SOME NEW RESULTS RELATED TO SUBGROUP COMMUTATIVITY DEGREES AND p-COMMUNTATIVITY DEGREES OF FINITE GROUPS

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Abstract. In 1970 Gallagher introduced the concept of commutativity degree of finite groups. Then some authors extend this concept to some variety of groups. In this paper, we define commutativity degree and p-commutativity degree with respect to Burnside variety of groups and study their subgroups and some properties of finite groups in this variety.
Keywords: commutativity degree, Burnside variety, p-commutativity degree.

1. Introduction

Through this paper we assume that all groups are finite. In [1] Gallagher introduced the concept of commutativity degree of finite groups. Lescot [4] defined commutativity degree on abelian variety and Moghaddam [5] defined commutativity degree with respect to the nilpotent variety. In this paper, we define the commutativity degree with respect to Burnside variety for any prime number p and study some results of Lescot in 1995 and Moghadam in 2005 (see [4,5]). Also we define subgroup commutativity degree with respect to Burnside variety of groups and investigate some properties of Tarnauceanu in [7]. Finally, we define p-commutativity degree and also subgroup p-commutativity degree of finite groups.

Let F be a free group freely generated by a countable set $X = \{x_1, x_2, ...\}$ and V a non-empty subset of F. Let $\mathcal{V}$ be a variety of groups defined by the set of laws V. There exists two important subgroups associated to a given group G with respect to a variety, as follows

$$V(G) =< v(g_1, g_2, ..., g_r)|g_i \in G, 1 \leq i \leq r, v \in V >$$

$$V^*(G) = \{g \in G|\ v(g_1, g_2, ..., g_ig, ..., g_r) = v(g_1, g_2, ..., g_r)\ |g_i \in G, 1 \leq i \leq r, v \in V\}$$

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which are called verbal subgroup and marginal subgroup of G with respect to the variety $\mathcal{V}$. Now, if we take $v(x) = x^p$, then this word corresponds to the Burnside variety $\mathcal{B}$ that means all groups of exponent p. The marginal subgroup corresponding to $\mathcal{B}$ is the set of all central element of exponent p, that is, $V^*(G) = \{x \in Z(G)|x^p = 1\}$ and the verbal subgroup is $V(G) = \{g^p|g \in G\}$.

Definition 1.1. Let $\mathcal{V}$ be a variety of groups defined by the set of laws V, and let G and H be two groups. Then the pair (ϕ_1, ϕ_2) is said to be $\mathcal{V}$-isologism between G and H, and we say that G and H are $\mathcal{V}$-isologic, if $\phi_1 : \frac{G}{V^*(G)} \to \frac{H}{V^*(H)}$ and $\phi_2 : V(G) \to V(H)$ are isomorphisms such that for all $v(x_1, x_2, ..., x_r) \in V$ and for all $(g_1, g_2, ..., g_r) \in G$, we have $\phi_2 v(g_1, g_2, ..., g_r) = v(h_1, ..., h_r)$ whenever $h_i \in \phi_1(g_i V^*(G))$, i=1,...,r.
It is easy to see that G and H are n-isoclinic, when $\mathcal{V}$ is the variety of all nilpotent groups of class atmost n.

2. Commutativity degrees of finite groups in Burnside variety

In this section we define the commutativity degree of finite groups with respect to Burnside variety. Let G be a group and $\mathcal{B}$ be Burnside variety, then commutativity degree of group G with respect to Burnside variety is called $\mathcal{B}$-degree and is as follows

$$d_{\mathcal{B}}(G) = \frac{1}{|G|}|\{x \in |x^p = 1\}|$$

obviously $0 < d_{\mathcal{B}}(G) \leq 1$.

Clearly if $G \in \mathcal{B}$, then $d_{\mathcal{B}}(G) = 1$.

Theorem 2.1. If G and H are two isologic groups, then $d_{\mathcal{B}}(G) = d_{\mathcal{B}}(H)$.

Proof. Let G and H be two isologic groups. Then there is a pair (α, β) with $\alpha : \frac{G}{V^*(G)} \longrightarrow \frac{H}{V^*(H)}$ and $\beta : V(G) \longrightarrow V(H)$ and we have $\beta(g^p) = (\alpha(gV^*(G))^p$, such that the following diagram is commutative

$$\begin{array}{ccc} \dfrac{G}{V^*(G)} & \longrightarrow & \dfrac{H}{V^*(H)} \\ a_G \downarrow & & a_H \downarrow \\ V(G) & \longrightarrow & V(H) \end{array} .$$

Where, $a_G(gV^*(G)) = g^p$ and $a_H(hV^*(H)) = h^p$,

$$\begin{aligned} \frac{|G|}{|V^*(G)|} d_{\mathcal{B}}(G) &= \frac{1}{|V^*(G)|}|\{x \in G|x^p = 1\}| \\ &= \frac{1}{V^*(G)|}|\{x \in G|a_G(xV^*(G)) = 1\}| \\ &= |\{xV^*(G) \in \frac{G}{V^*(G)}|\beta a_G(xV^*(G)) = 1\}| \\ &= |\{xV^*(G) \in \frac{G}{V^*(G)}|a_H\alpha(xV^*(G)) = 1\}| \end{aligned}$$

$$
\begin{aligned}
&= |\{hV^*(H) \in \frac{H}{V^*(H)} | a_H(hV^*(H)) = 1\}| \\
&= |\{hV^*(H) \in \frac{H}{V^*(H)}) | h^p = 1\}| \\
&= \frac{1}{|V^*(H)|} |\{h \in H | h^p = 1\}| = \frac{|H|}{|V^*(H)|} d_{\mathcal{B}}(H)
\end{aligned}
$$

so $d_{\mathcal{B}}(G) = d_{\mathcal{B}}(H)$.

In order to simplify our notation, we define the Burnside sign function by

$$\phi_{\mathcal{B}} : G \to \{0, 1\}$$

$$\phi_{\mathcal{B}} = \begin{cases} 1, & x^p = 1_G \\ 0, & x^p \neq 1_G. \end{cases}$$

So, $d_{\mathcal{B}}(G) = \frac{1}{|G|} \sum_{x \in G} \phi_{\mathcal{B}}(x)$.

Theorem 2.2. Let G be a group and N be a normal subgroup of G, then

$$d_{\mathcal{B}}(G) \leq d_{\mathcal{B}}(N) d_{\mathcal{B}}(\frac{G}{N}).$$

Proof. If $x \notin N$ and $x^p \in N$ then $\phi_{\mathcal{B}}(x) = 0$ but $\phi_{\mathcal{B}}(xN) = 1$,

$$
\begin{aligned}
|G| d_{\mathcal{B}}(G) &= |\{x \in G | x^p = 1\}| \\
&= \sum_{x \in G} \phi_{\mathcal{B}}(x) \\
&= \sum_{s \in \frac{G}{N}} \sum_{x \in s} \phi_{\mathcal{B}}(x) \leq \sum_{y = xN \in \frac{G}{N}} \phi_{\mathcal{B}}(y) \sum_{x \in N} \phi_{\mathcal{B}}(x) \\
&= |\frac{G}{N}| \phi_{\mathcal{B}}(\frac{G}{N}) |N| \phi_{\mathcal{B}}(N)
\end{aligned}
$$

which complete the proof.

Corollary 2.3. Let G be a group and N is a normal subgroup of G, then

$$d_{\mathcal{B}}(G) \geq d_{\mathcal{B}}(N)$$

and

$$d_{\mathcal{B}}(G) \leq d_{\mathcal{B}}(\frac{G}{N}).$$

In the following theorem we show that $\mathcal{B}$-degree of a direct product of groups is the same as the product of those of direct product.

Theorem 2.4. If G and H are two groups, then

$$d_{\mathcal{B}}(H \times G) = d_{\mathcal{B}}(H) \times d_{\mathcal{B}}(G).$$

Proof. By definition we have

$$\begin{aligned} d_{\mathcal{B}}(H\times G) &= \frac{1}{|H\times G|}|\{(h,g)\in H\times G|(h,g)^p=1\}| \\ &= \frac{1}{|H||G|}|\{(h,g)\in H\times G|(h^p,g^p)=(1_H,1_G)\}| \\ &= \frac{|\{h\in H|h^p=1\}|}{|H|}\frac{|\{g\in G|g^p=1\}|}{|G|} \\ &= d_{\mathcal{B}}(H)\times d_{\mathcal{B}}(G), \end{aligned}$$

which complete the proof.

Corollary 2.5. If q is a prime number and $(p,q)=1$ and G is a q-group, then

$$|G|d_{\mathcal{B}}(G)=1.$$

Now we introduce subgroup commutativity degree with respect to Burnside variety. The lattice formed by all subgroups of a group G will be denoted by $L(G)$ and will be called the subgroup lattice of the group G (see also [8]). Recall that $L(G)$ is a complete bounded lattices with respect to set inclusion, having initial element the trivial subgroup $\{1\}$ and final element G, and its binary operations $\wedge$, $\vee$ are defined by $H\wedge K=H\cap K$, $H\vee K=<H\cup K>$, for all $H,K\in L(G)$. Now, we define the subgroup Burnside commutativity degree by

$$sd_{\mathcal{B}}(G)=\frac{1}{|L(G)|}|\{H\in L(G)|H^p=1\}|$$

and the Burnside sign function will be define as following:

$$\phi_{\mathcal{B}}:L(G)\to\{0,1\},$$
$$\phi_{\mathcal{B}}(H)=\begin{cases}1, & H^p=1\\ 0, & H^p\neq 1\end{cases}$$

so

$$sd_{\mathcal{B}}(G)=\frac{1}{|L(G)|}\sum_{H\in L(G)}\phi_{\mathcal{B}}(H).$$

Corollary 2.6. If G is a q-group where $q\mid p$, then $sd_{\mathcal{B}}(G)=1$.

Theorem 2.7. Let N be a normal subgroup of a group G, then

$$sd_{\mathcal{B}}(G)\geq\frac{1}{|L(G)|}(sd_{\mathcal{B}}(\frac{G}{N})|L(\frac{G}{N})|+sd_{\mathcal{B}}(N)|L(N)|-1).$$

Proof. Let N be a normal subgroup of G and $A_1=\{H\in L(G)|N\subseteq H\}$ and$A_2=\{H\in L(G)|H\subset N\}$. but $A_1\cup A_2\subseteq L(G)$ so

$$\begin{aligned} sd_{\mathcal{B}}(G) &\geq \frac{1}{|L(G)|}\sum_{H\in A_1\cup A_2}\phi_{\mathcal{B}}(H) \\ &= \frac{1}{|L(G)|}(\sum_{H\in A_1}\phi_{\mathcal{B}}(H)+\sum_{H\in A_2}\phi_{\mathcal{B}}(H). \qquad (1) \end{aligned}$$

We can calculute the right side of (1) as follows

$$\sum_{H\in A_1}\phi_{\mathcal{B}}(H) = sd_{\mathcal{B}}\frac{G}{N}|L(\frac{G}{N})|, \tag{2}$$

$$\sum_{H\in A_2}\phi_{\mathcal{B}}(H) = \sum_{H\in A_2\cup N}\phi_{\mathcal{B}}(H) - \phi_{\mathcal{B}}(N) \geq sd_{\mathcal{B}}(N)|L(N)| - 1 \tag{3}$$

by insertion (2) and (3) in (1), the result hold.

Corollary 2.8. (i) If N and $\frac{G}{N}$ are q- groups where $q \mid p$, then

$$sd_{\mathcal{B}}(G) \geq \frac{1}{|L(G)|}(|L(\frac{G}{N}| + |L(N)| - 1),$$

(ii) If N is a normal subgroup of prime index in G, then

$$sd_{\mathcal{B}}(G) \geq \frac{1}{|L(G)|}(2|sd_{\mathcal{B}}(\frac{G}{N}) + sd_{\mathcal{B}}(N)|L(N)| - 1);$$

(iii) If N is a normal subgroup of index q in G where $q \mid p$, then

$$sd_{\mathcal{B}}(G) \geq \frac{1}{|L(G)|}(sd_{\mathcal{B}}(N)|L(N)| + 1);$$

(iv) If $(G_i)_{i=1}^{k}$ is a family of finite groups with coprime orders, then

$$sd_{\mathcal{B}}(\times_{i=1}^{k}G_i) = \pi_{i=1}^{k}sd_{\mathcal{B}}(G_i);$$

(v) If G is a finite nilpotent group and $(p_i)_{i=1,\ldots,k}$ are the sylow subgroups of G, then

$$sd_{\mathcal{B}}(G) = \pi_{i=1}^{k}sd_{\mathcal{B}}(G_i).$$

3. p-commutativity degrees of finite groups

In this section we define p-commutativity degree and subgroup p-commutativity degree of finite groups. Let G be a finite group then the p-commutativity degree, $d_p(G)$, is defined as follows

$$\begin{aligned} d_p(G) &= \frac{1}{|G|^2}|\{(x,y)\in G^2|[x^p,y^p]=1\}| \\ &= \frac{1}{|G|^2}|\{(x,y)\in G^2|x^py^p=y^px^p\}| \end{aligned}$$

obviously if $G \in \mathcal{B}$ then $d_p(G) = 1$, otherwise $0 < d_p(G) \leq 1$.

In the following Lemma we state some properties of $d_p(G)$.

Lemma 3.1. Let G and H be two isoclinic groups then $d_p(G) = d_p(H)$.

Proof. Since G and H are isoclinic, the following diagram is commutative

$$\begin{array}{ccc} \frac{G}{Z(G)} \times \frac{G}{Z(G)} & \longrightarrow & \frac{H}{Z(H)} \times \frac{H}{Z(H)} \\ \downarrow a_G & & \downarrow a_H \\ G' & \longrightarrow & H' \end{array}$$

$$\begin{aligned} |\frac{G}{Z(G)}|^2 d_p(G) &= \frac{1}{|Z(G)|^2}\{(x,y) \in G^2 | [x^p, y^p] = 1\} \\ &= \{(xZ(G), yZ(G)) \in (\frac{G}{Z(G)})^2 | \phi_2 a_G(x^p Z(G), y^p Z(G)) = 1\} \\ &= \{(\gamma, \eta) \in (\frac{G}{Z(G)})^2 | \phi_2(a_G(\gamma^p, \eta^p)) = 1\} \\ &= \{(\gamma, \eta) \in (\frac{G}{Z(G)})^2 | a_H(\phi_1^2((\gamma^p, \eta^p))) = 1\} \\ &= \{(\alpha, \beta) \in (\frac{H}{Z(H)})^2 | a_H(\alpha^p Z(H), \beta^p Z(H)) = 1\} \\ &= \frac{1}{|Z(H)|^2}\{(\alpha, \beta) \in H^2 | [\alpha^p, \beta^p] = 1\} \\ & |\frac{H}{Z(H)}|^2 d_p(H) \end{aligned}$$

so $d_p(G) = d_p(H)$.

Definition 3.2. We define the function f_p by

$$f_p : G \times G \to \{0, 1\}$$

$$f_p(x, y) = \begin{cases} 1, & [x^p, y^p] = 1 \\ 0, & [x^p, y^p] \neq 1 \end{cases}$$

so $d_p(G) = \frac{1}{|G|^2} \sum_{(x,y) \in G \times G} f_p(x, y)$.

In 1970 Gallagher [1] proved that if G is a finite group with a normal subgroup N, then $d(G) \leq d(N) d(\frac{G}{N})$. Now we generalize this result for the class of all group G in which the centralizer p-element is normal in G. Such group is called CN-group.

Lemma 3.3. Let G be a CN-group or centralizer p-element with a normal subgroup N, then

$$d_p(G) = \frac{1}{|G|} \sum_{x \in G} d_{\mathcal{B}}(\frac{G}{C_G(x^p)}).$$

Proof. Let $y^{-p} = y^p C_G(x^p)$ for any $x, y \in G$,

$$|G|^2 d_p(G) = |\{(x,y) \in G^2 | [x^p, y^p] = 1\}| = \sum_{x \in G} |\{y \in G | y^p \in C_G(x^p)\}|$$
$$= \sum_{x \in G} |\{\bar{y} \in \frac{G}{C_G(x^p)} | \bar{y}^p = \bar{1}\}||C_G(x^p)|$$
$$= \sum_{x \in G} |C_G(x^p)||\frac{G}{C_G(x^p)}|d_{\mathcal{B}}(\frac{G}{C_G(x^p)})$$
$$= \sum_{x \in G} |G| d_{\mathcal{B}}(\frac{G}{C_G(x^p)}),$$

which complete the proof.

Theorem 3.4. Let G be a CN-group then for any normal subgroup N

$$d_p(G) \leq d_p(N) d_p(\frac{G}{N}).$$

Proof. It is clear that if two elements x^p and y^p are commutative, then $x^p N$ and $y^p N$ are commutative. Now for any $x \in G$

$$\frac{C_G(x^p)}{N} \subseteq C_{\frac{G}{N}}(x^p N) \tag{4}$$

on the other hand we know

$$d_{\mathcal{B}}(\frac{G}{N}) \geq d_{\mathcal{B}}(G) \tag{5}$$

by Lemma 1.4,

$$d_p(G) = \frac{1}{|G|} \sum_{x \in G} d_{\mathcal{B}}(\frac{G}{C_G(x^p)}) \tag{6}$$

it is obvious $C_N(x^p) = C_G(x^p) \cap N$ for any $x \in G$, so

$$\frac{N}{C_N(x^p)} \cong \frac{C_G(x^p)N}{C_G(x^p)} \lhd \frac{G}{C_G(x^p)} \tag{7}$$

and so,

$$\frac{\frac{G}{C_G(x^p)}}{\frac{N}{C_N(x^p)}} \cong \frac{\frac{G}{N}}{\frac{C_G(x^p)N}{N}}. \tag{8}$$

Now by (1),

$$\frac{\frac{(\frac{G}{N})}{(\frac{C_G(x^p)N}{N})}}{\frac{C_{\frac{G}{N}}(x^p N))}{(\frac{C_G(x^p)N}{N})}} \cong \frac{\frac{G}{N}}{C_{\frac{G}{N}}(x^p N)} \tag{9}$$

by (3) and (4),

$$|G|d_p(G) = \sum_{x\in G} d_{\mathcal{B}}(\frac{G}{C_G(x^p)}) \leq \sum_{x\in G} d_{\mathcal{B}}(\frac{N}{C_N(x^p)})d_{\mathcal{B}}(\frac{(\frac{G}{C_G(x^p)})}{(\frac{N}{C_N(x^p)})}$$

by (5),

$$= \sum_{x\in G} d_{\mathcal{B}}(\frac{N}{C_N(x^p)})d_{\mathcal{B}}(\frac{(\frac{G}{N})}{(\frac{C_G(x^p)N}{N})}$$

by (2) and (6),

$$\leq \sum_{x\in G} d_{\mathcal{B}}(\frac{N}{C_N(x^p)}d_{\mathcal{B}}(\frac{\frac{G}{N}}{C_{\frac{G}{N}}(x^pN)})$$

by (3),

$$\begin{aligned}|G|d_p(G) &\leq \sum_{x\in N} d_{\mathcal{B}}(\frac{N}{C_N(x^p)})d_{\mathcal{B}}(\frac{\frac{G}{N}}{C_{\frac{G}{N}}(x^pN)}) \\ &\sum_{y=xN\in\frac{G}{N}} d_{\mathcal{B}}(\frac{N}{C_N(x^p)})d_{\mathcal{B}}(\frac{\frac{G}{N}}{C_{\frac{G}{N}}(x^pN)}) \\ &\leq \sum_{x\in N} d_{\mathcal{B}}\frac{N}{C_N(x^p)} \sum_{y\in\frac{G}{N}} d_{\mathcal{B}}(\frac{\frac{G}{N}}{C_{\frac{G}{N}}(x^pN)}) \\ &= |N|d_p(N)|\frac{G}{N}|d_p(\frac{G}{N})\end{aligned}$$

and we know $|G| = |N||\frac{G}{N}|$ so the proof is complete.

In the following theorem, we show that p-degree of a direct product of groups is the same as the product of those of direct product.

Theorem 3.5. If G and H are two groups, then

$$d_p(H \times G) = d_p(H) \times d_p(G).$$

Proof. By theorem 1.5, we have

$$\begin{aligned}d_p(H \times G) &= \frac{1}{|H \times G|} \sum_{(x,y)\in G\times H} d_{\mathcal{B}}(\frac{G \times H}{C_{G\times H}(x^p, y^p)}) \\ &= \frac{1}{|H||G|} \sum_{(x,y)\in G\times H} d_{\mathcal{B}}(\frac{G}{C_G(x^p)} \times \frac{H}{C_G(y^p)}) \\ &= \frac{1}{|H||G|} \sum_{(x,y)\in G\times H} d_{\mathcal{B}}(\frac{G}{C_G(x^p)}) \times d_{\mathcal{B}}(\frac{H}{C_G(y^p)}) \\ &= \frac{1}{|G|} \sum_{x\in G} d_{\mathcal{B}}(\frac{G}{C_G(x^p)}) \frac{1}{|H|} \sum_{y\in H} d_{\mathcal{B}}(\frac{H}{C_G(y^p)}) \\ &= d_p(H) \times d_p(G)\end{aligned}$$

which complete the proof.

Lemma 3.6. If r is number of the conjugacy classes of $V(G)$, then

$$d_p(G) = \frac{r|V(G)|}{|G|^2}$$

Proof. Let $c_1, c_2, ..., c_r$ be conjugacy classes of $V(G)$ for $i \in \{1, ..., r\}$ and $x_i^p \in c_i$. For any $y_i^p \in c_i$ there is $g^p \in V(G)$ such that $g^{-p}x_i^p g^p = y_i^p$ so

$$C_{V(G)}(y_i^p) = C_{V(G)}(x_i^p)$$

now we have

$$|G|^2 d_p(G) = |\{(x, y) \in G^2 | x^p y^p = y^p x^p\}| \geq |\{(x, y) \in V(G) \times V(G) | x^p y^p = y^p x^p\}|$$
$$= \sum_{x^p} \in V(G)|C_{V(G)}(x^p)| = \sum_{i=1}^{r} \sum_{x^p \in c_i} |C_{V(G)}(x^p)| = \sum_{i=1}^{r} |c_i|[V(G) : c_i]$$
$$\sum_{i=1}^{r} |V(G)| = r|V(G)|$$

and the proof is complete.

Corollary 3.7. If $K(V(G))$ is number of the conjugacy classes of $V(G)$, then

$$K(V(G)) \leq \frac{|G|^2}{|V(G)}.$$

Corollary 3.8. If G is free of initial member of order p, then the above equality is pointed.

Corollary 3.9. If G is q-group where $(p, q) = 1$, then

$$d_p(N)d_p(\frac{G}{N}) = \frac{K(V(N))K(V(\frac{G}{N}))}{|G|}.$$

Proof. G is a q-group that $(p, q) = 1$ so, $V(N) = (N^p), V(\frac{G}{N}) = (\frac{G}{N})^p$

$$d_p(N)d_p(\frac{G}{N}) = \frac{|N^p|}{|N|^2}K(N^p)\frac{|(\frac{G}{N})^p|}{|\frac{G}{N}|^2}K(V(\frac{G}{N}))$$
$$\frac{|V(N)||V(\frac{G}{N})}{|G|^2}K(V(N))K(V(\frac{G}{N}))$$

since $|G| = |V(G)| = |V(N)||V(\frac{G}{N})|$ so the result hold.

Definition 3.10. For two subgroups H and K of finite group G, $HK \in L(G)$ if and only if $KH = HK$ it means H and K are commutative. So $H^pK^p \in L(G)$ if and only if $H^pK^p = K^pH^p$ for any prime number of p. We define the subgroup p-commutativity degree $sd_p(G)$ of a finite group G, by

$$sd_p(G) = \frac{|\{(H,K) \in L(G) \times L(G)|H^pK^p = K^pH^p\}|}{|L(G)^2|}.$$

It is clear that, $0 < sd_p(G) \leq 1$. Now for any subgroup H of G we define

$$sd_p(G) = \frac{1}{|L(G)^2|} \sum_{H \in L(G)} |C_p(H)|,$$

where

$$C_p(H) = \{k \in L(G)|H^pK^p = K^pH^p\}.$$

In order to simplify our notation, we define the following function

$$f_p : L(G)^2 \to \{0,1\}$$

$$f(H,K) = \begin{cases} 1, & H^pK^p = K^pH^p \\ 0, & H^pK^p \neq K^pH^p \end{cases}$$

obviously $|C_p(H) = \sum_{K \in L(G)} f(H,K)$, for any $H \in L(G)$, so

$$sd_p(G) = \frac{1}{|L(G)^2|} \sum_{H,K \in L(G)} f(H,K).$$

Theorem 3.11. If $(G_i)_{i=1}^k$ be a family of finite groups having coprime orders, then

$$sd_p(\times_{i=1}^k G_i) = \pi_{i=1}^k sd_p(G_i).$$

Corollary 3.12. If G is a finite nilpotent group and $(p_i)_{i=1,\dots,k}$ are the sylow subgroups of G, then

$$sd_{\mathcal{B}}(G) = \pi_{i=1}^k sd_{\mathcal{B}}(G_i).$$

Theorem 3.13. Let N be a normal subgroup of a group G, then

$$sd_p(G) \geq \frac{1}{|L(G)|^2}[(|L(N)| + |L(\frac{G}{N})| - 1)^2 + (sd_p(N) - 1)|L(N)|^2 + (sd_p(\frac{G}{N}) - 1)|L(\frac{G}{N})|^2].$$

Proof. Let N be a normal subgroup of G and $A_1 = \{H \in L(G) | N \subseteq H\}$ and$A_2 = \{H \in L(G) | H \subset N\}$ but $A_1 \cup A_2 \subseteq L(G)$ so

$$\begin{aligned} sd_p(G) &\geq \frac{1}{|L(G)|^2} \sum_{H,K \in A_1 \cup A_2} f(H,K) \\ &= \frac{1}{|L(G)|^2} (\sum_{H,K \in A_1} f(H,K) \\ &+ \sum_{H,K \in A_2} f(H,K) + 2 \sum_{H \in A_1} \sum_{K \in A_2} f(H,K)). \end{aligned} \tag{10}$$

We can calculute the right side of (1) as follows

$$\begin{aligned} \sum_{H,K \in A_1} f(H,K) &= sd_p(\frac{G}{N})|L(\frac{G}{N})|^2 \\ \sum_{H,K \in A_2} f(H,K) &= \sum_{H,K \in A_2 \cup N} f(H,K) - 2 \sum_{H,K \in A_2 \cup N} f(H,N) + 1 \\ &= sd_p(N)|L(N)|^2 - 2|L(N)| + 1 \end{aligned}$$

and

$$2 \sum_{x \in A_1} \sum_{x \in A_2} f(H,K) = 2|A_1||A_2| = 2|L(\frac{G}{N})|(|L(N)| - 1)$$

so by (1) the reuslt is hold.

Corollary 3.14. If N is a normal subgroup of G such that N and $\frac{G}{N}$ are p-abelian groups, then

$$sd_p(G) =\geq (\frac{|L(N)| + |L(\frac{G}{N})| - 1}{|L(G)|})^2.$$

Now we can expand the definition of $ssd(G)$ to $ssd_p(G)$. Let, $G^p = V(G) = \{g^p | g \in G\}$, then

$$\begin{aligned} d_p(G) &= \frac{|(x,y) \in G^2 | [x^p, y^p] = 1|}{|G|^2} \\ &= \frac{1}{|G|^2} \sum_{x \in G} |C_{V(G)}(x^p)|, \end{aligned}$$

where

$$C_{V(G)}(x^p) = \{y^p \in V(G), |[x^p, y^p] = 1\}.$$

If we take a subgroup H in G, denoted by $d_p(H,G)$ as follows

$$\begin{aligned} d_p(H,G) &= \frac{|\{(x,y) \in H \times G | [x^p, y^p] = 1\}|}{|H||G|} \\ &= \frac{1}{|G||H|} \sum_{h \in H} |C_{V(G)}(h^p)| \end{aligned}$$

where $d_p(G,G) = d_p(G)$, so we define the relative subgroup p-commutativity degree of a subgroup H of G

$$sd_p(H,G) = \frac{1}{|L(G)||L(H)|}|\{(H_1,G_1) \in L(H_1) \times L(G_1)|H_1^p G_1^p = G_1^p H_1^p\}|$$

$sd_p(G) = sd_p(G,G)$ and also we can define $sd_p(H,K)$ for any group of G. But we know that by given subgroups H and K of G, the product $H^p K^p$ is not always a subroup of G.
Now H and K p-permute if $H^p K^p = K^p H^p$ and must be p-permutable in G, if H^p permutes with every power of p of subgroup of G. We will define strong subgroup p-commutativity degree of G by

$$ssd_p(G) = \frac{|\{(H,K) \in L(G) \times L(G)|[H^p, K^p] = 1\}|}{|L(G)|^2}$$

it is easy to see that

$$ssd_p(G) = \frac{1}{|L(G)|^2} \sum_{H \in L(G)} |comm_{p-G}(H)|.$$

Where

$$comm_{p-G}(H) = \{k \in L(G)|[H^p, K^p] = 1\}.$$

It is obviuos that $sd_p(G) = 1$ if and only if any subgroups of G be p-permutable. for a finite group G, $ssd_p(G) \leq sd(G)$ because we have

$$ssd_p(G) = \frac{1}{|L(G)|^2} \sum_{H \in L(G)} |comm_{p-G}(H)| \leq \frac{1}{|L(G)|^2} \sum_{H \in L(G)} |C_p(H)| = sd_p(G)$$

and it means

$$|\{(H,K) \in L(G) \times L(G)|H^p K^p = K^p H^p\}| \geq |\{(H,K) \in L(G) \times L(G)|[H^p, K^p] = 1\}|.$$

Theorem 3.15. If H and K are two subgroups of G, then

$$ssd_p(G) < \frac{|G|^2}{|L(G)|^2} \sum_{H,K \in L(G)} d_p(H,K).$$

Proof.

$$\begin{aligned} ssd_p(G) &= \frac{1}{|L(G)|^2}\{(H,K) \in L(G) \times L(G)|[H^p, K^p] = 1\}| \\ &\leq \frac{1}{|L(G)|^2} \sum_{H,K \in L(G)} |H||K| d_p(H,K) \\ &< \frac{1}{|L(G)|^2} \sum_{H,K \in L(G)} |G|^2 d_p(H,K) \end{aligned}$$

and the result is obtained.

In general, $L(G_1 \times G_2) \neq (G_1) \times L(G_2)$, and if G_1 and G_2 having coprime orders then the equality is obtained and if If $(G_i)_{i=1}^k$ be a family of finite groups having coprime orders, then

$$ssd_p(\times_{i=1}^k G_i) = \pi_{i=1}^k ssd_p(G_i).$$

Theorem 3.16. If H and K are two groups of G, then

$$sd_p(G) \geq \frac{1}{|L(G)|^2} \sum_{H \in L(G)} |\bigcap_{h \in H} C_{V(K)}(h^p)|.$$

Proof.

$$\bigcap_{h \in H} C_{V(K)}(h^p) = C_{V(K)}(V(H)) \subseteq C_p(H)$$

so

$$\sum_{H \in L(G)} |\bigcap_{h \in H} C_{V(K)}(h^p)| \leq \sum_{H \in L(G)} |C_p(H)|$$

and the result hold.

Theorem 3.17. If H and K are two groups of G, then

$$\sum_{H,K \in L(G)} d_p(H,K)|H||K| \geq \sum_{H,K \in L(G)} |\bigcap_{h \in H} C_{V(K)}(h^p)|.$$

Proof. The proof is obviouse with the following inequality

$$\sum_{h \in H} |C_{V(K)}(h^p)| \geq |\bigcap_{h \in H} C_{V(K)}(h^p)|.$$

Corollary 3.18. For any subgroups L and M of H that H is a subgroup of G,

$$\frac{1}{|L(G)|^2} \sum_{L \in L(H)} |\bigcap_{l \in L} C_{V(M)}(l^p)| \leq sd_p(H) \leq sd_p(G).$$

References

[1] P.X. Gallagher, *The number of conjugacy classes in a finite group*, Math. Z. 118 (1970), 175-179.

[2] A. Gholami, M.R. Mollaei, *Some Inequalities of Subgroup Commutativity Degree of Finite Groups*, Southeast Asian Bull. Math. 37 (2013), no. 6, 845–858.

[3] N.S. Hekster, *Varieties of groups and isologisms*, J. Austral. Math. 46 (1989), 22-60.

[4] P. Lescot, *Isoclinism classes and commutativity degrees of finite groups*, J. Algebra. 177 (1995), 847-869.

[5] R. Moghadam, *n-Isoclinism and n-Nilpotency degree of finite groups*, J. Algebra Colloquim. 12:2 (2005), 255-261.

[6] G. Russo, *Subgroup s-commutativity degrees of finite groups*, J. Algebra. 19 (2012), no. 2, 373-382.

[7] M. Tarnauceanu, *Subgroup commutativity degrees of finite groups*, J. Algebra. 321 (2009), 2508-2520.

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