OFF-STEP DISCRETIZATION FOR SYSTEM OF NONLINEAR SINGULAR BOUNDARY VALUE PROBLEMS USING VARIABLE MESH

Arshad Khan*
Department of Mathematics
Faculty of Natural Sciences
Jamia Millia Islamia
New Delhi-25
India
akhan1234in@rediffmail.com

Sucheta Nayak
Department of Mathematics
Lady Shri Ram College for Women
University of Delhi
New Delhi-24
India
suchetanayak@ lsr.edu.in

R. K. Mohanty
Department of Mathematics
South Asian University
New Delhi- 21
India
rmohanty@sau.ac.in

Abstract. In this paper, we propose two generalized variable mesh schemes based on off-step points to solve the system of nonlinear singular two point boundary value problems . Theoretical analysis proves that the proposed methods have second, third and fourth order convergence. Both the methods are applicable to singular boundary value problems. Numerical results are also provided to show the accuracy and efficiency of the proposed methods.
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1. Introduction

In this paper, we present some effective numerical techniques using variable mesh based on off-step points to solve system of M nonlinear singular boundary value problems(BVPs) of the following type:

$$(1) \quad {u_{xx}}^{(i)} = F^{(i)}(x, u^{(1)}, ..., u^{(i)}, ..., u^{(M)}, {u_x}^{(1)}, ..., {u_x}^{(i)}, ..., {u_x}^{(M)}), a \leq x \leq b,$$

*. Corresponding author

subject to boundary conditions

$$(2)\quad u^{(i)}(a) = A_i, u^{(i)}(b) = B_i, \text{where } {u_x}^{(i)} = \frac{du^{(i)}}{dx}, {u_{xx}}^{(i)} = \frac{d^2u^{(i)}}{dx^2}, i = 1(1)M$$

We assume that for $-\infty < a \leq x \leq b < \infty$ and $-\infty < u^{(i)}, {u_x}^{(i)} < \infty$, we have

(i) $F^{(i)}(x, u^{(1)}, u^{(2)}, .., u^{(i)}, .., u^{(M)}, {u_x}^{(1)}, {u_x}^{(2)}, .., {u_x}^{(i)}, ., {u_x}^{(M)})$ is continuous;

(ii) $\frac{\partial F^{(i)}}{\partial u^{(j)}}$ and $\frac{\partial F^{(i)}}{\partial {u_x}^{(j)}}$ exist and are continuous;

(iii) $\frac{\partial F^{(i)}}{\partial u^{(j)}} > 0$ and $|\frac{\partial F^{(i)}}{\partial {u_x}^{(j)}}| \leq C$, for some positive constant C and $i, j = 1(1)M$.

These conditions, proved by Keller [10] assure us the existence of a unique solution of the system of boundary value problem (1)-(2).

In the present paper, we have derived generalized schemes of second and third order using variable mesh based on off-step points for solving system of two point boundary value problems (1)-(2). Such systems decomposes several higher order problems and then solves them efficiently. The higher order boundary value problems models various phenomena in the field of astrophysics, hydrodynamics, fluid dynamics, astronomy, beam and wave theory, engineering and applied physics (see [2], [3], [6], [12], [19]). The existence and uniqueness of such higher order boundary value problems are discussed by Aftabizadeh [1], Regan [4], Agarwal [20]. Several authors have also solved such boundary value problems using numerical techniques. To name a few, Akram et. al. [8] used kernel space method to solve eighth order boundary value problems and also used non-polynomial spline technique [9] to solve sixth order boundary value problems. Talwar et. al. [13] developed finite difference method to solve fourth order BVPs on uniform and variable mesh respectively. Noor et. al. [14] and Pandey [17] used homotopy perturbation method and finite difference method respectively to solve sixth order boundary value problems. So, previous work motivated us to solve higher order boundary value problems using system of boundary value problems.

Presently, the higher order boundary value problems are decomposed into system of second order boundary value problems (1). The boundary conditions are also modified accordingly and when incorporated into the scheme containing system of discretized second order boundary value problems, we get a block tri-diagonal Jacobian. In case of linear boundary value problem, we have used block Gauss elimination method to solve the Jacobian and in case of nonlinear problem we have used block Newton's method .

The sections of this paper are organized as follows. In section 2,we give details of derivation of the scheme using second order linear boundary value problem and in section 3, we provide generalization of the scheme. In section 4, we present the application of the proposed schemes on a fourth order singular boundary value problem.In section 5, we discuss convergence analysis of variable as well as uniform mesh schemes and in section 6, we provide numerical illustrations to demonstrate the accuracy of the proposed schemes. Finally in section 7, we provide concluding remarks.

2. Derivation of the schemes

We consider a second order nonlinear boundary value problem of the following type:

$$u_{xx} = F(x,u,u_x), \text{ such that } u(a)=A, u(b)=B \tag{3}$$

We discretise the solution region [a,b] such that $a = x_0 < x_1 < x_2....x_{N-1} < x_N = b$. Let $h_j = x_j - x_{j-1}, j = 1(1)N$ be the mesh size and the mesh ratio be $\sigma_j = \frac{h_{j+1}}{h_j} > 0, j = 1(1)N-1$. When $\sigma_j = 1$ the mesh reduces to a uniform mesh i.e., $h_{j+1} = h_j = h$. Now, without loss of generality, we choose $\sigma_j = \sigma$ a constant $\forall j$. Also, assume u_j and U_j be the approximate and exact solution of (3) at the grid points $x_j, j = 1,2,....,N$. Now,we start with following approximations at the grid points ,

$$S_j = \sigma(\sigma+1), \tag{4}$$

$$\bar{u}_{j+\frac{1}{2}} = \frac{u_{j+1}+u_j}{2}, \tag{5}$$

$$\bar{u}_{j-\frac{1}{2}} = \frac{u_{j-1}+u_j}{2}, \tag{6}$$

$$\bar{u}_{x_{j+\frac{1}{2}}} = \frac{u_{j+1}-u_j}{h_j\sigma}, \tag{7}$$

$$\bar{u}_{x_{j-\frac{1}{2}}} = \frac{u_j-u_{j-1}}{h_j}, \tag{8}$$

$$\bar{u}_{x_j} = \frac{u_{j+1}+(\sigma^2-1)u_j-\sigma^2u_{j-1}}{h_jS_j}. \tag{9}$$

Using (3), we define the following:

$$\bar{F}_{j+\frac{1}{2}} = f(x_{j+\frac{1}{2}}, \bar{u}_{j+\frac{1}{2}}, \bar{u}_{x_{j+\frac{1}{2}}}), \tag{10}$$

$$\bar{F}_{j-\frac{1}{2}} = f(x_{j-\frac{1}{2}}, \bar{u}_{j-\frac{1}{2}}, \bar{u}_{x_{j-\frac{1}{2}}}), \tag{11}$$

$$\bar{F}_j = f(x_j, u_j, \bar{u}_{x_j}). \tag{12}$$

Now, to get a higher order approximation for u_j and u_{x_j}, we define the following approximations:

$$\hat{u}_j = u_j + \delta h_j^2(\bar{F}_{j+\frac{1}{2}} + \bar{F}_{j-\frac{1}{2}}), \tag{13}$$

$$\hat{u}_{x_j} = \bar{u}_{x_j} + \gamma h_j(\bar{F}_{j+\frac{1}{2}} - \bar{F}_{j-\frac{1}{2}}), \tag{14}$$

where δ, γ are parameters to be determined. Therefore, the modified $\bar{F}_j$ is as follows:

$$\hat{F}_j = f(x_j, \hat{u}_j, \hat{u}_{x_j}). \tag{15}$$

It is easy to see that

$$\hat{u}_j = u_j + \delta h_j^2 (2u_{xx_j}) + O(h_j^3), \sigma \neq 1 \tag{16}$$

$$\hat{u}_{x_j} = u_{x_j} + \frac{h_j^2}{6} u_{xxx_j} (\sigma + 3\gamma(1+\sigma)) + O(h_j^3), \sigma \neq 1 \tag{17}$$

$$\bar{F}_{j+\frac{1}{2}} = F_{j+\frac{1}{2}} + \frac{(h_j\sigma)^2}{8} u_{xx_j} G + \frac{(h_j\sigma)^2}{24} u_{xxx_j} H + O(h_j^3), \tag{18}$$

$$\bar{F}_{j-\frac{1}{2}} = F_{j-\frac{1}{2}} + \frac{h_j^2}{8} u_{xx_j} G + \frac{h_j^2}{24} u_{xxx_j} H + O(h_j^3), \tag{19}$$

$$\hat{F}_j = F_j + 2h_j^2 \delta u_{xx_j} G + \frac{h_j^2}{6} (\sigma + 3\gamma(1+\sigma)) u_{xxx_j} H + O(h_j^3), \tag{20}$$

where

$$G = \frac{\partial f}{\partial u_j}, H = \frac{\partial f}{\partial u_{x_j}}.$$

Now, by Taylor's expansion we derive the following off-step discretization schemes for $\sigma \neq 1, j = 1(1)N-1$:

$$u_{j+1} - (1+\sigma)u_j + \sigma u_{j-1} = \frac{h_j^2}{6} (A_j F_{j+\frac{1}{2}} + B_j F_{j-\frac{1}{2}}), \tag{21}$$

$$u_{j+1} - (1+\sigma)u_j + \sigma u_{j-1} = \frac{h_j^2}{6} (P_j F_{j+\frac{1}{2}} + R_j F_{j-\frac{1}{2}} + Q_j F_j), \tag{22}$$

where $A_j, B_j, P_j, Q_j$and R_j are given as follows:

$$A_j = \sigma(2\sigma+1), B_j = \sigma(2+\sigma), \tag{23}$$

$$P_j = 2\sigma^2, R_j = 2\sigma, Q_j = \sigma(\sigma+1). \tag{24}$$

Further, we discretize the boundary value problem(3)by using the first scheme (21) as :

$$u_{j+1} - (1+\sigma)u_j + \sigma u_{j-1} = \frac{h_j^2}{6} (A_j \bar{F}_{j+\frac{1}{2}} + B_j \bar{F}_{j-\frac{1}{2}}) + T_j^2. \tag{25}$$

Here, we can easily show that $T_j^2 = O(h_j^4)$ by using the approximations (18)-(19). Similarly, using the second scheme(22) along with the approximation (18)-(20) we discretize (3) at each grid points x_j:

$$\begin{aligned} \sigma u_{j-1} - (1+\sigma)u_j + u_{j+1} &= \frac{h_j^2}{6} (P_j \bar{F}_{j+\frac{1}{2}} + Q_j \hat{F}_j + R_j \bar{F}_{j-\frac{1}{2}}) \\ &- h_j^4 \big[(\frac{P_j \sigma^2}{24} + \frac{Q_j}{6} (\sigma + 3\gamma(\sigma+1)) + \frac{R_j}{24}) u_{xxx} H \\ &+ (\frac{P_j \sigma^2}{8} + 2\delta Q_j + \frac{R_j}{8}) u_{xx} G \big] + T_j^3. \end{aligned} \tag{26}$$

To make the local truncation error T_j^3 of proposed scheme as $O(h_j^5)$, the coefficients of h_j^4 is equated to zero. Hence, we get $\delta = -\frac{(\sigma^2+1-\sigma)}{8}, \gamma = -\frac{(\sigma+1+\sigma^2)}{6(1+\sigma)}$. The same truncation error for uniform mesh becomes $O(h^6)$. Note that the coefficients $A_j, B_j, P_j, Q_j$and R_j are positive for $\sigma > 0$, which is a necessary condition for convergence of the methods[16].

Hence, both the proposed three point discretization schemes for $j = 1(1)N-1$ are as follows:

$$u_{j+1} - (1+\sigma)u_j + \sigma u_{j-1} = \frac{h_j^2}{6}(A_j\bar{F}_{j+\frac{1}{2}} + B_j\bar{F}_{j-\frac{1}{2}}) \tag{27}$$

and

$$u_{j+1} - (1+\sigma)u_j + \sigma u_{j-1} = \frac{h_j^2}{6}(P_j\bar{F}_{j+\frac{1}{2}} + Q_j\hat{F}_j + R_j\bar{F}_{j-\frac{1}{2}}). \tag{28}$$

3. Generalization of the schemes

We generalize our method for the solution of the system of M nonlinear boundary value problems (1). At the grid point $x_j, j = 1(1)N-1$ and for $i = 1(1)M$, we use the following approximations and schemes:

$$S_j = \sigma(\sigma+1), \tag{29}$$

$$\bar{u}^{(i)}_{j+\frac{1}{2}} = \frac{u^{(i)}_{j+1} + u^{(i)}_j}{2}, \tag{30}$$

$$\bar{u}^{(i)}_{j-\frac{1}{2}} = \frac{u^{(i)}_{j-1} + u^{(i)}_j}{2}, \tag{31}$$

$$\bar{u}^{(i)}_{x_{j+\frac{1}{2}}} = \frac{u^{(i)}_{j+1} - u^{(i)}_j}{h_j\sigma}, \tag{32}$$

$$\bar{u}^{(i)}_{x_{j-\frac{1}{2}}} = \frac{u^{(i)}_j - u^{(i)}_{j-1}}{h_j}, \tag{33}$$

$$\bar{u}^{(i)}_{x_j} = \frac{u^{(i)}_{j+1} + (\sigma^2-1)u^{(i)}_j - \sigma^2 u^{(i)}_{j-1}}{h_j S_j}, \tag{34}$$

$$\begin{aligned}\bar{F}^{(i)}_{j+\frac{1}{2}} = f^{(i)}(&x_{j+\frac{1}{2}}, \bar{u}^{(1)}_{j+\frac{1}{2}}, \bar{u}^{(2)}_{j+\frac{1}{2}}, ..., \bar{u}^{(i)}_{j+\frac{1}{2}}, ..., \bar{u}^{(M)}_{j+\frac{1}{2}},\\ &\bar{u}^{(1)}_{x_{j+\frac{1}{2}}}, \bar{u}^{(2)}_{x_{j+\frac{1}{2}}}, ..., \bar{u}^{(i)}_{x_{j+\frac{1}{2}}}, ..., \bar{u}^{(M)}_{x_{j+\frac{1}{2}}}),\end{aligned} \tag{35}$$

$$\begin{aligned}\bar{F}^{(i)}_{j-\frac{1}{2}} = f^{(i)}(&x_{j-\frac{1}{2}}, \bar{u}^{(1)}_{j-\frac{1}{2}}, \bar{u}^{(2)}_{j-\frac{1}{2}}, ..., \bar{u}^{(i)}_{j-\frac{1}{2}}, ..., \bar{u}^{(M)}_{j-\frac{1}{2}},\\ &\bar{u}^{(1)}_{x_{j-\frac{1}{2}}}, \bar{u}^{(2)}_{x_{j-\frac{1}{2}}}, ..., \bar{u}^{(i)}_{x_{j-\frac{1}{2}}}, ..., \bar{u}^{(M)}_{x_{j-\frac{1}{2}}}),\end{aligned} \tag{36}$$

$$\bar{F}^{(i)}_j = f^{(i)}(x_j, u^{(1)}_j, u^{(2)}_j, ..., u^{(i)}_j, ..., u^{(M)}_j, \bar{u}^{(1)}_{x_j}, \bar{u}^{(2)}_{x_j}, ..., \bar{u}^{(i)}_{x_j}, ..., \bar{u}^{(M)}_{x_j}), \tag{37}$$

$$(38)\quad \hat{u}_j^{(i)} = u_j^{(i)} + \delta_i h_j^2(\bar{F}^{(i)}_{j+\frac{1}{2}} + \bar{F}^{(i)}_{j-\frac{1}{2}}),$$

$$(39)\quad \hat{u}_{x_j}^{(i)} = \bar{u}_{x_j}^{(i)} + \gamma_i h_j(\bar{F}^{(i)}_{j+\frac{1}{2}} - \bar{F}^{(i)}_{j-\frac{1}{2}}),$$

$$(40)\quad \hat{F}_j^{(i)} = f^{(i)}(x_j, \hat{u}_j^{(1)}, \hat{u}_j^{(2)}, ..., \hat{u}_j^{(i)}, ..., \hat{u}_j^{(M)}, \hat{u}_{x_j}^{(1)}, \hat{u}_{x_j}^{(2)}, ..., \hat{u}_{x_j}^{(i)}, ..., \hat{u}_{x_j}^{(M)}),$$

$$(41)\quad u_{j+1}^{(i)} - (1+\sigma)u_j^{(i)} + \sigma u_{j-1}^{(i)} = \frac{h_j^2}{6}(A_j\bar{F}^{(i)}_{j+\frac{1}{2}} + B_j\bar{F}^{(i)}_{j-\frac{1}{2}}),$$

$$(42)\quad u_{j+1}^{(i)} - (1+\sigma)u_j^{(i)} + \sigma u_{j-1}^{(i)} = \frac{h_j^2}{6}(P_j\bar{F}^{(i)}_{j+\frac{1}{2}} + Q_j\hat{F}_j^{(i)} + R_j\bar{F}^{(i)}_{j-\frac{1}{2}}).$$

where

$$(43)\quad A_j = \sigma(2\sigma+1), B_j = \sigma(2+\sigma)$$

$$(44)\quad P_j = 2\sigma^2, R_j = 2\sigma, Q_j = \sigma(\sigma+1).$$

4. Application to fourth order singular boundary value problem

We consider a fourth order singular boundary value problem of the following type:

$$(45)\quad u_{xxxx}(x) = a(x)u_{xxx}(x) + b(x)u_{xx}(x) + d(x), 0 \leq x \leq 1$$

subject to boundary conditions:

$$(46)\quad u(0) = \alpha_1, u_{xx}(0) = \alpha_2, u(1) = \beta_1, u_{xx}(1) = \beta_2,$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are real constants and $a(x)$ is singular at $x = 0$. Using (1), we write the problem $(45) - (46)$ as follows:

$$(47)\quad u_{xx}(x) = v(x),$$

$$(48)\quad v_{xx}(x) = a(v)v_x(x) + b(x)v(x) + d(x),$$

subject to

$$(49)\quad u(0) = \alpha_1, v(0) = \alpha_2, u(1) = \beta_1, v(1) = \beta_2.$$

Applying the difference scheme (41) to the coupled second order boundary value problem $(47) - (48)$ we obtain the following difference scheme:

$$(50)\quad \sigma_j u_{j-1} - (1+\sigma_j)u_j + u_{j+1} = \frac{h_j^2}{6}(A_j\bar{v}_{j+\frac{1}{2}} + B_j\bar{v}_{j-\frac{1}{2}}),$$

$$(51)\quad \sigma_j v_{j-1} - (1+\sigma_j)v_j + v_{j+1} = \frac{h_j^2}{6}\big[A_j(a_{j+\frac{1}{2}}\bar{v}_{x_{j+\frac{1}{2}}} + b_{j+\frac{1}{2}}\bar{v}_{j+\frac{1}{2}} + d_{j+\frac{1}{2}}) + B_j(a_{j-\frac{1}{2}}\bar{v}_{x_{j-\frac{1}{2}}} + b_{j-\frac{1}{2}}\bar{v}_{j-\frac{1}{2}} + d_{j-\frac{1}{2}})\big].$$

Then, we define the following relations for $a_{j\pm\frac{1}{2}}$ for the coupled finite difference scheme (47) – (48)

$$a^*_{j-\frac{1}{2}} = a_j - \frac{h_j}{2}a_{x_j} + O(h_j^2), \tag{52}$$

$$a^*_{j+\frac{1}{2}} = a_j + \frac{\sigma h_j}{2}a_{x_j} + O(h_j^2), \tag{53}$$

$$a^{**}_{j-\frac{1}{2}} = a_j - \frac{h_j}{2}a_{x_j} + \frac{(h_j)^2}{8}a_{xx_j} + O(h_j^3), \tag{54}$$

$$a^{**}_{j+\frac{1}{2}} = a_j + \frac{\sigma h_j}{2}a_{x_j} + \frac{(\sigma h_j)^2}{8}a_{xx_j} + O(h_j^3). \tag{55}$$

Similar relations for $b_{j\pm\frac{1}{2}}, d_{j\pm\frac{1}{2}}$ can be defined. Using the relations (52) – (53) in (50) – (51) we get,

$$\sigma u_{j-1} - (1+\sigma)u_j + u_{j+1} = \frac{h_j^2}{6}(A_j\bar{v}_{j+\frac{1}{2}} + B_j\bar{v}_{j-\frac{1}{2}}), \tag{56}$$

$$\begin{aligned}\sigma v_{j-1} - (1+\sigma)v_j + v_{j+1} = \frac{h_j^2}{6}\big[&A_j(a^*_{j+\frac{1}{2}}\bar{v}_{x_{j+\frac{1}{2}}} + b^*_{j+\frac{1}{2}}\bar{v}_{j+\frac{1}{2}} + d^*_{j+\frac{1}{2}})\\ &+ B_j(a^*_{j-\frac{1}{2}}\bar{v}_{x_{j-\frac{1}{2}}} + b^*_{j-\frac{1}{2}}\bar{v}_{j-\frac{1}{2}} + d^*_{j-\frac{1}{2}})\big].\end{aligned} \tag{57}$$

Finally, substituting (29)-(36) in (56)-(57) we obtain the vector difference equation of boundary value problem (45)-(46) as follows:

$$\begin{aligned}&\begin{bmatrix} sub_j^{11} & sub_j^{12} \\ sub_j^{21} & sub_j^{22}\end{bmatrix}\begin{bmatrix} u_{j-1} \\ v_{j-1}\end{bmatrix} + \begin{bmatrix} diag_j^{11} & diag_j^{12} \\ diag_j^{21} & diag_j^{22}\end{bmatrix}\begin{bmatrix} u_j \\ v_j\end{bmatrix}\\ &+\begin{bmatrix} sup_j^{11} & sup_j^{12} \\ sup_j^{21} & sup_j^{22}\end{bmatrix}\begin{bmatrix} u_{j+1} \\ v_{j+1}\end{bmatrix} = \begin{bmatrix} \phi_j^1 \\ \phi_j^2\end{bmatrix},\end{aligned} \tag{58}$$

where

$$\begin{cases} sub_j^{11} = -\sigma, \quad sub_j^{12} = \dfrac{h_j^2\sigma(\sigma+2)}{12}, \\ sub_j^{21} = 0, sub_j^{22} = -\sigma + \dfrac{h_j\sigma(\sigma+2)}{12}\big(-2a_j + h_j(a_{x_j} + b_j) - \dfrac{h_j^2 b_{x_j}}{2}\big). \end{cases} \tag{59}$$

$$\begin{cases} diag_j^{11} = (1+\sigma), diag_j^{12} = \dfrac{3h_j^2\sigma(\sigma+1)}{12}, \\ diag_j^{21} = 0, diag_j^{22} = (1+\sigma) + \dfrac{h_j(2\sigma+1)}{12}(-2a_j - h_j\sigma(a_{x_j} - b_j) \\ \qquad + \dfrac{(h_j\sigma)^2 b_{x_j}}{2}) \end{cases} \tag{60}$$

$$\begin{cases} sup_j^{11} = -1, \quad sup_j^{12} = \dfrac{h_j^2\sigma(2\sigma+1)}{12}, \\ sup_j^{21} = 0, sup_j^{22} = -1 + \dfrac{h_j(2\sigma+1)}{12}\big(2a_j + h_j\sigma(a_{x_j} + b_j) + \dfrac{(h_j\sigma)^2 b_{x_j}}{2}\big) \end{cases} \tag{61}$$

$$\begin{cases} \phi_j^1 = 0, \quad \phi_j^2 = -\dfrac{h_j^2}{6}\big[d_j(3\sigma^2 + \sigma) + d_{x_j}h_j(\sigma^3 + \sigma)\big] \end{cases} \cdot \tag{62}$$

Similarly, using the difference scheme (42) for the boundary value problem (45) – (46), we obtain the following difference scheme :

$$\sigma u_{j-1} - (1+\sigma)u_j + u_{j+1} = \frac{h_j^2}{6}(P_j\bar{v}_{j+\frac{1}{2}} + Q_j\hat{v}_j + R_j\bar{v}_{j-\frac{1}{2}}), \tag{63}$$

$$\begin{aligned}\sigma v_{j-1} - (1+\sigma)v_j + v_{j+1} &= \frac{h_j^2}{6}(G_j[a^{**}_{j+\frac{1}{2}}\bar{v}_{x_{j+\frac{1}{2}}} + b^{**}_{j+\frac{1}{2}}\bar{v}_{j+\frac{1}{2}} + d^{**}_{j+\frac{1}{2}}] \\ &\quad + Q_j[a_j\hat{v}_{x_j} + b_j\hat{v}_j + d_j] \\ &\quad + H_j[a^{**}_{j-\frac{1}{2}}\bar{v}_{x_{j-\frac{1}{2}}} + b^{**}_{j-\frac{1}{2}}\bar{v}_{j-\frac{1}{2}} + d^{**}_{j-\frac{1}{2}}]),\end{aligned} \tag{64}$$

where $G_j = P_j + Q_j(a_j\gamma_2 h_j + b_j\delta_2 h_j^2)$, $H_j = R_j - Q_j(a_j\gamma_2 h_j + b_j\delta_2 h_j^2)$, $j = 1(1)N-1$. This scheme can be simplified upto $O(h_j^4)$ terms by using (29)-(40).

5. Convergence analysis

The convergence analysis of scalar singular boundary value problem has been given by R.K. Mohanty [18]. We consider the vector form of convergence analysis of scheme (41) for the coupled second order boundary value problem (47) – (49). Now, once the boundary conditions are incorporated in the the vector difference equation (58), it can be written in matrix form as:

$$D\hat{u} + \hat{\phi} + T(\hat{h}_j) = \begin{bmatrix} sub_j & diag_j & sup_j \end{bmatrix} \begin{bmatrix} \hat{u}_{j-1} \\ \hat{u}_j \\ \hat{u}_{j+1} \end{bmatrix} + \hat{\phi}_j + T(\hat{h}_j) = \hat{0}, \tag{65}$$

where D is a block tridiagonal matrix of order $N-1$;$sub_j, sup_j, diag_j$ are block matrices of order 2×2 in D,

$$\begin{aligned} \hat{u} &= [\hat{u}_1, \hat{u}_2, ..., \hat{u}_j, ...\hat{u}_{N-1}]^T, \text{ where } \hat{u_j} = [u_j, v_j]^T \\ \hat{\phi} &= [\hat{\phi}_1 + sub_1[\alpha_1, \alpha_2]^T, \hat{\phi}_2, ..., \hat{\phi}_j, ..., \hat{\phi}_{N-1} + sup_{N-1}[\beta_1, \beta_2]^T]^T, \\ &\quad \text{where } \hat{\phi_j} = [\phi_j^1, \phi_j^2]^T \\ T(\hat{h}_j) &= [{T_1}^2, {T_2}^2, ..., {T_{N-1}}^2]^T \\ \hat{0} &= [[0,0]^T, [0,0]^T, ...[0,0]^T]^T. \end{aligned}$$

Let

$$U = [[U_1, V_1]^T, \quad [U_2, V_2]^T, [U_j, V_j]^T, ...[U_{N-1}, V_{N-1}]^T]^T \cong \hat{u} \text{ satisfy}$$

$$DU + \hat{\phi} = 0, \text{ where D is defined in (65) .} \tag{66}$$

Let $\hat{e_j} = [U_j - u_j, V_j - v_j]^T \equiv [e_{j_u}, e_{j_v}]^T$ be the discretization error in absence of round off error, then $U - \hat{u} = E = [\hat{e_1}, \hat{e_2}, ..., \hat{e_{N-1}}]^T$.

Subtracting (65) from (66), we obtain the error equation as follows

$$DE = T. \tag{67}$$

Let $|a_j| \leq K_1, |a_{x_j}| \leq K_2, |b_j| \leq K_3$ and $|b_{x_j}| \leq K_4$ for some positive constants K_1, K_2, K_3, K_4 , then using (59) and (61) we get,

$$\|sup_j\|_\infty \leq \max_{1\leq j\leq N-2} \begin{cases} 1 + \dfrac{h_j^2\sigma(1+2\sigma)}{12} \\ 1 + \dfrac{(1+2\sigma)}{12}[2h_jK_1 + h_j^2\sigma(K_2+K_3) \\ +\frac{h_j^3\sigma^2}{2}K_4] + O(h_j^4) \end{cases} \tag{68}$$

$$\|sub_j\|_\infty \leq \max_{2\leq j\leq N-1} \begin{cases} \sigma + \dfrac{h_j^2\sigma(2+\sigma)}{12} \\ \sigma + \dfrac{\sigma(2+\sigma)}{12}[2h_jK_1 + h_j^2(K_2+K_3) \\ +\frac{h_j^3}{2}K_4] + O(h_j^4) \end{cases} . \tag{69}$$

Thus, for sufficiently small h_j, we get $\|sub_j\|_\infty \leq \sigma$ and $\|sup_j\|_\infty \leq 1$. Hence, D is irreducible. Now, let $sum_{row\,j}$ be the sum of elements of row_j of D

$$sum_{row\,j} = \begin{cases} \sigma + \dfrac{h_j^2}{12}\sigma(4+5\sigma), \quad j=1 \\ \sigma + \dfrac{h_j\sigma(2+\sigma)a_j}{6} + \dfrac{h_j^2\sigma}{12}\big[-(2+\sigma)a_{x_j} + (4+5\sigma)b_j\big] \\ +O(h_j^3), j=2 \end{cases} \tag{70}$$

$$sum_{row\,j} = \begin{cases} \dfrac{h_j^2}{2}S_j, \quad j=3,5..N-4 \\ \dfrac{h_j^2}{2}b_jS_j, j=4,6..N-3 \end{cases} \tag{71}$$

$$sum_{row\,j} = \begin{cases} 1 + \dfrac{h_j^2}{12}\sigma(5+4\sigma), j=N-2 \\ 1 - \dfrac{h_j(1+2\sigma)a_j}{6} + \dfrac{h_j^2}{12}\big[-\sigma(1+2\sigma)a_{x_j} + \sigma(5+4\sigma)b_j\big] \\ +O(h_j^3), j=N-1 \end{cases} \tag{72}$$

Let

$$0 < K_{min} \leq min(K_1, K_2, K_3, K_4) \leq K_{max}. \tag{73}$$

Using (70)-(72) and for sufficiently small h_j, we can easily prove that D is Monotone. Therefore, D^{-1} exist and $D^{-1} \geq 0$. Hence by (67) we have,

$$\|E\| = \|D^{-1}\|\|T\|. \tag{74}$$

Now, for sufficiently small h_j, by (70) – (73) we can say that:

$$sum_{row\,j} > \begin{cases} \frac{h_j^2}{12}\sigma(4+5\sigma), & j=1 \\ \frac{h_j^2}{12}\sigma(4+5\sigma)K_{min}, j=2 \end{cases} \tag{75}$$

$$sum_{row\,j} > \begin{cases} \frac{h_j^2}{2}S_j,\ j=3,5...N-4 \\ \frac{h_j^2}{2}S_jK_{min}, & j=4,6...N-3 \end{cases} \tag{76}$$

$$sum_{row\,j} > \begin{cases} \frac{h_j^2}{12}\sigma(5+4\sigma), & j=N-2 \\ \frac{h_j^2}{12}\sigma(5+4\sigma)K_{min}, j=N-1 \end{cases}. \tag{77}$$

Since $\sigma > 0$ we can say that:

$$\begin{aligned} sum_{row\,j} &> \max[\frac{h_j^2}{12}\sigma(4+5\sigma) \quad , \frac{h_j^2}{12}\sigma(4+5\sigma)K_{min}] \\ &= \frac{h_j^2}{12}\sigma(4+5\sigma)K_{min}, \text{for} \quad j=1,2 \end{aligned} \tag{78}$$

$$sum_{row\,j} > \max[\frac{h_j^2}{2}S_j \quad , \frac{h_j^2}{2}K_{min}S_j] = \frac{h_j^2}{2}S_jK_{min}, \quad \text{for } j=3,4...,N-3 \tag{79}$$

$$\begin{aligned} sum_{row\,j} &> \max[\frac{h_j^2}{12}\sigma(5+4\sigma) \quad , \frac{h_j^2}{12}\sigma(5+4\sigma)K_{min}] \\ &= \frac{h_j^2}{12}\sigma(5+4\sigma)K_{min}, \text{for} \quad j=N-2,N-1. \end{aligned} \tag{80}$$

Let ${D_{i,j}}^{-1}$ be the $(i,j)^{th}$ element of D^{-1}, then by theory of matrices [21] for $i = 1(1)N-1$

$${D_{i,j}}^{-1} \le \frac{1}{sum_{row\,j}}. \tag{81}$$

By using (78)-(80), we have

$$\frac{1}{sum_{row\,j}} \le \begin{cases} \frac{12}{h_j^2\sigma(4+5\sigma)K_{min}}, & j=1,2 \\ \frac{2}{h_j^2S_jK_{min}}, & j=3,4,5...,N-3 \\ \frac{12}{h_j^2\sigma(5+4\sigma)K_{min}}, & j=N-2,N-1 \end{cases}. \tag{82}$$

Now let us define

$$\parallel {D_{i,j}}^{-1} \parallel = \max_{1\leq i\leq N-1} \sum_{j=1}^{N-1} \mid {D_{i,j}}^{-1} \mid \textit{ and } \parallel T_j \parallel = \max_{1\leq j\leq N-1} \sum_{j=1}^{N-1} \mid T_j^2 \mid . \tag{83}$$

Therefore,using (67) and (81) − (83) we get,

$$\parallel E \parallel \leq \frac{12}{h_j^2 K_{min}\sigma}\left(\frac{1}{4+5\sigma} + \frac{1}{5+4\sigma} + \frac{1}{6(1+\sigma)}\right)O(h_j^4) = O(h_j^2). \tag{84}$$

Hence, the second order convergence of scheme (41) for boundary value problems of the type (45) − (46) follows. Therefore, without loss of generality we can say that finite difference scheme (41) has second order convergence for boundary value problems (1)-(2). Similarly, we can prove the third order convergence of the difference scheme (42).

Theorem 1. The scheme (41) for the numerical solution of system of nonlinear singular boundary value problem (1) − (2) with sufficiently small $h_j, 0 < \sigma \neq 1$ has second order convergence under appropriate conditions.

5.1 Convergence Analysis of Fourth Order Method

Let us consider a fourth order singular boundary value problem of the following type:

$$u_{xxxx}(x) = F(x, u(x), u_x(x), u_{xx}(x), u_{xxx}(x)), 0 \leq x \leq 1 \tag{85}$$

subject to boundary conditions:

$$u(0) = \alpha_1, u_{xx}(0) = \alpha_2, u(1) = \beta_1, u_{xx}(1) = \beta_2 \tag{86}$$

where $F = a(x)u(x) + g(x)$ and $a(x)$ is singular at $x = 0$. Using (1), we write the problem (85) − (86) as follows:

$$u_{xx}(x) = v(x), \tag{87}$$

$$v_{xx}(x) = a(x)u(x) + g(x), \tag{88}$$

subject to

$$u(0) = \alpha_1, v(0) = \alpha_2, u(1) = \beta_1, v(1) = \beta_2. \tag{89}$$

Next,we convert the approximations (30) $-$ (40) and variable mesh scheme (42) by putting $\sigma = 1$ into uniform mesh:

$$\bar{u}^{(i)}_{j+\frac{1}{2}} = \frac{u^{(i)}_{j+1} + u^{(i)}_{j}}{2} \tag{90}$$

$$\bar{u}^{(i)}_{j-\frac{1}{2}} = \frac{u^{(i)}_{j-1} + u^{(i)}_{j}}{2} \tag{91}$$

$$\bar{u}^{(i)}_{x_{j+\frac{1}{2}}} = \frac{u^{(i)}_{j+1} - u^{(i)}_{j}}{h} \tag{92}$$

$$\bar{u}^{(i)}_{x_{j-\frac{1}{2}}} = \frac{u^{(i)}_{j} - u^{(i)}_{j-1}}{h} \tag{93}$$

$$\bar{u}^{(i)}_{x_j} = \frac{u^{(i)}_{j+1} - u^{(i)}_{j-1}}{2h} \tag{94}$$

$$\begin{aligned}\bar{F}^{(i)}_{j+\frac{1}{2}} &= f^{(i)}(x_{j+\frac{1}{2}}, \bar{u}^{(1)}_{j+\frac{1}{2}}, \bar{u}^{(2)}_{j+\frac{1}{2}}, ..., \bar{u}^{(i)}_{j+\frac{1}{2}}, ..., \bar{u}^{(M)}_{j+\frac{1}{2}}, \\ &\qquad \bar{u}^{(1)}_{x_{j+\frac{1}{2}}}, \bar{u}^{(2)}_{x_{j+\frac{1}{2}}}, ..., \bar{u}^{(i)}_{x_{j+\frac{1}{2}}}, ..., \bar{u}^{(M)}_{x_{j+\frac{1}{2}}})\end{aligned} \tag{95}$$

$$\begin{aligned}\bar{F}^{(i)}_{j-\frac{1}{2}} &= f^{(i)}(x_{j-\frac{1}{2}}, \bar{u}^{(1)}_{j-\frac{1}{2}}, \bar{u}^{(2)}_{j-\frac{1}{2}}, ..., \bar{u}^{(i)}_{j-\frac{1}{2}}, ..., \bar{u}^{(M)}_{j-\frac{1}{2}}, \\ &\qquad \bar{u}^{(1)}_{x_{j-\frac{1}{2}}}, \bar{u}^{(2)}_{x_{j-\frac{1}{2}}}, ..., \bar{u}^{(i)}_{x_{j-\frac{1}{2}}}, ..., \bar{u}^{(M)}_{x_{j-\frac{1}{2}}})\end{aligned} \tag{96}$$

$$\begin{aligned}\bar{F}^{(i)}_{j} &= f^{(i)}(x_j, u^{(1)}_{j}, u^{(2)}_{j}, ..., u^{(i)}_{j}, ..., u^{(M)}_{j}, \\ &\qquad \bar{u}^{(1)}_{x_j}, \bar{u}^{(2)}_{x_j}, ..., \bar{u}^{(i)}_{x_j}, ..., \bar{u}^{(M)}_{x_j})\end{aligned} \tag{97}$$

$$\hat{u}^{(i)}_{j} = u^{(i)}_{j} - \frac{h^2}{8}(\bar{F}^{(i)}_{j+\frac{1}{2}} + \bar{F}^{(i)}_{j-\frac{1}{2}}) \tag{98}$$

$$\hat{u}^{(i)}_{x_j} = \bar{u}^{(i)}_{x_j} - \frac{h}{4}(\bar{F}^{(i)}_{j+\frac{1}{2}} - \bar{F}^{(i)}_{j-\frac{1}{2}}) \tag{99}$$

$$\begin{aligned}\hat{F}^{(i)}_{j} &= f^{(i)}(x_j, \hat{u}^{(1)}_{j}, \hat{u}^{(2)}_{j}, ..., \hat{u}^{(i)}_{j}, ..., \hat{u}^{(M)}_{j}, \\ &\qquad \hat{u}^{(1)}_{x_j}, \hat{u}^{(2)}_{x_j}, ..., \hat{u}^{(i)}_{x_j}, ..., \hat{u}^{(M)}_{x_j})\end{aligned} \tag{100}$$

$$u^{(i)}_{j+1} - 2u^{(i)}_{j} + u^{(i)}_{j-1} = \frac{h^2}{3}(\bar{F}^{(i)}_{j+\frac{1}{2}} + \hat{F}^{(i)}_{j} + \bar{F}^{(i)}_{j-\frac{1}{2}}). \tag{101}$$

Now, we use the uniform mesh scheme(101) to solve the coupled second order boundary value problem (87)-(89) and obtain the following difference scheme:

$$u_{j-1} - 2u_j + u_{j+1} = \frac{h^2}{3}(\bar{v}_{j+\frac{1}{2}} + \hat{v}_j + \bar{v}_{j-\frac{1}{2}}) \tag{102}$$

$$\begin{aligned}v_{j-1} - 2v_j + v_{j+1} &= \frac{h^2}{3}\big[(a_{j+\frac{1}{2}}\bar{u}_{j+\frac{1}{2}} + g_{j+\frac{1}{2}}) \\ &+ (a_j\hat{u}_j + g_j) + (a_{j-\frac{1}{2}}\bar{u}_{j-\frac{1}{2}} + g_{j-\frac{1}{2}})\big].\end{aligned} \tag{103}$$

Also, we define the following relations for $a_{j\pm\frac{1}{2}}$ in the coupled finite difference scheme (102)-(103)

$$a^*_{j-\frac{1}{2}} = a_j - \frac{h}{2}a_{x_j} + \frac{h^2}{8}a_{xx_j} + O(h^3), \tag{104}$$

$$a^*_{j+\frac{1}{2}} = a_j + \frac{h}{2}a_{x_j} + \frac{h^2}{8}a_{xx_j} + O(h^3). \tag{105}$$

Similar relations for $g_{j\pm\frac{1}{2}}$can also be defined. Using the relations (104) – (105) in (102) – (103) we get,

$$(106)\qquad u_{j-1} - 2u_j + u_{j+1} = \frac{h^2}{3}(\bar{v}_{j+\frac{1}{2}} + \hat{v}_j + \bar{v}_{j-\frac{1}{2}})$$

$$\begin{aligned} v_{j-1} - 2v_j + v_{j+1} &= \frac{h^2}{3}\big[(a^*_{j+\frac{1}{2}}\bar{u}_{j+\frac{1}{2}} + g_{j+\frac{1}{2}}) \\ (107)\qquad &+ (a_j\hat{u}_j + g_j) + (a^*_{j-\frac{1}{2}}\bar{u}_{j-\frac{1}{2}} + g_{j-\frac{1}{2}})\big]. \end{aligned}$$

Finally, we simplify the difference scheme (106) – (107) upto order $O(h^5)$ terms and obtain the analogue of vector difference equation (58). The components of the vector difference equation are as follows

$$(108)\qquad \begin{cases} sub^{11}_j = -1 - h^4\frac{a_j}{48} + h^5\frac{a_{x_j}}{96}, \quad sub^{12}_j = \frac{h^2}{6}, \\ sub^{21}_j = h^2\frac{a_j}{6} - h^3\frac{a_{x_j}}{12} + h^4\frac{a_{xx_j}}{48}, \quad sub^{22}_j = -1 - h^4\frac{a_j}{48}. \end{cases}$$

$$(109)\qquad \begin{cases} diag^{11}_j = 2 - h^4\frac{a_j}{24}, \quad diag^{12}_j = \frac{2}{3}h^2, \\ diag^{21}_j = h^2\frac{2a_j}{3} + \frac{h^4 a_{xx_j}}{24}, \quad diag^{22}_j = 2 - h^4\frac{a_j}{24} \end{cases}$$

$$(110)\qquad \begin{cases} sup^{11}_j = -1 - h^4\frac{a_j}{48} - h^5\frac{a_{x_j}}{96}, \quad sup^{12}_j = \frac{h^2}{6}, \\ sup^{21}_j = h^2\frac{a_j}{6} - h^3\frac{a_{x_j}}{12} + h^4\frac{a_{xx_j}}{48}, \quad sup^{22}_j = -1 - h^4\frac{a_j}{48}. \end{cases}$$

$$(111)\qquad \begin{cases} \phi^1_j = -2h^4 g_j, \qquad \phi^2_j = -h^2\big(g_j + \frac{h^2}{12}g_{xx_j}\big). \end{cases}$$

Once we incorporate the boundary conditions in the vector difference equation (58) we get the matrix form (65) and as done in section 5 we similarly obtain the error equation (67). Further,let $|a_j| \le K_1, |a_{x_j}| \le K_2, |a_{xx_j}| \le K_3$ for some positive constant K_1, K_2, K_3. Now, using (108) and (110) we get,

$$(112)\qquad \|sup_j\|_\infty \le \max_{1\le j\le N-2} \begin{cases} 1 + \frac{h^2}{6} + h^4\frac{K_1}{48} + h^5\frac{K_2}{96}, \\ 1 + h^2\frac{K_1}{6} + h^3\frac{K_2}{12} + h^4\frac{K_1+K_3}{48} \end{cases}$$

$$(113)\qquad \|sub_j\|_\infty \le \max_{2\le j\le N-1} \begin{cases} 1 + \frac{h^2}{6} + h^4\frac{K_1}{48} + h^5\frac{K_2}{96}, \\ 1 + h^2\frac{K_1}{6} + h^3\frac{K_2}{12} + h^4\frac{K_1+K_3}{48} \end{cases} .$$

Thus for sufficiently small h, we get $\|sub_j\|_\infty \leq \sigma$ and $\|sup_j\|_\infty \leq 1$. Hence, D is irreducible.

Let $sum_{row\, j}$ be the sum of elements of row_j of D

$$(114)\quad sum_{row\, j} = \begin{cases} 1 + h^2\dfrac{5}{6} - h^4\dfrac{a_j}{16} - h^5\dfrac{a_{x_j}}{96}, & j = 1 \\ 1 + h^2\dfrac{5a_j}{6} + h^3\dfrac{a_{x_j}}{12} + h^4(\dfrac{-a_j}{16} + \dfrac{3a_{xx_j}}{48}), & j = 2 \end{cases}$$

$$(115)\quad sum_{row\, j} = \begin{cases} h^2 - h^4\dfrac{a_j}{12}, & j = 3, 5..N-4 \\ h^2 a_j + h^4(\dfrac{a_{xx_j}}{12} - \dfrac{a_j}{12}), & j = 4, 6..N-3 \end{cases}$$

$$(116)\quad sum_{row\, j} = \begin{cases} 1 + h^2\dfrac{5}{6} - h^4\dfrac{a_j}{16} + h^5\dfrac{a_{x_j}}{96}, & j = N-2 \\ 1 + h^2\dfrac{5a_j}{6} - h^3\dfrac{a_{x_j}}{12} - h^4(\dfrac{-a_j}{16} + \dfrac{3a_{xx_j}}{48}), & j = N-1 \end{cases}.$$

Let

$$(117)\quad 0 < K_{min} \leq min(K_1, K_2, K_3) \leq K_{max}.$$

Using (114) – (117) and for sufficiently small h, we can easily prove that D is Monotone. Therefore, D^{-1} exist and $D^{-1} \geq 0$. Hence by (67) we have,

$$(118)\quad \|E\| = \|D^{-1}\|\|T\|,$$

where $T = O(h^6)$ as discussed in section 2. Now for sufficiently small h, by (114) – (116) we can say that:

$$(119)\quad sum_{row\, j} > \begin{cases} h^2\dfrac{5}{6}, & j = 1 \\ h^2\dfrac{5}{6}K_{min}, j = 2 \end{cases}$$

$$(120)\quad sum_{row\, j} \geq \begin{cases} h^2, & j = 3, 5...N-4 \\ h^2 K_{min}, & j = 4, 6...N-3 \end{cases}$$

$$(121)\quad sum_{row\, j} > \begin{cases} h^2\dfrac{5}{6}, & j = N-2 \\ h^2\dfrac{5}{6}K_{min}, j = N-1 \end{cases}.$$

Since $\sigma > 0$ we can say that:

$$(122)\quad sum_{row\, j} > \max[h^2\frac{5}{6} \quad , h^2\frac{5}{6}K_{min}] = h^2\frac{5}{6}K_{min}, \text{for} \quad j = 1, 2$$

$$(123)\quad sum_{row\, j} \geq \max[h^2 \quad , h^2 K_{min}] \; = h^2 K_{min}, \text{ for } j = 3, 4..., N-3$$

$$sum_{row\, j} > \max[h^2\frac{5}{6} \quad , h^2\frac{5}{6}K_{min}]$$

$$(124)\quad = h^2\frac{5}{6}K_{min}, \text{for} \quad j = N-2, N-1.$$

Let ${D_{i,j}}^{-1}$ be the $(i,j)^{th}$ element of D^{-1}, then as discussed in section 5,

$$ {D_{i,j}}^{-1} \leq \frac{1}{sum_{row\,j}}. \tag{125} $$

By using (122)-(124), we have

$$ \frac{1}{sum_{row\,j}} \leq \begin{cases} \dfrac{6}{5h^2K_{min}}, & j = 1,2 \\ \dfrac{1}{h^2K_{min}}, & j = 3,4,5...,N-3 \\ \dfrac{6}{5h^2K_{min}}, & j = N-2, N-1 \end{cases}. \tag{126} $$

Now let us define,

$$ \| {D_{i,j}}^{-1} \| = \max_{1\leq i\leq N-1} \sum_{j=1}^{N-1} | {D_{i,j}}^{-1} | \tag{127} $$

Therefore,using (67),(125) - (127) we get,

$$ \| E \| \leq \frac{6}{h^2K_{min}}(\frac{2}{5}+\frac{1}{6})O(h^6) = \frac{17}{5K_{min}}O(h^4). \tag{128} $$

Hence, the fourth order convergence of uniform mesh scheme for system of boundary value problems of the type $(87)-(89)$ follows. Therefore, without loss of generality we can say that fourth order convergence follows for system of boundary value problems of the type $(1)-(2)$.

6. Numerical illustration

To illustrate the proposed methods, we solve following six problems. The root mean square (RMS) errors in case of variable mesh and maximum absolute(MA) error for uniform mesh are tabulated in the Tables 1-6. For simplicity, we have chosen $\sigma_j = \sigma$ and hence $h_1 = \frac{(\sigma-1)}{(\sigma^{N-1}-1)}, \sigma \neq 1$. Therefore, the rest of the h_j 's can be obtained as $h_{j+1} = \sigma h_j, j = 1(1)N-1$ .In case of the presence of boundary layer near the left or right end of the domain, take $h_1 = \begin{cases} \frac{\sigma-1}{\sigma^N-1}, \sigma > 1 \\ \frac{1-\sigma}{1-\sigma^N}, \sigma < 1 \end{cases}$. This ensures the mesh points in the boundary layer region near the left or right end of the interval. The linear system of difference equations have been solved by Block Gauss Elimination method and the nonlinear system of difference equations by Block Newton's method in which we have considered $u_0 = 0$ as the initial approximation. The computational order of convergence (COC^*) is also given for fourth order uniform mesh method. All calculations have been done in Matlab 07. Also, in the following questions $u^{(i)}(x)$ means i^{th} order derivative of $u(x)$.

Example 5.1. We consider fourth order nonlinear boundary value problem [5]: $u^{(4)}(x) = 6\exp(-4\ u(x)) - \frac{12}{(1+x)^4}, 0 < x < 1,\ u(0) = 0, u^{(2)}(0) = -1,$ $u(1) = .6931, u^{(2)}(1) = -.25$. The exact solution is given by $u(x) = \log(1+x)$. The RMS errors for $\sigma = 0.9$ and MA error for $\sigma = 1$ are tabulated in Table 1.

Table 1: Example 5.1

	RMS error			MA error
N	$O(h_j{}^2)method$	$O(h_j{}^3)method$	Twizell[5]	$O(h^4)method$
8	5.2564e-05	4.8610e-06	.37e-05	7.2499e-07
16	6.5587e-05	1.2961e-06	.29e-06	4.6937e-08
32	6.0144e-05	6.7628e-07	.19e-07	2.9600e-09

Example 5.2. Consider a sixth order nonlinear boundary value problem ([7],[11]): $u^{(6)}(x) = \exp(-x)u(x), 0 < x < 1,\ u(0) = u^{(2)}(0) = u^{(4)}(0) = 1, u(1) = u^{(2)}(1) = u^{(4)}(1) = e$. The exact solution is given by $u(x) = \exp(x)$.The RMS errors for $\sigma = 0.9$ and MA error for $\sigma = 1$ are tabulated in Table 2.

Table 2: Example 5.2

	RMS error				MA Error
N	$O(h_j{}^2)$ method	$O(h_j{}^3)method$	Haq et. al.[7]	Inayat et.al.[11]	$O(h^4)method$
.1	2.5623e-04	7.5612e-08	-1.2e-04	1.1106e-07	6.7273e-09
.2	4.0612e-04	1.1260e-07	-2.3e-04	2.1138e-07	1.2324e-08
.3	4.7519e-04	1.2429e-07	-3.2e-04	2.9128e-07	1.6686e-08
.4	4.8429e-04	1.2000e-07	-3.8e-04	3.4229e-07	7.3657e-08
.5	4.5014e-04	1.0613e-07	-4.0e-04	3.6143e-07	2.1209e-08
.6	3.8597e-04	8.6985e-08	-3.9e-04	3.4461e-07	2.1072e-08
.7	3.0201e-04	6.5347e-08	-3.3e-04	2.9390e-07	1.9102e-08
.8	2.0606e-04	4.2992e-08	-2.4e-04	2.1404e-07	1.5094e-08
.9	1.0397e-04	2.1001e-08	-1.2e-04	1.1271e-07	8.8147e-09

Example 5.3. Consider fourth order linear boundary value problem of the form [15]: $u^{(4)}(x) - u(x) = -8x\cos(x) - 12\sin(x), 0 \le x \le 1,\ u(0) = u(1) = 0, u^{(2)}(0) = 0, u^{(2)}(1) = 2\sin(1) + 4\cos(1)$. The exact solution is given by $u(x) = (x^2 - 1)\sin(x)$. The RMS errors for a fixed value $\sigma = 0.9$ and MA error for $\sigma = 1$ are tabulated in Table 3.

Table 3: Example 5.3

	RMS error		MA error	
N	$O(h_j{}^2)method$	$O(h_j{}^3)method$	$O(h^4)method$	Ramadan[15]
8	6.4952e-04	8.1257e-06	7.8386e-07	3.010 e-05
16	5.9397e-04	2.1152e-06	4.9504e-08	1.8318 e-06
32	5.1433e-04	1.0688e-06	3.0919e-09	1.1179e-07

Example 5.4. We consider the sixth order linear boundary value problem ([9],[22]): $(\frac{d^6}{dx^6}+1)u(x)=6(2x\cos(x)+5\sin(x)), 0\le x\le 1,\ u(0)=u(1)=0, u^{(2)}(0)=0, u^{(2)}(1)=2\sin(1)+4\cos(1),\ u^{(4)}(0)=0, u^{(4)}(1)=-12\sin(1)-8\cos(1)$. The exact solution is given by $u(x)=(x^2-1)\sin(x)$.The RMS errors for a fixed value $\sigma=0.9$ and MA error for $\sigma=1$ are tabulated in Table 4.

Table 4: *Example 5.4*

	RMS error		MA error		
N	$O({h_j}^2)method$	$O({h_j}^3)method$	$O(h^4)method$	Akram et.al.[9]	Siddiqi et.al.[22]
8	6.4952e-04	4.2946e-06	6.5904e-07	1.5379 e-06	8.1514e-05
16	5.9397e-04	9.9183e-07	4.1834e-08	1.9790 e-07	2.1052 e-05
32	5.1433e-04	4.7874e-07	2.6137e-09	4.0596 e-08	5.3084 e-06

Example 5.5. Consider a fourth order nonlinear singular boundary value problem of the form: $(\frac{d^4}{dx^4}+\frac{4}{x}\frac{d^3}{dx^3})u=u^2+sin(x)-4\frac{\cos(x)}{x}, 0<x\le 1$. The exact solution is given by $u(x)=\sin(x)$.The boundary conditions are obtained from the exact solution by test procedure .The RMS errors for a fixed value $\sigma=0.9$ and MA error for $\sigma=1$ are tabulated in Table 5 .

Table 5: *Example 5.5*

	RMS error		MA error	
N	$O({h_j}^2)method$	$O({h_j}^3)method$	$O(h^4)method$	COC^*
8	2.1632e-04	5.1244e-06	1.8261e-06	-
16	1.3961e-04	1.3673e-06	1.2672e-07	3.8490
32	1.0763e-04	7.0981e-07	8.3028e-09	3.9320
64	1.0184e-04	6.2357e-07	5.3206e-10	3.9639

Example 5.6. We consider a sixth order nonlinear singular boundary value problem of the form: $(\frac{d^6}{dx^6}+\frac{6}{x}\frac{d^5}{dx^5}+2)u=e^u+6e^x(\frac{1+x}{x}), 0<x\le 1$. The exact solution is given by $u(x)=\exp(x)$.The boundary conditions are obtained from the exact solution by test procedure .The RMS errors for a fixed value $\sigma=0.9$ and MA error for $\sigma=1$ are tabulated in Table 6 .

Table 6: *Example 5.6*

	RMS error		MA error	
N	$O({h_j}^2)method$	$O({h_j}^3)method$	$O(h^4)method$	COC^*
8	6.1864e-04	6.4701e-07	5.7095e-07	-
16	3.1623e-04	1.6324e-07	4.5354e-08	3.654
32	2.2084e-04	8.3600e-08	3.4251e-09	3.727
64	2.0512e-04	7.3184e-08	2.5097e-10	3.770

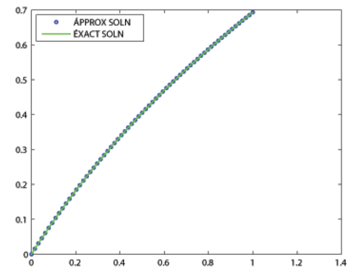

Figure 1: Graph of the exact solution $u(x) = \log(1 + x)$versus the approximate solution in fourth order uniform mesh method for N = 64 and $\sigma = 1$ for Example 5.1

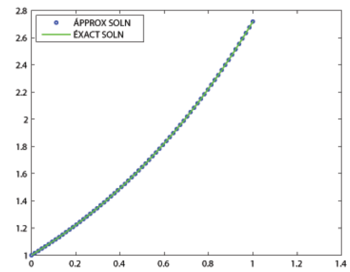

Figure 2: Graph of the exact solution $u(x) = \exp(x)$ versus the approximate solution in fourth order uniform mesh method for N = 64 and $\sigma = 1$ for Example 5.2

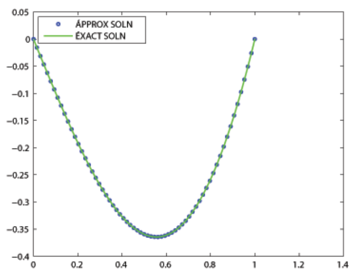

Figure 3: Graph of the exact solution $u(x) = (x^2 - 1)\sin(x)$ versus the approximate solution in fourth order uniform mesh method for N = 64 and $\sigma = 1$ for Example 5.3

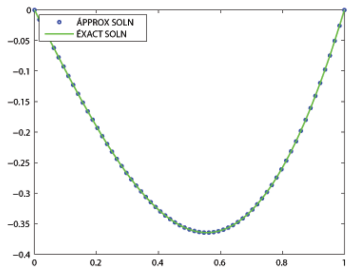

Figure 4: Graph of the exact solution $u(x) = (x^2 - 1)\sin(x)$ versus the approximate solution in fourth order uniform mesh method for N = 64 and $\sigma = 1$ for Example 5.4

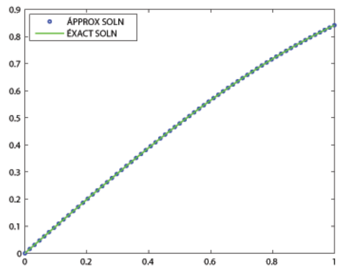

Figure 5: Graph of the exact solution $u(x) = \sin(x)$ versus the approximate solution in fourth order uniform mesh method for N = 64 and $\sigma = 1$ for Example 5.5

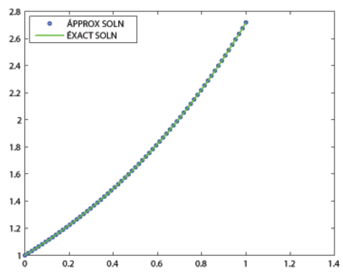

Figure 6: Graph of the exact solution $u(x) = \exp(x)$ versus the approximate solution in fourth order uniform mesh method for N = 64 and $\sigma = 1$ for Example 5.6

7. Conclusion

We derived second and third order variable mesh schemes using off-step points for solving linear and nonlinear even order boundary value problems. Although in this paper, only fourth order and sixth order nonlinear and linear boundary value problems are considered, but the method is general enough to be implemented in case of higher even order linear and nonlinear boundary value problems. Table $1-4$ shows presence of refinement in results when compared with other nonlinear and linear boundary value problems solved by using extrapolation, collocation method using Haar wavelets,iterative method and non polynomial splines. Computationally our methods seems to be more viable due to usage of only three grid points at a time which leads to solving of a tri-diagonal matrix. Also, in the end we have solved fourth and sixth order nonlinear singular boundary value problems. Due to the usage of off-step points presence of singularity is overcomed. As per the literature available, such class of boundary value problems has not been solved so far. Therefore, due to unavailability of any prior results we were unable to present a comparative study. Hence we have compared our own results in Table 5 and 6. We have also provided the computational order of convergence (COC^*) for the uniform mesh method. Our methods are applicable to problems in cartesian as well as polar coordinates and even higher order singularly perturbed boundary value problems can be solved easily due to usage of variable mesh.

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