermodule of M. □

Lemma 3.16. *Let M be an R-hypermodule, N a secondary subhypermodule of M and $r \in R$. Then $r \circ N$ is a secondary subhypermodule of M.*

Proof. Let $r \circ N \subsetneq N$. Since N is secondary, then there exists $n \in \mathbb{N}$ such that $\{0_M\} = r^n \circ N$. So $r^{n-1} \circ (r \circ N) = \{0_M\}$. Thus $r \circ N$ is a secondary subhypermodule of M. □

We call a subhypermodule N of an R-hypermodule M a minimal secondary subhypermodule of M if N is a secondary subhypermodule which is not strictly contains any other secondary subhypermodule of M.

Corollary 3.17. *Let M be an R-hypermodule and N be a minimal secondary subhypermodule of M. Then N is second and $W(N) = Ann(N)$.*

Proof. Let $r \in W(N)$. Then $r \circ N$ is a secondary subhypermodule of M (by Lemma 3.16). Since N is minimal secondary and $r \circ N \subseteq N$, then $r \circ N = \{0_M\}$. Hence N is a second subhypermodule of M and $r \in Ann(N)$ and $W(N) = Ann(N)$. □

4. Top hypermodules

In this section, we topologize $Spec^s(M)$ and investigate the properties of the induced topology. First we recall some definitions.

An R-module M is called uniserial, if for any submodules K, L of M, either $K \subseteq L$ or $L \subseteq K$. Also, A submodule N of M is called strongly hollow, if for any submodules K, L of M, $N \subseteq K + L$ implies that $N \subseteq K$ or $N \subseteq L$ (see [2]). So we can define uniserial R-hypermodule and strongly hollow subhypermodule in the same way.

Example 4.1. Let M be a uniserial R-module and N be a submodule of M. We define $\circ : R \times M \longrightarrow P^*(M)$ by $r \circ x = rx + N$, for all $r \in R$ and $x \in M$. Then $(M, +, \circ)$ is a uniserial R-hypermodule.

Example 4.2. Let M be an R-module and N a submodule of M. We define $\circ : R \times M \longrightarrow P^*(M)$ by $r \circ x = rx + N$, for all $r \in R$ and $x \in M$. Then $(M, +, \circ)$ is an R-hypermodule and N is a strongly hollow subhypermodule of M.

By $SH(M)$, we mean the set of all strongly hollow subhypermodules of M.

For every subhypermodule N of an R-hypermodule M, we set

$$V^s(N) = \{K \in Spec^s(M) | K \subseteq N\}$$
$$\chi^s(N) = \{K \in Spec^s(M) | K \nsubseteq N\}.$$

Also, for an R-hypermodule M we set

$$\zeta^s(M) = \{V^s(N) | N \textit{ is a subhypermodule of } M\}$$
$$\tau^s(M) = \{\chi^s(N) | N \textit{ is a subhypermodule of } M\}$$
$$Z^s(M) = (Spec^s(M), \tau^s(M)).$$

Theorem 4.3. *Let M be an R-hypermodule and $\{N_i\}_{i\in I}$ be a family of subhypermodules of M. Then the following statements hold:*

(*i*) $V^s(\{0_M\}) = \emptyset$, $V^s(M) = Spec^s(M)$;

(*ii*) $\bigcap_{i\in I} V^s(N_i) = V^s(\bigcap_{i\in I} N_i)$;

(*iii*) $V^s(N) \cup V^s(K) \subseteq V^s(N+K)$ *for subhypermodules* N, K.

Proof. Immediate. □

In general, $\zeta^s(M)$ is not closed under finite unions. If $\zeta^s(M)$ is closed under finite unions, then we call M is top hypermodule.

For example, if $Spec^s(M) = \emptyset$, then M is a top hypermodule. Also, if M is a uniserial R-hypermodule, then M is a top hypermodule.

Lemma 4.4. *Let M be an R-hypermodule. If $Spec^s(M) \subseteq SH(M)$, then M is a top hypermodule.*

Proof. It is obvious. □

Lemma 4.5. *Let M be an R-hypermodule. If M is a top hypermodule, then the closure of any subset $A \subseteq Spec^s(M)$ is*

$$\overline{A} = V^s(\sum\{K|K \in A\}).$$

Proof. Let $A \subseteq Spec^s(M)$. Since $A \subseteq V^s(\sum\{K|K \in A\})$, then

$$\overline{A} \subseteq V^s(\sum\{K|K \in A\}).$$

Conversely, Let $N \in V^s(\sum\{K|K \in A\}) \setminus A$ and $\chi^s(L)$ be a neighborhood of N. Then $N \nsubseteq L$. So there exists $T \in A$ such that $T \nsubseteq L$. That is $T \in \chi^s(L) \cap (A \setminus \{N\})$ and N is a cluster point of A. Thus

$$V^s(\sum\{K|K \in A\}) \subseteq \overline{A}.$$

□

Theorem 4.6. *Let M be an R-hypermodule. If M is a top hypermodule, then*

$$B = \{\chi^s(N)\big|N = \langle X\rangle, X \textit{ is a finite subset of } M\}$$

is a basis of open sets for $Z^s(M)$.

Proof. Any $K \in Spec^s(M)$ is contained in some $\chi^s(N)$ for some subhypermodule N of M such that $N = \langle X\rangle$ and X is a finite subset of M.

Now let $\{\chi^s(N_1), \chi^s(N_2)\} \subseteq B$ and L a subhypermodule of M such that $L \in \chi^s(N_1) \cap \chi^s(N_2)\}$. Since M is a top hypermodule, then

$$\chi^s(N_1) \cap \chi^s(N_2) = \chi^s(N_1 + N_2).$$

By Corollary 2.9, $N_1 + N_2 = \langle X + Y\rangle$ and so $\chi^s(N_1 + N_2) \in B$. □

A nonempty topological space X is called irreducible if X is not the union of two proper closed subsets (see [3]).

Lemma 4.7 ([2]). *Let X be a nonempty topological space and $A \subseteq X$. Then the following are equivalent:*

(1) *A is irreducible;*

(2) *for any closed subsets A_1, A_2 of X, $A \subseteq A_1 \cup A_2$ implies that $A \subseteq A_1$ or $A \subseteq A_2$;*

(3) *for any open subsets U_1, U_2 of X, $U_1 \cap A \neq \emptyset \neq U_2 \cap A$ implies that $(U_1 \cap U_2) \cap A \neq \emptyset$.*

Proposition 4.8. *Let M be an R-hypermodule and $A \subseteq Spec^s(M)$. If M is a top hypermodule and A is irreducible, then $\sum_{N \in A} N$ is a second subhypermodule of M.*

Proof. Let $r \in R$ and suppose that

$$r \circ (\sum_{N \in A} N) \neq \sum_{N \in A} N$$

and

$$r \circ (\sum_{N \in A} N) \neq \{0_M\}.$$

We set

$$A_1 = \{K \in A \big| r \circ K = K\} \text{ and}$$
$$A_2 = \big\{K \in A \big| r \circ K = \{0_M\}\big\}.$$

So $A \subseteq V^s(\sum_{N \in A_1} N) \cup V^s(\sum_{N \in A_2} N)$.
But $A \nsubseteq V^s(\sum_{N \in A_1} N)$ and $A \nsubseteq V^s(\sum_{N \in A_2} N)$ which is a contradiction. □

Proposition 4.9. *Let M be an R-hypermodule and*

$$A \subseteq Spec^s(M) \subseteq SH(M).$$

If $\sum_{N \in A} N$ is a nonzero strongly hollow subhypermodule of M, then A is irreducible.

Proof. Let $A \subseteq V^s(N_1) \cup V^s(N_2)$. Then $\sum_{N \in A} N \subseteq N_1 + N_2$.
Since $(\sum_{N \in A} N) \in SH(M)$, then $\sum_{N \in A} N \subseteq N_1$ or $\sum_{N \in A} N \subseteq N_2$. Hence

$$A \subseteq V^s(N_1) \text{ or } A \subseteq V^s(N_2).$$

□

Definition 4.10. *A nonzero R-hypermodule M is simple, if the only subhypermodules of M are $\{0_M\}$ and M.*

Example 4.11. Let $R = \{0, 1\}$ with

$$\begin{aligned} 0 + 0 &= \{0\} \\ 0 + 1 = 1 + 0 &= \{0, 1\} \\ 1 + 1 &= \{1\} \\ 0 \cdot 0 &= \{0\} \\ 0 \cdot 1 = 1 \cdot 0 &= \{0\} \\ 1 \cdot 1 &= \{0, 1\}. \end{aligned}$$

Then R is a simple R-hypermodule.

By $S(M)$, we mean the set of all simple subhypermodules of an R-hypermodule M.

We say that an R-hypermodule M has the min-property, if for any simple subhypermodule L of M, $L \nsubseteq \sum_{K \in S(M) \setminus \{L\}} K$.

For example, every R-hypermodule M with at most one simple subhypermodule has the min-property.

Definition 4.12. *Let M be an R-hypermodule. Then*

(i) *M is called semisimple, if for every subhypermodule K of M, there exists a subhypermodule P of M such that $M = K \oplus P$,*

(ii) *M is called atomic, if any nonzero subhypermodule N of M contains a simple subhypermodule,*

(iii) *M is called uniform, if for any nonzero subhypermodules N_1, N_2 of M, we have $N_1 \cap N_2 \neq \{0_M\}$.*

For example, any simple R-hypermodule is semisimple and any uniserial R-hypermodule is uniform.

Definition 4.13 ([2])**.** *We say that a nonempty topological space X*

(i) *is countably compact if every open cover of X has a countable finite subcover;*

(ii) *is ultraconnected if the intersection of any two nonempty closed subsets of X is nonempty.*

Proposition 4.14. *Let M be an R-hypermodule and atomic top hypermodule. M is uniform if and only if M is ultraconnected.*

Proof. Assume that M is uniform and $V^s(N_1), V^s(N_2)$ are nonempty closed subsets of $Spec^s(M)$. Then $N_1 \neq \{0_M\}$ and $N_2 \neq \{0_M\}$. Since M is uniform, then $N_1 \cap N_2 \neq \{0_M\}$. But $V^s(N_1) \cap V^s(N_2) = V^s(N_1 \cap N_2)$. Since M is an atomic R-hypermodule, then there exists a simple subhypermodule L of M such that $L \subseteq N_1 \cap N_2$. So $V^s(N_1 \cap N_2) \neq \emptyset$.

Conversely, let L_1, L_2 be nonzero subhypermodules of M. So $V^s(L_1) \neq \emptyset$ and $V^s(L_2) \neq \emptyset$. Since M is ultraconnected and $V^s(L_1) \cap V^s(L_2) = V^s(L_1 \cap L_2)$, then $L_1 \cap L_2 \neq \{0_M\}$ and hence M is uniform. □

Theorem 4.15. *Let M be an R-hypermodule and atomic top hypermodule. If $S(M)$ is countable, then $Spec^s(M)$ is countably compact.*

Proof. Assume that $S(M) = \{N_{\lambda_k}\}_{k\geq 1}$ is countable. Let $\{\chi^s(L_\alpha)\}_{\alpha\in I}$ be an open cover of $Spec^s(M)$. Since $S(M) \subseteq Spec^s(M)$, then for some $\alpha_k \in I$, $N_{\lambda_k} \nsubseteq L_{\alpha_k}$ for each $k \geqslant 1$.

Let $\bigcap_{k\geqslant 1} L_{\alpha_k} \neq \{0_M\}$. Since M is an atomic R-hypermodule, then there exists a simple subhypermodule N of M such that $N \subseteq \bigcap_{k\geqslant 1} L_{\alpha_k}$ which is contradiction.

Hence $\bigcap_{k\geqslant 1} L_{\alpha_k} = \{0_M\}$ and so

$$Spec^s(M) = \chi^s(\{0_M\}) = \chi^s(\bigcap_{k\geqslant 1} L_{\alpha_k}) = \bigcup_{k\geqslant 1} \chi^s(L_{\alpha_k}).$$

□

Proposition 4.16. *Let M be an R-top hypermodule and*

$$Spec^s(M) = S(M).$$

(*i*) *If M has the min-property, then $Spec^s(M)$ is discrete.*

(*ii*) *M has a unique simple subhypermodule if and only if M has the min-property and $Spec^s(M)$ is connected.*

Proof. (i) Let M has the min-property. Then for every $K \in S(M)$,

$$\chi^s(\sum_{N\in S(M)\setminus\{K\}} N) = \{K\}.$$

Since every singleton set is open, then $Spec^s(M)$ is discrete.

(ii) Let M has a unique simple subhypermodule. Clearly, M has the min-property and $Spec^s(M)$ is connected.

Conversely, by (i) $Spec^s(M)$ is discrete. Since $Spec^s(M)$ is connected, then $Spec^s(M)$ has only one point. □

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