ON PROPERTIES OF VARIOUS MORPHISMS IN THE CATEGORIES OF GENERAL KRASNER HYPERMODULES

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Abstract. In a recent paper entitled *Pre-semihyperadditive Categories*, we introduced some categories in which for objects A and B, the class of all morphisms from A to B denoted by $Mor(A, B)$, admits an algebraic hyperstructures such as semihypergroup or hypergroup. Then after defining and fixing a general Krasner hyperring R, we introduced and studied the categories of general Krasner R-hypermodules, $_R\mathcal{G}.\mathbf{mod}$, $_{R_s}\mathcal{G}.\mathbf{mod}$, $_{R_w}\mathcal{G}.\mathbf{mod}$ and etc. In this paper we present some properties of multi-valued homomorphisms as morphisms of these categories and study various concepts related to these morphisms in connection with the fundamental relation of their domain or codomain.
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1. Introduction

Hypergroup as a generalization of group in Algebra was introduced by Marty during the 8th Congress of the Scandinavian Mathematicians [15] in 1934. This concept has resulted in a new branch of mathematics science named Hyperstructures Theory. This theory as a new field of modern Algebra has been developed in the view points of theory and applications by many researchers, see e.g. [2, 8, 9, 10, 11, 12]. Among others, Ameri [1], Corsini [3], Corsini and Leoreanu [4], Cristea et al. [5,6,7], Novák [16,17], Massouros [13,14], Vougiouklis

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[20], studied hypercompositional structures defined in terms of binary relations. In [19], the authors described the interaction between hyperstructure theory and category theory. As defined in [19], in a pre-semihyperadditive category $\mathcal{C}$ for two objects A and B the class of morphisms from A into B denoted by $Mor_{\mathcal{C}}(A, B)$ is at least a semihypergroup.
As two examples of such categories, letting R be a general Krasner hyperring, we introduced the category ${}_R\mathcal{G}.\mathbf{mod}$ (resp., ${}_R\mathbf{G}.\mathbf{mod}$) consisting of general Krasner hypermodules as objects and $R-mv$-homomorphism (resp., R-homomorphism) as morphisms. Every morphism of ${}_R\mathbf{G}.\mathbf{mod}$ is a function that satisfies some conditions and can be considered as a special morphism of ${}_R\mathcal{G}.\mathbf{mod}$. Behavior of a morphism of ${}_R\mathcal{G}.\mathbf{mod}$ differs from that of one (usual) function and we naturally do not expect to be the same. For example, an $R-mv$-homomorphism has various types of injectivity and monicness. In this regards we introduce some kinds of these concepts and obtain some results explaining the related properties of a morphism of ${}_R\mathcal{G}.\mathbf{mod}$.

2. Preliminary

Let H be a nonempty set and $P^*(H)$ denotes the set of all nonempty subsets of H. A hyperoperation $\cdot$ on H is a map $\cdot : H \times H \longrightarrow P^*(H)$ sending (a, b) to $a \cdot b \subseteq H$. In this case $(H, \cdot)$ is called a *hypergroupoid*. The hyperoperation $\cdot$ is extended to subsets of H in a natural way. Indeed, $A \cdot B$ or AB is given by

$$A \cdot B = \bigcup_{(a,b)\in A\times B} a \cdot b. \tag{2 .1}$$

A hypergroupoid $(H, \cdot)$ is said to be a semihypergroup if $\cdot$ is associative. A hypergroupoid $(H, \cdot)$ is a hypergroup if it is a semihypergroup satisfying (*reproductivity* property) $Hx = xH = H$, for every $x \in H$. A semihypergroup or hypergroup H is called commutative if $xy = yx$ for every $x, y \in H$. An element y of semihypergroup $(H, +)$ is called identity if for all $x \in H$, $y \in x + y \cap y + x$.

Let x be an element of semihypergroup $(H, +)$ (resp., $(H, \cdot)$) such that $x+y = y$ (resp., $x \cdot y = y$). Then x is called a left scalar identity (resp., unit). Similarly, a right scalar identity (resp., unit) is defined with the affection on the right.

An element x of semihypergroup $(H, +)$ (resp., $(H, \cdot)$) is called a scalar identity (resp., unit) if it is a left and right scalar identity (resp., unit). We denote the scalar identity (resp., unit) of H by 0_H (resp., 1_H). Every scalar identity or scalar unit in a semihypergroup H is unique.

Definition 2.1. A nonempty set H together with the hyperoperation "+" is called a *canonical hypergroup* if the following axioms hold:

1. $(H, +)$ is a commutative semihypergroup,

2. there is a scalar identity 0_H,

3. for every $x \in H$, there is a unique element denoted by $-x$ such that $0_H \in x + (-x)$ which for simplicity we write $0_H \in x - x$,

4. $x \in y + z$ implies $y \in x - z$ (and thus $z \in -y + x$).

Definition 2.2 ([12])**.** A non-empty set R together with the hyperoperation $+$ and the operation $\cdot$ is called a *Krasner hyperring* if the following axioms hold:

1. $(R, +)$ is a canonical hypergroup;

2. $(R, \cdot)$ is a semigroup including 0_R as a bilaterally absorbing element, that is $0_R \cdot x = x \cdot 0_R = 0_R$ for all $x \in R$;

3. $(y + z) \cdot x = (y \cdot x) + (z \cdot x)$ and $x \cdot (y + z) = x \cdot y + x \cdot z$ for all $x, y, z \in R$.

Definition 2.3 ([18])**.** Let $(R, +, \cdot)$ be a hyperring. A canonical hypergroup $(A, +)$ together with left external multiplication $* : R \times A \longrightarrow A$ is called a *left Krasner hypermodule* over R if for all $r_1, r_2 \in R$ and for all $a_1, a_2 \in A$ the following axioms hold:

1. $r_1 * (a_1 + a_2) := \bigcup_{b \in a_1 + a_2} r_1 * b = r_1 * a_1 + r_1 * a_2$;

2. $(r_1 + r_2) * a_1 := \bigcup_{r \in r_1 + r_2} r * a_1 = r_1 * a_1 + r_2 * a_1$;

3. $(r_1 \cdot r_2) * a_1 = r_1 * (r_2 * a_1)$;

4. $0_R * a_1 = 0_A$.

Definition 2.4 ([19])**.** A nonempty set R together with two hyperoperations $+$ and $\cdot$ is called a *general Krasner hyperring* if the following axioms hold:

1. $(R, +)$ is a canonical hypergroup (with scalar identity 0_R),

2. $(R, \cdot)$ is a semihypergroup including 0_R as a bilaterally absorbing element, i.e., $0_R \cdot a = a \cdot 0_R = 0_R$ for all $a \in A$,

3. $(y + z) \cdot x \subseteq (y \cdot x) + (z \cdot x)$ and $x \cdot (y + z) \subseteq x \cdot y + x \cdot z$ for all $x, y, z \in R$.

We say a general Krasner hyperring $(R, +, \cdot)$ has the scalar unit 1_R if $1_R \cdot r = r \cdot 1_R = r$ for all $r \in R$.

Definition 2.5 ([19])**.** Let R be a general Krasner hyperring. A nonempty set A is called a *left general Krasner hypermodule over* R, for short a left general Krasner R-hypermodule, if $(A, +)$ is a canonical hypergroup together with the map $* : R \times A \longrightarrow P^*(A)$ satisfying the following axioms for all $r_1, r_2 \in R$ and $a_1, a_2 \in A$:

1. $r_1 * (a_1 + a_2) := \bigcup_{b \in a_1 + a_2} r_1 * b \subseteq r_1 * a_1 + r_1 * a_2$;

2. $(r_1 + r_2) * a_1 := \bigcup_{r \in r_1 + r_2} r * a_1 \subseteq r_1 * a_1 + r_2 * a_1$;

3. $(r_1 \cdot r_2) * a_1 \subseteq r_1 * (r_2 * a_1)$ in which $(r_1 \cdot r_2) * a_1 := \bigcup_{r \in r_1 \cdot r_2} r * a_1$ and $r_1 * (r_2 * a_1) := \bigcup_{a \in r_2 * a_1} r * a$;

4. $0_R * a_1 = 0_A$.

A left general Krasner R-hypermodule A is called unitary if R has the scalar unit 1_R with $1_R * a = a$ for all $a \in A$. Every general Krasner hyperring R with 1_R is a unitary left general Krasner R-hypermodule.
In an obvious way, one can consider the external multiplication map $*\colon A \times R \longrightarrow P^*(A)$ to define the right general Krasner R-hypermodule. From now on, R denotes a general Krasner hyperring. Also, for convenience, by a hyperring R and an R-hypermodule we mean a general Krasner hyperring and a left general Krasner R-hypermodule, respectively.
In order to have a category whose objects are the class of all R-hypermodules, we need morphisms. For this, we start with the following concept.

Definition 2.6. For two R-hypermodules A and B, let f be a function from A into $P^*(B)$, that is a multi-valued function from A to B, that satisfies the conditions

1. $f(x+y) \subseteq f(x) + f(y)$,

2. $f(r * x) \subseteq r * f(x)$,

for all $r \in R$ and all $x, y \in A$. In this case, f is said to be a *multi-valued R-homomorphism*, for short we write *$R-mv$-homomorphism* from A to B. Sometimes an $R-mv$-homomorphism is called an *inclusion* $R-mv$-homomorphism.

Note that $+$ in Definition 2.6 is given by (2 .1). If f satisfies the conditions

1. $f(x+y) = f(x) + f(y)$,

2. $f(r * x) = r * f(x)$,

for all $r \in R$ and all $x, y \in A$, then it is called a *strong $R-mv$-homomorphism*. Also, if f satisfies the conditions

1. $f(x+y) \cap [f(x) + f(y)] \neq \emptyset$,

2. $f(r * x) \cap r * f(x) \neq \emptyset$,

for all $r \in R$ and all $x, y \in A$, then f is said to be a weak $R-mv$-homomorphism. The class of all $R-mv$-homomorphisms, strong $R-mv$-homomorphisms and weak $R-mv$-homomorphisms from A to B as morphisms from A to B is denoted by $Hom_R(A,B)$, $Hom_R^s(A,B)$ and $Hom_R^w(A,B)$, respectively.

Let $f \in Hom_R(A,B)$ and $g \in Hom_R(B,C)$. Define the composition $g \circ f$ as:

$$(g \circ f)(a) = \bigcup_{b \in f(a)} g(b), \quad \forall a \in A. \tag{2 .2}$$

Following [19], ${}_R\mathcal{G}.\mathbf{mod}$, ${}_{R_s}\mathcal{G}.\mathbf{mod}$ and ${}_{R_w}\mathcal{G}.\mathbf{mod}$ denote the categories formed by the class of all R-hypermodules together with the class of all R-*mv*-homomorphisms, strong R-*mv*-homomorphisms and weak $R-mv$-homomorphisms, respectively, with the composition of morphisms as (2 .2).

One can consider a function f from A into B satisfying two conditions in Definition 2.6 as a morphism. We call such morphism an (inclusion) R-homomorphism. Similarly, we can define strong R-homomorphisms and weak R-homomorphisms from A to B. We use $hom_R(A,B)$, $\hom^s_R(A,B)$ and $hom^w_R(A,B)$ for the class of (inclusion) R-homomorphisms, strong R-homomorphisms and weak R-homomorphisms from A to B, respectively. Also, we denote the corresponding categories by ${}_R\mathbf{G}.\mathbf{mod}$, ${}_{R_s}\mathbf{G}.\mathbf{mod}$ and ${}_{R_w}\mathbf{G}.\mathbf{mod}$, respectively. Any singleton set is identified with its element. Thus we may write $f(a)=b$ instead of $f(a)=\{b\}$. Therefore, any single-valued $f\in Hom_R(A,B)$ is an element of $hom_R(A,B)$, and conversely, any element of $hom_R(A,B)$ is a single-valued element of $Hom_R(A,B)$.

Remark 2.7. It is necessary to emphasize that according to [18], the external multiplication $*$ in a Krasner hypermodule over a Krasner hyperring R is single-valued and for every morphism between two Krasner hypermodules such as $f: A\longrightarrow P^*(B)$ and $f: A\longrightarrow B$, we have $f(r*x)=r*f(x)$ for every $x\in A$ and $r\in R$, while in the categories of general Krasner hypermodules over a general Krasner hyperring R, $*$ is a multi-valued map and for an $R-mv$-homomorphism f, we have $f(r*x)\subseteq r*f(x)$ for every $x\in A$ and $r\in R$.

In the sequel, we write ra instead of $r*a$ if there is no confusion.
For more details about hyperrings and hypermodules, see e.g. [1, 11, 13, 14]. For some concepts relatrd to category theory the reader can refer to [3].

3. Some properties of morphisms of ${}_R\mathcal{G}.\mathbf{mod}$

From now on, fixing a general Krasner hyperring $(R,+,\cdot)$ and following [20], let Γ^* be the smallest equivalence relation such that $(R/\Gamma^*,\oplus,\otimes)$ is a ring with

$$\forall x,y\in R:\quad \Gamma^*(x)\oplus\Gamma^*(y)=\Gamma^*(z)\quad \forall z\in\Gamma^*(x)+\Gamma^*(y),$$
$$\forall x,y\in R:\quad \Gamma^*(x)\otimes\Gamma^*(y)=\Gamma^*(z)\quad \forall z\in\Gamma^*(x)\cdot\Gamma^*(y).$$

Also, let A be a general Krasner R-hypermodule and ϵ^*_A be the smallest equivalence relation such that firstly, $(A/\epsilon^*_A,\oplus)$ is a (commutative) group with

$$\forall x,y\in R:\quad \Gamma^*(x)\oplus\Gamma^*(y)=\Gamma^*(z)\quad \forall z\in\Gamma^*(x)+\Gamma^*(y),$$

and secondly, A/ϵ^*_A is an R/Γ^*-module with the external multiplication

$$*': R/\Gamma^* *' A/\epsilon^*_A\longrightarrow A/\epsilon^*_A,\ \ \Gamma^*(r)*'\epsilon^*_A(a)=\epsilon^*_A(z)\quad \forall z\in\Gamma^*(r)*\epsilon^*_A(a)$$

for all $(r,a)\in R\times A$. Here, Γ^*, ϵ^*_A, R/Γ^* and A/ϵ^*_A are called *fundamental relation of* R, *fundamental relation of* A, *fundamental ring of* R and *fundamental module of* A, respectively.

In this section, we study some properties of an arbitrary $R-mv$-homomorphism $f \in Hom_R(A,B)$ in connection with the equivalence relations ϵ_A^* and ϵ_B^*.

Definition 3.1. A nonempty subset $B \subseteq A$ is called an R-subhypermodule of the R-hypermodule A if B is an R-hypermodule itself.

Let $f \in Hom_R(A,B)$. Set

$$Ker(f) := \{x \in A \mid f(x) = 0_B\},$$
$$K_f := \{x \in A \mid 0_B \in f(x)\},$$
$$Im(f) := \{y \in B \mid \exists x \in A \quad y \in f(x)\}.$$

Remark 3.2. Clearly K_f and $Im(f)$ are not necessarily R-subhypermodules of A and B, respectively. It is easy to see that the set $Ker(f)$ is always an R-subhypermodule of A.

Example 3.3. Clearly every Krasner hyperring R is a Krasner R-hypermodule. Also, every Krasner hypermodule is a general Krasner hypermodule. Consider the set of nonnegative real numbers denoted by $\mathbb{R}^{+0}$. It is easy to verify that $\mathbb{R}^{+0}$ under the hyperoperation $+$ defined by $x+y = max\{x,y\}$ if $x \neq y$, and $[0,x]$ if $x=y$, is a canonical hypergroup. Considering the usual multiplication $*\colon \mathbb{R}^{+0} \times \mathbb{R}^{+0} \longrightarrow \mathbb{R}^{+0}$, we can check that $(\mathbb{R}^{+0},+,*)$ is a Krasner hyperring, and indeed $A := \mathbb{R}^{+0}$ is a Krasner A-hypermodule. Consequently, A is a unitary general Krasner A-hypermodule. Now define $f : A \longrightarrow P^*(A)$ with $f(x) = [0,x]$ for every $x \in A$. Then $f \in Hom_A^s(A,A)$ with $Ker(f) = \{0\}$ and $K_f = Im(f) = A$.

In the sequel, we introduce few kinds of equality. First note that for an R-hypermodule A and $X \subseteq A$, we define $\overline{\epsilon_A^*(X)} = \{\epsilon_A^*(x) \mid x \in X\}$ as a subset of $\frac{A}{\epsilon_A^*}$. Note that $\overline{\epsilon_A^*(X)}$ is different from the subset $\bigcup_{x\in X} \epsilon_A^*(x) \subseteq A$.

Definition 3.4.

(i) We say $a_1, a_2 \in A$ are ϵ_A^*-*equal* if $\epsilon_A^*(a_1) = \epsilon_A^*(a_2)$ and *product-equal*, for short pro-equal, if $\overline{\epsilon_A^*(ra_1)} \cap \overline{\epsilon_A^*(ra_2)} \neq \emptyset, \quad \forall r \in R$.

(ii) We say $a_1, a_2 \in A$ are R-*equal* if $\overline{\epsilon_A^*(Ra_1)} \cap \overline{\epsilon_A^*(Ra_2)} \neq \emptyset$.

It is necessary to emphasize ra_1 and ra_2 are two subsets of A in Definition 3.4.

Proposition 3.5. *Let R be a hyperring with 1_R and A be a unitary R-hypermodule.*

1. *If $a_1, a_2 \in A$ are pro-equal, then they are ϵ_A^*-equal.*

2. *If $a_1, a_2 \in A$ are pro-equal, then they are R-equal.*

Proof. 1. Let for all $r \in R$, $\overline{\epsilon_A^*(ra_1)} \cap \overline{\epsilon_A^*(ra_2)} \neq \emptyset$. If 1_R is the scalar unit of R, then $1_R * a_1 = \{a_1\}$ and $1_R * a_2 = \{a_2\}$. Thus

$$\overline{\epsilon_A^*(\{a_1\})} \cap \overline{\epsilon_A^*(\{a_2\})} \neq \emptyset \Longrightarrow \{\epsilon_A^*(a_1)\} \cap \{\epsilon_A^*(a_2)\} \neq \emptyset.$$

Hence $\epsilon_A^*(a_1) = \epsilon_A^*(a_2)$.

2. Let $a_1, a_2 \in A$ be pro-equal. Then for all $r \in R$, $\overline{\epsilon_A^*(ra_1)} \cap \overline{\epsilon_A^*(ra_2)} \neq \emptyset$. So $\forall r \in R, \exists x \in ra_1, \exists y \in ra_2 : \epsilon_A^*(x) = \epsilon_A^*(y)$. Clearly, $x \in Ra_1$ and $y \in Ra_2$ imply $\epsilon_A^*(x) \in \overline{\epsilon_A^*(Ra_1)}$ and $\epsilon_A^*(y) \in \overline{\epsilon_A^*(Ra_2)}$. Thus the result is followed. □

Definition 3.6. Let $g, h \in Hom_R(A, B)$.

(i) (equality of morphisms) We say g and h are *equal* if $g(a) = h(a), \forall a \in A$. In this case we write $g = h$.

(ii) (*weak equality* of morphisms) We say g and h are *weakly equal* if $g(a) \cap h(a) \neq \emptyset, \forall a \in A$. In this case we write $g \approx h$.

(iii) (ϵ_B^**-equality* of morphisms) We say g and h are ϵ_B^**-equal* if $\overline{\epsilon_B^*(g(a))} = \overline{\epsilon_B^*(h(a))}, \forall a \in A$. In this case we write $g \doteq h$.

(iv) (*weak* ϵ_B^**-equality* of morphisms) We say g and h are *weakly* ϵ_B^**-equal* if $\overline{\epsilon_B^*(g(a))} \cap \overline{\epsilon_B^*(h(a))} \neq \emptyset, \forall a \in A$. In this case we write $g \ddot{=} h$.

Clearly $g = h \Longrightarrow g \approx h \Longrightarrow g \ddot{=} h$, and $g \doteq h \Longrightarrow g \ddot{=} h$.
In the case that both f and g are single-valued, the above types of equality is as the following:

1. $g = h$ if and only if $g \approx h$.

2. $g \doteq h$ if and only if $g \ddot{=} h$.

Definition 3.7. A morphism $f \in Hom_R(A, B)$ is said

(i) *injective* if for all $a_1, a_2 \in A$, $f(a_1) = f(a_2)$ implies $a_1 = a_2$.

(ii) *strongly injective* if for all $a_1, a_2 \in A$, $f(a_1) \cap f(a_2) \neq \emptyset$ implies $a_1 = a_2$.

For a single-valued $f \in Hom_R(A, B)$, injectivity is equivalent to strong injectivity.

Example 3.8. The morphism f mentioned in Example 3.3 is injective.

Definition 3.9. A morphism $f \in Hom_R(A, B)$ is said

(i) ϵ_A^**-injective* if for all $a_1, a_2 \in A$, $f(a_1) = f(a_2) \Longrightarrow \epsilon_A^*(a_1) = \epsilon_A^*(a_2)$.

(ii) *strongly* ϵ_A^**-injective* if for all $a_1, a_2 \in A$, $f(a_1) \cap f(a_2) \neq \emptyset \Longrightarrow \epsilon_A^*(a_1) = \epsilon_A^*(a_2)$.

Every (strongly) injective $f \in Hom_R(A, B)$ is (strongly) ϵ_A^*-injective. It is clear that if f is strongly ϵ_A^*-injective, then it is ϵ_A^*-injective. Also a single-valued $f \in Hom_R(A, B)$ is strongly ϵ_A^*-injective if and only if it is ϵ_A^*-injective.

Definition 3.10. $f \in Hom_R(A, B)$ is said to be

(i) $\epsilon_{B,A}^*$*-injective* if for all $a_1, a_2 \in A$,

$$\overline{\epsilon_B^*(f(a_1))} = \overline{\epsilon_B^*(f(a_2))} \Longrightarrow \epsilon_A^*(a_1) = \epsilon_A^*(a_2).$$

(ii) *strongly $\epsilon^*_{B,A}$-injective* if for all $a_1, a_2 \in A$,

$$\overline{\epsilon^*_B(f(a_1))} \cap \overline{\epsilon^*_B(f(a_2))} \neq \emptyset \Longrightarrow \epsilon^*_A(a_1) = \epsilon^*_A(a_2).$$

Obviously, each strongly $\epsilon^*_{B,A}$-injective is $\epsilon^*_{B,A}$-injective. A single-valued $f \in Hom_R(A,B)$ is $\epsilon^*_{B,A}$-injective if and only if it is strongly $\epsilon^*_{B,A}$-injective.

In every category $\mathcal{C}$, a morphism $f \in Mor_{\mathcal{C}}(B,C)$ is said to be *mono* if for every $g,h \in Mor_{\mathcal{C}}(A,B)$, the following implication holds $f \circ g = f \circ h \Longrightarrow g = h$.

Definition 3.11. Let $f \in Hom_R(B,C)$. Then f is said to be

(i) *monic* or $R - mv$-monomorphism if f is a mono of ${}_R\mathcal{G}$.**mod** (in the sense of category theory).

(ii) $\epsilon^*_{C,B}$-*monic* if for all $g,h \in Hom_R(A,B)$, $f \circ g \doteq f \circ h \Longrightarrow g \doteq h$.

(iii) *partially $\epsilon^*_{C,B}$-monic* if for all $g,h \in Hom_R(A,B)$, $f \circ g \ddot{=} f \circ h \Longrightarrow g \ddot{=} h$.

Proposition 3.12. *Every strongly $\epsilon^*_{C,B}$-injective $R - mv$-homomorphism $f \in Hom_R(B,C)$ is partially $\epsilon^*_{C,B}$-monic.*

Proof. Let $f \in Hom_R(B,C)$ be strongly $\epsilon^*_{C,B}$-injective and $g,h \in Hom_R(A,B)$. Assume $f \circ g \ddot{=} f \circ h$. Then

$$\begin{aligned} &\forall a \in A: \quad \overline{\epsilon^*_C[(f\circ g)(a)]} \cap \overline{\epsilon^*_C[(f\circ h)(a)]} \neq \emptyset \\ &\Longrightarrow \forall a \in A: \quad \overline{\epsilon^*_C[f(g(a))]} \cap \overline{\epsilon^*_C[f(h(a))]} \neq \emptyset. \end{aligned}$$

Hence $\forall a \in A : \exists x \in g(a), \exists y \in h(a) : \overline{\epsilon^*_C(f(x))} \cap \overline{\epsilon^*_C(f(y))} \neq \emptyset$. Now by assumption, $\epsilon^*_B(x) = \epsilon^*_B(y)$. On the other hand, $x \in g(a)$ and $y \in h(a)$ implies $\overline{\epsilon^*_B(g(a))} \ni \epsilon^*_B(x) = \epsilon^*_B(y) \in \overline{\epsilon^*_B(h(a))}$. So $\overline{\epsilon^*_B(g(a))} \cap \overline{\epsilon^*_B(h(a))} \neq \emptyset$. Thus we have $g \ddot{=} h$. □

Proposition 3.13. *Let R be a hyperring with 1_R and $f \in Hom^s_R(A,B)$ be partially $\epsilon^*_{B,A}$-monic. If for $a_1, a_2 \in A$, $f(a_1) \cap f(a_2) \neq \emptyset$, then a_1 and a_2 are pro-equal.*

Proof. Assume $f(a_1) \cap f(a_2) \neq \emptyset$. Then for an arbitrary $r \in R$, $f(ra_1) \cap f(ra_2) = rf(a_1) \cap rf(a_2) \neq \emptyset$. Now consider $j_1, j_2 \in Hom_R(R,A)$, with $j_1(r) = ra_1$ and $j_2(r) = ra_2$ for all $r \in R$. Thus

$$\begin{aligned} f(j_1(r)) \cap f(j_2(r)) \neq \emptyset &\Longrightarrow \overline{\epsilon^*_C(f(j_1(r)))} \cap \overline{\epsilon^*_C(f(j_2(r)))} \neq \emptyset \\ &\Longrightarrow \overline{\epsilon^*_C(f \circ j_1)(r)]} \cap \overline{\epsilon^*_C[(f \circ j_2)(r)]} \neq \emptyset \end{aligned}$$

for all $r \in R$. So $f \circ j_1 \ddot{=} f \circ j_2$. According to the assumption, we obtain $j_1 \ddot{=} j_2$, i.e., $\overline{\epsilon^*_B(j_1(r))} \cap \overline{\epsilon^*_B(j_2(r))} \neq \emptyset$ for all $r \in R$. Therefore $\overline{\epsilon^*_A(ra_1)} \cap \overline{\epsilon^*_A(ra_2)} \neq \emptyset$ for all $r \in R$ and the proof is complete. □

Proposition 3.14. *Let R be a hyperring with 1_R and A be a unitary R-hypermodule. If $f \in Hom^s_R(A,B)$ is partially $\epsilon^*_{B,A}$-monic, then f is strongly ϵ^*_A-injective.*

Proof. Suppose $f(a_1)\cap f(a_2)\neq\emptyset$. Now Proposition 3.13 implies $\overline{\epsilon^*_A(ra_1)}=\overline{\epsilon^*_A(ra_2)}$. Now it immediately follows from Case 1 of Proposition 3.5, $\epsilon^*_A(a_1)=\epsilon^*_A(a_2)$. □

Clearly, every (strongly) $\epsilon^*_{B,A}$-injective $R-mv$-homomorphism $f\in Hom_R(A,B)$ is (strongly) ϵ^*_A-injective.

Proposition 3.15. *Let R be a hyperring. Suppose R has 1_R and B is a unitary R-hypermodule. If $f \in Hom^s_R(A,B)$ is monic in ${}_R\mathcal{G}.\mathbf{mod}$, then f is injective.*

Proof. Let $f(a) = f(a')$ and consider $\bar{a}, \bar{a}' \in Hom_R(R,A)$ with $\bar{a}(r) = \{ra\}$ and $\bar{a}'(r) = \{ra'\}$. Then $(f \circ \bar{a})(r) = f(ra) = rf(a) = rf(a') = f(ra') = (f \circ \bar{a}')(r)$. Hence $\bar{a}(r) = \bar{a}'(r)$. Since $\bar{a}(1_R) = \bar{a}'(1_R)$ and A is unitary, we have $a = a'$. □

Proposition 3.16. *If $f \in Hom_R(B,C)$ is strongly injective, then f is monic in ${}_R\mathcal{G}.\mathbf{mod}$.*

Proof. Suppose $f \circ g = f \circ h$ for $g, h \in Hom_R(A,B)$. Let $a \in A$ and $b \in g(a)$. Clearly, $f(b) \subseteq f(g(a)) = f(h(a))$. This means $f(b) \cap f(b')\neq\emptyset$ for some $b'\in h(a)$. Thus $b = b'$. So $g(a)\subseteq h(a)$. Similarly, $g(a) = h(a)$. Hence $g = h$. □

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