CHAPMAN-JOUGUET TRAVELLING WAVE FOR A TWO-STEPS REACTION SCHEME

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Abstract. In this paper, a version of the Majda's model for a two-steps reaction is studied. Then by studying the stable and unstable manifolds of the resulted system of ordinary differential equations, the existence of a heteroclinic orbit is proved.
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1. Introduction

Majda [7] wrote the Navier-Stokes equations of reacting flow for a one-step reaction as a model which is expected to retain important aspects of the strongly coupled interactions between the nonlinear motion of a gas mixture and chemical reactions involving the different species of gas making up the mixture. This model in the abstract form is as follows.

$$\begin{cases} (U+qZ)_t + f(U)_x = BU_{xx}, \\ Z_t = -k\phi(U)Z \end{cases} \tag{1}$$

where the unknown function $U = U(x,t)$ is real valued(should be thought of as a stand–in for density, velocity, and temperature), $0 \leq Z = Z(x,t) \leq 1$ measures the fraction by mass of reactant (fuel) in a simple one-step reaction scheme, the flux f is a nonlinear convex function, ϕ is the ignition function and finally k, q, and B are positive constants measuring reaction rate, heat release, and viscosity, respectively.

Following [3], we restrict attention to the Majda's model for a two-steps reaction scheme $R \to I, I \to P$ which is modeled by

$$\begin{cases} \beta T_{xx} = \frac{d}{d\tau}T + \{f(T)\}_x - q_0 Z_{0x} - q_1 Z_{1x}, \\ Z_{0x} = \kappa_1 \Phi_1(T) Z_0, \\ Z_{1x} = \kappa_2 \Phi_2(T)(Z_0 - Z_1), \end{cases} \tag{2}$$

where $\kappa_j > 0,\ j = 1, 2$ are constant, T temperature, Z_0 mass fraction of unburned gas (note that the completely unburned gas is $Z_0 = 1$ and a totally burned gas is $Z_0 = 0$), Z_1 mass fraction of intermediate gas (note that the completely intermediate gas is $Z_1 = 1$ and a totally intermediate gas is $Z_1 = 0$), τ a variable proportional to the arc length along the characteristics, the coordinate x is not the space coordinate but is determined as a scaled space-time coordinate representing distance to the reaction zone (the x-differentiation occurs because Z_0 in (2) is convected at the much slower fluid velocity rather than the much faster reacting shock speed (see [1] or [9] for details)), q_j for $j = 1, 2$ denote heats of reactions with either (i) $q_0, q_1 > 0$ (exothermic/exothermic), or (ii) $q_0 > 0, q_1 < 0$ (exothermic/endothermic), and finally $\beta > 0$ is a lumped viscosity-thermal-conductivity coefficient. The function $\Phi_1(T)$, which is called the "reaction rate function" is defined by:

$$\Phi_1(T) = \begin{cases} 0, & \text{for } T < T_i, \\ \Phi'(T), & \text{for } T \geq T_i, \end{cases} \tag{3}$$

where $\Phi'(T)$ is a smooth positive function and T_i is the "ignition temperature" of the reaction. A typical example for $\Phi'(T)$ is the Arrhenius law, *i.e.* $\Phi'(T) = T^{\gamma} e^{-\frac{A}{T}}$ for some positive constants γ and A. Notice that $\Phi_1(T)$ is discontinues at the point T_i. Moreover, $\Phi_2(T)$ is ignition function of the usual form, so $\Phi_2(T) > 0$ and is continuous. Also $f(T)$ is a convex and strongly nonlinear function satisfying (see [7])

$$\begin{array}{l} \frac{\partial f}{\partial T} = a(T) > 0, \frac{\partial^2 f}{\partial T^2} > \delta > 0, \\ \lim\limits_{T \to +\infty} f(T) = +\infty, \end{array} \tag{4}$$

and for example you can choose $f(T) = \frac{1}{2} a T^2, (a > 0)$ (see [9, page 1100]).

We study the existence of Chapman-Jouguet detonation wave for the system (2) when this model involving two reactions which are exothermic, i.e. $q_0, q_1 > 0$. In order to prove the existence of travelling wave solution for the system (2), the system (2) is reduced to a system of differential equations and the rest points of the resulted system are studied, in Section 2. Finally, in Section 3 the existence of CJ detonation wave for (2) is proved.

2. Travelling wave solution

First, the definition of the travelling wave solution of the system (2) is recalled.

Definition 2.1. A travelling wave solution between two states $(T_l, Z_{0l}, Z_{1l})^T$ and $(T_r, Z_{0r}, Z_{1r})^T$ of the system (2) is a solution $(T(x,\tau), Z_0(x,\tau), Z_1(x,\tau))^T$ of the system (2), if there is a constant $s \in \mathbb{R}$, which is called the speed of combustion shock wave, satisfying "the jump and entropy conditions", moreover this solution depends only on the variable $\xi = x - s\tau$ [13].

This means that a travelling wave solution of (2) has the following form:

$$(T(x - s\tau), Z_0(x - s\tau), Z_1(x - s\tau))^T.$$

Thus the system (2) reduces to the following system of equations (for travelling wave solutions):

$$(5)\qquad \begin{cases} \beta T_{\xi\xi} = -sT_\xi + \{f(T)\}_\xi - q_0 Z_{0\xi} - q_1 Z_{1\xi}, \\ Z_{0\xi} = \kappa_1 \Phi_1(T) Z_0, \\ Z_{1\xi} = \kappa_2 \Phi_2(T)(Z_0 - Z_1). \end{cases}$$

Integrate the first equation in (5), to get $\beta T_\xi = f(T) - sT - q_0 Z_0 - q_1 Z_1 + C$, where C is the constant of integration. Thus the system (5) reduces to the following system:

$$(6)\qquad \begin{cases} \beta T_\xi = f(T) - sT - q_0 Z_0 - q_1 Z_1 + C := g(T, Z_0, Z_1), \\ Z_{0\xi} = \kappa_1 \Phi_1(T) Z_0, \\ Z_{1\xi} = \kappa_2 \Phi_2(T)(Z_0 - Z_1). \end{cases}$$

In order to find the rest points of the system (6), we have

$$(7)\qquad \begin{cases} f(T) - sT - q_0 Z_0 - q_1 Z_1 + C = 0, \\ \kappa_1 \Phi_1(T) Z_0 = 0, \\ \kappa_2 \Phi_2(T)(Z_0 - Z_1) = 0. \end{cases}$$

Since $\Phi_1(T) = 0$ for $T < T_i$, where T_i is the ignition temperature and this set is contained in the region $0 < Z_0 \le 1$. Thus from the second equation of (7), at a rest point, we must have $Z_0 = 0$ or $T < T_i$. Thus we have two cases as follows:

Case 1: $Z_0 = 0$. Since $\Phi(T) > 0$, from the last equation of (7), we have $Z_1 = 0$. Also the first equation of (7), at a rest point, we have

$$(8)\qquad g_{00}(T) := g(T, 0, 0) = f(T) - sT + C = 0.$$

Case 2: $\Phi_1(T) = 0$. This means $T < T_i$. Thus the second equation of (7) gives $Z_0 = m_0$, for $0 < m_0 \le 1$. Also the last equation of (7), implies $Z_1 = m_0$, since $\phi_2(T) > 0$. Finally, the first equation of (7) gives a set of rest points. This set is

$$(9)\qquad g_{m_0 m_0}(T) := g(T, m_0, m_0) = f(T) - sT - q_0 m_0 - q_1 m_0 + C = 0.$$

Now we have the following lemmas.

Lemma 2.2. *$g_{m_0 m_0}(T)$ has an absolute minimum, where $0 < m_0 \le 1$, i.e. $\frac{dg_{m_0 m_0}(T)}{dT} = 0$ for precisely one value of T.*

Proof. By computing the first and second derivative $g_{m_0 m_0}(T)$ with respect to T, we have

$$\frac{dg_{m_0 m_0}}{dT} = f'(T) - s = a(T) - s, \quad \frac{d^2 g_{m_0 m_0}}{dT^2} = f''(T) > \delta > 0.$$

Notice that $a(T) - s = 0$ for precisely one value of T and $\frac{d^2 g_{m_0 m_0}}{dT^2} > \delta > 0$. Thus $g_{m_0 m_0}(T)$ has an absolute minimum. ☐

Lemma 2.3. *There is a number $C_{00} \in \mathbb{R}$ such that for $C > C_{00}$, the system* (8) *admits no solutions. For $C = C_{00}$ it admits one solution, and for $C < C_{00}$ it admits two solutions.*

Proof. It is trivial to see. □

Remark 2.4. With respect to Lemma 2.3 for $C = C_{00}$ we have one rest point $(T_{CJ}, 0, 0)$ which represent the Chapman-Jouguet state.

Now consider $C = C_{00}$, for $m_0 = 0$, Lemma 2.2 along with (4) show that the equation (8) has exactly one root T_{CJ} with $s = f'(T_{CJ})$.

Consider now all the rest points of the system (6) corresponding to unburned gas states. These points are described by $g(T_{m_0}, m_0, m_0) = 0$, $T_{m_0} \leq T_i$, where $0 < m_0 \leq 1$ and T_i is the ignition temperature which will be obtained later. By considering the above results the rest points of the system (6) are

$$u_{CJ} = (T_{CJ}, 0, 0), \quad u_{m_0 0} = (T_{m_0 0}, m_0, m_0), 0 < m_0 \leq 1, T_{m_0 0} \leq T_i, \tag{10}$$

where $T_i < T_{CJ}$.

In the present work it is assumed that the rest point u_{CJ} exists.

Corollary 2.5. *If the rest point u_{CJ} exists, then the rest point $u_{m_0 0}$ exists for some $0 < m_0 \leq 1$.*

Now, we recall the definition of Chapman-Jouguet detonation wave.

Definition 2.6. A combustion shock wave between u_{CJ} and $u_{m_0 0}$ is called a Chapman-Jouguet detonation wave.

From mathematical point of view, the existence of Chapman-Jouguet detonation wave corresponds to the existence of some complete orbit of the system (6) which is running from the rest point u_{CJ} to $u_{m_0 0}$ for some $0 < m_0 \leq 1$. Such an orbit is called a travelling wave solution for the system (2).

3. Chapman-Jouguet detonation

We consider the linearized system of (6) at the rest point u_{CJ}, which can be written as $\dot{u} = M_{CJ}(u - u_{CJ})$, where

$$M_{CJ} = \begin{bmatrix} \frac{1}{\beta}(f'(T_{CJ}) - s) & -\frac{q_0}{\beta} & -\frac{q_1}{\beta} \\ 0 & \kappa_1 \Phi'(T_{CJ}) & 0 \\ 0 & \kappa_2 \Phi_2(T_{CJ}) & -\kappa_2 \Phi_2(T_{CJ}) \end{bmatrix},$$

where the entries of the matrix must be considered at the rest point u_{CJ}. The next lemma says about the sign of the eigenvalues and its proof is easy.

Lemma 3.1. *Let $\lambda_k, k = 1, 2, 3$, be the eigenvalues of the matrix M_{CJ} at the rest point $u_{CJ} = (T_{CJ}, 0, 0)$. Then at the rest point $u_{CJ}, \lambda_1 = 0, \lambda_2 > 0, \lambda_3 < 0$.*

About the eigenvectors at this rest point we have the following theorem.

Theorem 3.2. *Let $(y_1, y_2, y_3)^T$ be an eigenvector corresponding to the positive eigenvalue λ_2. At u_{CJ}, either $y_1 < 0$ and $y_2 > 0, y_3 > 0$, or the reveres inequalities hold.*

Proof. Notice that

$$\begin{bmatrix} 0 & -\frac{q_0}{\beta} & -\frac{q_1}{\beta} \\ 0 & \kappa_1\Phi'(T_{CJ}) & 0 \\ 0 & \kappa_2\Phi_2(T_{CJ}) & -\kappa_2\Phi_2(T_{CJ}) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \lambda_2 y_1 \\ \lambda_2 y_2 \\ \lambda_2 y_3 \end{bmatrix}.$$

Thus

$$\begin{cases} \lambda_2 y_1 = -\dfrac{q_0}{\beta} y_2 - \dfrac{q_1}{\beta} y_3, \\ \kappa_1 \Phi'(T_{CJ}) y_2 = \lambda_2 y_2, \\ \kappa_2 \Phi_2(T_{CJ}) y_2 - \kappa_2 \Phi_2(T_{CJ}) y_3 = \lambda_2 y_3. \end{cases} \tag{11}$$

The last equation of (11) implies $\kappa_2\Phi_2(T_{CJ})y_2 = (\kappa_2\Phi_2(T_{CJ}) + \lambda_2)y_3$. Since $\lambda_2 > 0$, this means that $\text{sgn} y_2 = \text{sgn} y_3$. Also, the first equation of (11) implies $\text{sgn} y_1 = -\text{sgn} y_2 = -\text{sgn} y_3$. □

Now, we define:

$$D = \{u \in \mathbb{R}^3 : \ 0 < Z_0 < 1, \ T < T_{CJ}, \ g(T, Z_0, Z_1) < 0, \ 0 < Z_0 - Z_1, \ Z_1 > 0\}.$$

Notice that the rest point u_{CJ} is located on ∂D. Moreover, by Lemma 3.1, the unstable manifold at u_{CJ}, is one dimensional.

Lemma 3.3. *Let D be as above, the unstable manifold at u_{CJ} intersects D on a curve.*

Proof. The linearized system of (6) at the rest point u_{CJ}, has the following form:

$$\begin{cases} \beta \dot{T} = -q_0 Z_0 - q_1 Z_1 = g_{1l}(T, Z_0, Z_1), \\ \dot{Z}_0 = \kappa_1 \Phi'(T_{CJ}) Z_0 = g_{2l}(Z_0, Z_1), \\ \dot{Z}_1 = \kappa_2 \Phi_2(T_{CJ}) Z_0 - \kappa_2 \Phi_2(T_{CJ}) Z_1 = g_{3l}(Z_0, Z_1), \end{cases} \tag{12}$$

Let $(y_1, y_2, y_3)^T$ be an eigenvector corresponding to the positive eigenvalue $\lambda_2 = \kappa_1\Phi'(T_{CJ})$, at the rest point u_{CJ}. Now consider the solution $u(\xi) = (T(\xi), Z_0(\xi), Z_1(\xi))^T = (y_1, y_2, y_3)^T e^{\lambda_2 \xi} + u_{CJ}$, of the linear system (12). Then $u(\xi) \in D_{CJ}$, where $D_{CJ} = \{u \in \mathbb{R}^3 : \ 0 < Z_0 < 1, \ 0 < Z_0 - Z_1, \ Z_1 > 0\}$. To see this notice that $(g_{1l}(T, Z_0, Z_1), g_{2l}(Z_0, Z_1), g_{3l}(Z_0, Z_1))^T = M(u - u_{CJ}) = MYe^{\lambda_2\xi} = \lambda_2 Y e^{\lambda_2 \xi} = (\lambda_2 y_1, \lambda_2 y_2, \lambda_2 y_3)^T e^{\lambda_2 \xi}$.

By Theorem 3.2, we may assume that $y_1 < 0$ and $y_2 > 0, y_3 > 0$. Since $\lambda_2 > 0$, it follows from the last equality that $(g_{1l}(T, Z_0, Z_1), g_{2l}(Z_0, Z_1), g_{3l}(Z_0, Z_1)) \in$

D_{CJ}. This means that the unstable manifold of (12), at the rest point u_{CJ}, which is the line $M_C J = \{u \in \mathbb{R}^3 : \quad u - u_{00} = (y_1, y_2, y_3)^T s, s \in \mathbb{R}\}$, lies in D_{CJ} for $s > 0$, and lies in D for $s > 0$ and small. Thus the unstable manifold of the system (12), at the rest point u_{CJ} intersect D on a curve. □

Now consider the following system of ordinary differential equations:

$$\begin{cases} \beta T_\xi = f(T) - sT - q_0 Z_0 - q_1 Z_1 + C = g(T, Z_0, Z_1), \\ Z_{0\xi} = \kappa_1 \Phi'(T) Z_0 := g_4(T, Z_0, Z_1), \\ Z_{1\xi} = \kappa_2 \Phi_2(T)(Z_0 - Z_1) := g_5(T, Z_0, Z_1). \end{cases} \tag{13}$$

where $\Phi'(T)$ is defined by (3). Notice that the above system leads us to the proof of the existence of travelling waves for Chapman-Jouguet detonation wave. Before doing this, we recall the following theorem from [8].

Theorem 3.4. *Suppose the function f in*

$$\frac{dx}{dt} = f(x), \; x = (x_1, x_2, \cdots, x_n)^T, \tag{14}$$

is locally Lipschitz in a neighborhood of the closure of a bounded open set D which is homeomorphic to the cylinder $\{x \in \mathbb{R}^n : \sum_{i=1}^{n-1} x_i^2 < 1, \; 0 < x_n < 1\}$, and (14) is gradient-like with respect to a function h in D. Moreover, suppose the following conditions hold:

- C_1: *The set $\{x \in \overline{D} : h(x) = c\}$ corresponds to the set $\{x \in \mathbb{R}^n : \sum_{i=1}^{n-1} x_i^2 \leq 1, x_n = c\}$ for $0 \leq c \leq 1$, under the homeomorphism.*
- C_2: *The system (14) has finitely many rest points which are located in the set $\{x \in \partial D : h(x) = 0\}$. Moreover this system has no other rest point in $\overline{D}$.*
- C_3: *The flow comes in D on $\{x \in \partial D : 0 < h(x) < 1\}$.*
- C_4: *Let $\tilde{x}$ be one of the rest points of the system (14), and the unstable manifold of this system at $\tilde{x}$ intersects D in a nonempty set.*

Then there is a point $p \in \{x \in \partial D : h(x) = 1\}$ such that $\lim_{t \to -\infty} p.t = \tilde{x}$. Moreover, if the intersection of D and the unstable manifold at $\tilde{x}$ is one dimensional, this point is unique. If this dimension is more than one, then there are infinitely many of such points.

Lemma 3.5. *Let D be as above. Then there is a unique orbit of the system (13) which lies in D, its α-limit set is u_{CJ}, and this orbit intersects the set $\Delta = \{u \in \overline{D} : g(T, Z_0, Z_1) < 0, T < T_{CJ}, Z_0 = 1, Z_1 = 1\}$. Along this orbit $T(\xi)$ is decreasing and $Z_0(\xi)$ and $Z_1(\xi)$ are increasing.*

Proof. First of all note that the system (13) is gradient like with respect to $h(u) = Z_0$ in D, and is locally Lipschitz in a neighborhood of $\overline{D}$. Now we shall show that the system (13) together with D(as D), u_{CJ}, (as the rest point $\tilde{x}$) and the real valued function $h(u) = Z_0$(as h) satisfy all of the conditions of Theorem 3.4.

Conditions C_1, C_2 and C_4 trivially hold. We shall show that Condition C_3 of Theorem 3.4 is fulfilled. To see this, let $u_0 \in \{u \in \partial D : 0 < Z_0 < 1\}$. Then $g(u_0) = 0$ or $T = T_{CJ}$ or $Z_0 = 0$ or $Z_0 = 1$ or $Z_1 = 0$ and $Z_0 - Z_1 = 0$.

Now, suppose $g(u_0) = 0$. If we differentiate $g(T, Z_0, Z_1)$ along the orbit of (13) we obtain $\frac{dg(u)}{d\xi} = (\frac{\partial f(T)}{\partial T} - s - (q_0)_T Z_0 - (q_1)_T Z_1)\dot{T} - q_0\dot{Z}_0 - q_1\dot{Z}_1$. Hence

$$\begin{aligned} \frac{dg(u)}{d\xi}_{|g(u_0)=0} &= \left(\left(\frac{\partial f(T)}{\partial T} - s - (q_0)_T Z_0 - (q_1)_T Z_1\right)\frac{g(u)}{\beta} - q_0\dot{Z}_0 - q_1\dot{Z}_1\right)_{|g(u_0)=0} \\ &= -q_0\dot{Z}_0 - q_1\dot{Z}_1 < 0. \end{aligned}$$

Thus the flow comes in $\overline{D}$ on $g(u_0) = 0$.

Let $T = T_{01}$ and differentiate T along the orbit to obtain:

$$\frac{dT}{d\xi}_{|T=T_{CJ}} = \frac{1}{\beta}(f(T) - sT - q_0Z_0 - q_1Z_1 + C)_{|T=T_{CJ}} = \frac{1}{\beta}g(T_{CJ}, Z_0, Z_1) < 0.$$

Thus the flow comes in $\overline{D}$ on $T = T_{CJ}$.

Let $Z_0 - Z_1 = 0$ and differentiate $Z_0 - Z_1$ along the orbit to obtain:

$$\begin{aligned} \frac{Z_0 - Z_1}{d\xi}_{|Z_0-Z_1=0} = \dot{Z}_0 - \dot{Z}_{1|Z_0-Z_1=0} &= [\kappa_1\Phi'(T)Z_0 - \kappa_2\Phi_2(T)(Z_0 - Z_1)]_{|Z_0-Z_1=0} \\ &= \kappa_1\Phi'(T)Z_0 > 0 \end{aligned}$$

Thus the flow comes in $\overline{D}$ on $Z_0 - Z_1 = 0$.

Let $Z_1 = 0$ and differentiate Z_1 along the orbit to obtain:

$$\frac{Z_1}{d\xi}_{|Z_1=0} = \dot{Z}_{1|Z_1=0} = \kappa_2\Phi_2(T)(Z_0 - Z_1)_{|Z_1=0} > 0.$$

Thus the flow comes in $\overline{D}$ on $Z_1 = 0$.

Let $Z_0 = 0$ and differentiate Z_0 along the orbit to obtain:

$$\frac{Z_0}{d\xi}_{|Z_0=0} = \dot{Z}_{0|Z_0=0} = \kappa_1\Phi'(T)Z_{0|Z_0=0} > 0.$$

Thus the flow comes in $\overline{D}$ on $Z_0 = 0$.

Let $Z_0 = 1$ and differentiate Z_0 along the orbit to obtain:

$$\frac{Z_0}{d\xi}_{|Z_0=1} = \dot{Z}_{0|Z_0=1} = \kappa_1\Phi'(T)Z_{0|Z_0=1} > 0.$$

Thus the flow goes out $\overline{D}$ on $Z_0 = 1$.

Hence Condition C_3 of Theorem 3.4 holds too. Thus by Theorem 3.4, there must be an orbit of the system (13) lying in D, initiating at a point on the surface $Z_0 = 1$ and running to the point u_{CJ} as $\xi \to -\infty$. Finally from the system (13) and the set D it follows that along this orbit $Z_0(\xi)$ and $Z_1(\xi)$ are increasing and $T(\xi)$ is decreasing. □

Let $\tilde{u}(\xi), \xi \in (-\infty, \xi_0]$ be the orbit which is given by the above lemma. Then $\tilde{u}(\xi_0) \in \{u \in \overline{D} : Z_0 = 1\}$ and $\lim_{\xi \to -\infty} \tilde{u}(\xi) = u_{CJ}$. About the orbit $\tilde{u}(\xi)$ we have the following lemma.

Lemma 3.6. *Let $\tilde{u}(\xi)$ be as above. Then there is $0 < \tilde{Z}_0 \leq 1$, such that for $\kappa_1\beta$ and $\kappa_2\beta$ small enough, the orbit $\tilde{u}(\xi)$ meets the line $T = T_i$, at $\tilde{u}(\tilde{\xi}) = (T_i, \tilde{Z}_0, \tilde{Z}_1)$, for some $\tilde{\xi} \in (-\infty, \xi_0)$.*

Proof. Let u_{CJ} be as (10). Choose the line $P : T - T_{CJ} = 0$, and $u_{CJ} \in \{u | T < T_{CJ}\}$. Let $(T_i, Z_{0i}, Z_{1i})^T$ be the unique solution of the equation $g(u) = 0,\ T = T_i,\ Z_{0i} - Z_{1i} = 0$, this shows $Z_{0i} = Z_{1i}$ and from $g(u) = 0$, we have $Z_{0i} = Z_{1i} := Z_i = \frac{1}{q_0+q_1}(f(T_i) - sT_i + C)$. Also from $T_m < T_i < T_{CJ}$, it follows that $0 < Z_i < 1$ and $\{u \in \overline{D} :\ g(u) = 0, Z_i < Z_0 < 1,\ Z_i < Z_1 < 1\} \subset \{u \in \overline{D} : T \leq T_i\}$. Now consider the line $P' : T - T_i = 0$, since $u_{m0} \in \{u \in \overline{D} : (T - T_i) < 0\}$, we can choose $Z_i < Z_0' < 1,\ Z_i < Z_1' < 1$ such that

$$\{u \in \overline{D} : g(u) = 0, Z_0' < Z_0 < 1, Z_1' < Z_1 < 1\} \subset \{u \in \overline{D} : T - T_i < 0\}.$$

Let $D_0 = \{u \in D : T_i < T < T_{CJ}, Z_0' < Z_0 < 1\} \cup \{u \in D : T_i < T < T_{CJ}, Z_1' < Z_1 < 1\}$ and $\delta = \max_{u \in \overline{D}_0} g(u)$. Then $\delta < 0$.

Now suppose the orbit $\tilde{u}(\xi), \xi \in (-\infty, \xi_0]$, does not meet the set $\{u \in D : T = T_i, 0 < Z_0 \leq 1, 0 < Z_1 \leq 1\}$. Let $\xi_1 < \xi_0$ and $\xi_2 < \xi_3 < \xi_0$ be the solutions of the equations $\tilde{Z}_0(\xi) = Z_0'$, $\tilde{Z}_1(\xi) = Z_1'$ and $\tilde{Z}_1(\xi) = 1$, respectively, where $\tilde{Z}_0(\xi)$ and $\tilde{Z}_1(\xi)$ are the second and third components of $\tilde{u}(\xi)$, respectively. Since $\frac{d}{d\xi}(T) = \frac{1}{\beta} g(u) < 0, T$ is decreasing along the orbit $\tilde{u}(\xi)$, it follows that $\tilde{u}(\xi)$ remains in D_0 for $\min\{\xi_3, \xi_1\} < \xi < \xi_0$. Now along the orbit $\tilde{u}(\xi)$ in D_0 we have:

$$
\begin{aligned}
-\frac{dT}{dZ_0} &= \frac{1}{\frac{dZ_0}{d\xi}}\left(-\frac{dT}{d\xi}\right) = \frac{1}{\kappa_1 Z_0 \Phi'(T)}\left(\frac{-1}{\beta} g(u)\right) \geq \frac{\sigma_1(-\delta)}{\kappa_1\beta} > 0,\\
-\frac{dT}{dZ_1} &= \frac{1}{\frac{dZ_1}{d\xi}}\left(-\frac{dT}{d\xi}\right) = \frac{1}{\kappa_2(Z_0 - Z_1)\Phi_2(T)}\left(\frac{-1}{\beta} g(u)\right) \geq \frac{\sigma_2(-\delta)}{\kappa_2\beta} > 0,
\end{aligned}
$$

where $\frac{1}{\sigma_1} = \max_{u \in \overline{D}} \Phi'(T) Z_0$ and $\frac{1}{\sigma_2} = \max_{u \in \overline{D}} \Phi_2(T)(Z_0 - Z_1)$, respectively.

Let $T_0 = T(\xi_0)$, then $T_i < T_0$, if $\tilde{u}(\xi)$ does not meet T_i. Therefore

$$
\begin{aligned}
T_{CJ} - T_i \geqslant & T_{CJ} - T_0 = -\int_{-\infty}^{\xi_0} \frac{1}{\beta} g(u) d\xi = \int_{-\infty}^{\xi_0} \frac{1}{\beta}(-g(u)) d\xi\\
& \geqslant \int_{\xi_1}^{\xi_0} \frac{1}{\beta}(-g(u)) d\xi = \int_{Z_0'}^{1} \frac{1}{\kappa_1 Z_0 \Phi'(T)}\left(\frac{-1}{\beta} g(u)\right) dZ_0 \geqslant \frac{\sigma_1(-\delta)}{\kappa_1\beta}(1 - Z_0'),
\end{aligned}
$$

which is impossible for $\kappa_1\beta$ small enough. Let $T_0 = T(\xi_0)$, then $T_i < T_0$, if $\tilde{u}(\xi)$ does not meet T_i. Therefore

$$\begin{aligned} T_{CJ} - T_i \geqslant & T_{CJ} - T_0 = -\int_{-\infty}^{\xi_0} \frac{1}{\beta} g(u) d\xi = \int_{-\infty}^{\xi_0} \frac{1}{\beta}(-g(u)) d\xi \\ \geqslant & \int_{\xi_2}^{\xi_3} \frac{1}{\beta}(-g(u)) d\xi = \int_{Z_1'}^{1} \frac{1}{\kappa_2(Z_0 - Z_1)\Phi_2(T)} (\frac{-1}{\beta} g(u)) dZ_1 \\ \geqslant & \frac{\sigma_2(-\delta)}{\kappa_2\beta}(1 - Z_1'), \end{aligned}$$

which is impossible for $\kappa_2\beta$ small enough. Thus there is a $\tilde{\xi} \in (-\infty, \xi_0)$ such that the orbit $\tilde{u}(\xi)$ meets the line $T = T_i$ at the point $\tilde{u}_i = (\tilde{T}_i, \tilde{Z}_{0i}, \tilde{Z}_{1i})^T$, where $\tilde{T}_i = T_i$, $\tilde{Z}_{0i} = \tilde{Z}_0(\tilde{\xi})$ and $\tilde{Z}_{1i} = \tilde{Z}_1(\tilde{\xi})$. □

From now on we assume that $\kappa_1\beta$ and $\kappa_2\beta$ are small enough, or the orbit $\tilde{u}(\xi)$ meets the line $T = T_i$ at the point $\tilde{u}_i = (\tilde{T}_i, \tilde{Z}_{0i}, \tilde{Z}_{1i})^T$, where $\tilde{T}_i = T_i$, $\tilde{Z}_{0i} = \tilde{Z}_0(\tilde{\xi})$ and $\tilde{Z}_{1i} = \tilde{Z}_1(\tilde{\xi})$. We call the point $\tilde{u}_i$ the ignition point. According to Lemma 3.5, this point for Chapman-Jouget detonation is unique.

Theorem 3.7. *Suppose that the system* (6) *admits the rest points* u_{CJ} *and* u_{m0}*, for some* $0 < m \leq 1$*. If* $\kappa_1\beta$ *and* $\kappa_2\beta$ *are small enough, then there is a unique orbit of the system* (6) *which is running from* u_{CJ} *to* u_{m0}*, for some* $0 < m \leq 1$*.*

Proof. In the region $T < T_i$, the second equation of (6) becomes $\dot{Z}_0 = 0$. Thus, in this region, along the orbit of this system $Z_0(\xi)$ is constant. Let $Z_0(\xi) = \tilde{Z}_{0i}$, where $\tilde{Z}_{0i}$ is the second component of $\tilde{u}_i$, (the ignition point). On the surface $Z_0 = \tilde{Z}_{0i}$, the system (6) reduces to the following two dimensional system of equations, in the region $T \leq T_i$:

$$\begin{cases} \beta\dot{T} = f(T) - sT - q_0\tilde{Z}_{0i} - q_1 Z_1 + C, \\ \dot{Z}_1 = \kappa_2\Phi_2(T)(\tilde{Z}_{0i} - Z_1). \end{cases} \tag{15}$$

By solving the second equation of (15), we have $\frac{dZ_1}{\tilde{Z}_{0i} - Z_1} = \kappa_2\Phi_2(T)d\xi$ and so $Z_1 = \tilde{Z}_{0i} - e^{-\int_\xi^{+\infty}\kappa_2\Phi_2(T(\eta))d\eta}$. Therefore the first equation of (15) is

$$\beta\dot{T} = f(T) - sT - q_0\tilde{Z}_{0i} - q_1(\tilde{Z}_{0i} - e^{-\int_\xi^{+\infty}\kappa_2\Phi_2(T(\eta))d\eta}) + C := F_1(T). \tag{16}$$

Now consider the region $D' = \{T \in \mathbb{R} : F_1(T) < 0, T < T_i\}$. Notice that $T = T_i \in \partial D'$. Also this is trivial to see that any orbit of (16) initiating at a point on $\partial D' \cap \{T : T = T_i\}$ approaches to the unique rest point of the system (16) which is located in the region $T < T_i$, as ξ tends to $+\infty$. We denote this rest point by $\overline{T_i}$.

Now, consider again the ignition point $\tilde{u}_i = (\tilde{T}_i, \tilde{Z}_{0i}, \tilde{Z}_{1i})$. By the above argument, there is a unique orbit of the system (6), say

$$\tilde{\tilde{u}}(\xi) = \begin{cases} (\tilde{\tilde{T}}(\xi), \tilde{\tilde{Z}}_0(\xi), \tilde{\tilde{Z}}_1(\xi)), & \tilde{\xi} < \xi < +\infty \\ (\tilde{T}_i, \tilde{Z}_{0i}, \tilde{Z}_{1i}), & \xi = \tilde{\xi} \end{cases}$$

with $\tilde{\tilde{Z}}_0(\xi) = \tilde{Z}_{0i}$ for $\tilde{\xi} \leq \xi$ and $\lim_{\xi \to +\infty} \tilde{\tilde{u}}(\xi) = (\overline{T}_i, \tilde{Z}_{0i}, \overline{Z}_{1i})$. Along this orbit $T(\xi)$ is decreasing, $Z_1(\xi)$ is increasing and $Z_0(\xi)$ is constant, respectively. This orbit lies in D, the domain which is used in the proof of Lemma 3.5.

Now define

$$u(\xi) = \begin{cases} \tilde{u}(\xi), & \xi < \tilde{\xi}, \\ \tilde{\tilde{u}}(\xi), & \xi \geq \tilde{\xi}. \end{cases}$$

Then $u(\xi)$ is a complete orbit of the system (6) lying in D and is running from u_{CJ} to u_{m0} for some $0 < m \leq 1$. □

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