LEFT ALMOST POLYGROUPS

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Abstract. In this introductory note we define the concept of left almost polygroups and provide several examples. We also discuss the quotient structure and isomorphism theorems for left almost polygroups.
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1. Introduction

The work on left almost structures starts with the notion of left almost semigroups (abbreviated as LA-semigroups) defined by Kazim and Naseeruddin [8] in 1972. This structure, known also as Abel Grassmann-groupoid (abbreviated as AG-groupoid), is a groupoid where the left invertive law holds. Later on, Mushtaq and Kamran [10] introduced a new concept of a non-associative group, called the left almost group (LA-group).

The idea of defining left almost hyperstructures belongs to Hila and Dine [7], who proposed the study of left almost semihypergroups, further explored by Yaqoob et al. [11] and Amjad et al. [1]. Recently, in 2015, Gulistan et al. [6]

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extended it in the form of H_v-LA-semigroups, where the weak left invertive law holds.

The theory of polygroups has been initiated by Comer [2, 3, 4], who applied these quasicanonical hypergroups in color schemes theory, graph theory or cylindric algebras. A comprehensive overview of the most relevant results on polygroups is summarized in Davvaz' book [5].

The aim of this introductory note is to define the concept of left almost polygroups (abbreviated LA-polygroups). In particular, we study some basic properties of them, related with their subhyperstructures, and propose new constructions of such polygroups. Results concerning the homomorphism problems related to the left almost polygroups are covered in Section 4. The paper' conclusions are gathered in the last section.

2. Preliminaries

In this section, we first recall some basic definitions and results concerning LA-semigroups, and then we focus on those related to polygroups.

A groupoid $(S, \cdot)$ is called a *left almost semigroup* (abbreviate LA-semigroup) [8], if it satisfies the *left invertive law*: $(a \cdot b) \cdot c = (c \cdot b) \cdot a$, for all $a, b, c \in S$.

It is well known [8] that, in an LA-semigroup $(S, \cdot)$ the *medial law* $(a \cdot b) \cdot (c \cdot d) = (a \cdot c) \cdot (b \cdot d)$, for all $a, b, c, d \in S$ holds, too. If the LA-semigroup S contains a left identity, the *paramedial law* $(a \cdot b) \cdot (c \cdot d) = (d \cdot c) \cdot (b \cdot a)$, for all $a, b, c, d \in S$, is valid. Moreover, is such an LA-semigroup, by using the medial law, we get $a \cdot (b \cdot c) = b \cdot (a \cdot c)$, for all $a, b, c, d \in S$.

Extending the left invertive property to hyperstructures, one obtained the following notion.

Definition 1 ([7, 11]). *A hypergroupoid $(S, \circ)$, which is left invertive, that is $(x \circ y) \circ z = (z \circ y) \circ x$, for all $x, y, z \in S$, is called an LA-semihypergroup.*

Example 1 ([11]). Let $S = \mathbb{Z}$. If we define on S the following hyperproduct $x \circ y = y - x + 3\mathbb{Z}$, where $x, y \in \mathbb{Z}$, then $(S, \circ)$ becomes an LA-semihypergroup.

Definition 2 ([2]). *A hypergroup $(H, \circ)$ is called a polygroup if*

(i) there exists $e \in H$ such that $e \circ x = x = x \circ e$, for all $x \in H$,

(ii) for all $x \in H$ there exists an unique element, say $x^{-1} \in H$, such that $e \in x \circ x^{-1} \cap x^{-1} \circ x$,

(iii) for all $x, y, z \in H$, $z \in x \circ y \Rightarrow x \in z \circ y^{-1} \Rightarrow y \in x^{-1} \circ z$.

3. Left almost polygroups

In this section, we introduce and study some properties of the concept of left almost polygroup (abbreviate LA-polygroup) and provide some examples on how to construct new such hyperstructures. We also discuss some useful results related to LA-polygroups.

Definition 3. *A multivalued system* $\langle P, \circ, e, ^{-1} \rangle$, *where* $e \in P$, $^{-1} : P \longrightarrow P$ *and* $\circ : P \times P \longrightarrow \mathcal{P}^*(P)$ *is called an LA-polygroup, if, for all* $x, y, z \in P$, *the following axioms hold:*

(*i*) *left invertive law:* $(x \circ y) \circ z = (z \circ y) \circ x$,

(*ii*) *reproducibility axiom:* $x \circ P = P \circ x = P$,

(*iii*) *there exists a left identity* $e \in P$ *such that* $e \circ x = x$,

(*iv*) $e \in x \circ x^{-1} \cap x^{-1} \circ x$ *(we call* x^{-1} *the inverse of* x*)*,

(*v*) $x \in y \circ z \Longrightarrow y \in x \circ z^{-1}$.

In the above definition $\mathcal{P}^*(P)$ is the set of all non-empty subsets of P and e is just the left identity. The following elementary facts about LA-polygroups follow easily from the above axioms $e^{-1} = e$ and $(x^{-1})^{-1} = x$.

Example 2. Let $P = \{e, x, y\}$ and the binary hyperoperation "$\circ$" be defined as in the following table:

$\circ$	e	x	y
e	e	x	y
x	y	$\{e, x, y\}$	$\{x, y\}$
y	x	$\{x, y\}$	$\{e, x, y\}$

Here all the elements of P satisfy the left invertive law and e is the left identity. $^{-1}$ is a unitary operation on P taken as

	e	x	y
$^{-1}$	e	x	y

Now, for the fifth condition in Definition 3, see the following calculations:

$$\begin{array}{lllcllllll}
e & = & e \circ e & \Longrightarrow & e & = & e \circ e^{-1} & = & e \circ e & = & e \\
x & = & e \circ x & \Longrightarrow & e & \in & x \circ x^{-1} & = & x \circ x & = & \{e, x, y\} \\
y & = & e \circ y & \Longrightarrow & e & \in & y \circ y^{-1} & = & y \circ y & = & \{e, x, y\} \\
y & = & x \circ e & \Longrightarrow & x & = & y \circ e^{-1} & = & y \circ e & = & x \\
e & \in & x \circ x & \Longrightarrow & x & = & e \circ x^{-1} & = & e \circ x & = & x \\
x & \in & x \circ x & \Longrightarrow & x & \in & x \circ x^{-1} & = & x \circ x & = & \{e, x, y\} \\
y & \in & x \circ x & \Longrightarrow & x & \in & y \circ x^{-1} & = & y \circ x & = & \{x, y\} \\
x & \in & x \circ y & \Longrightarrow & x & \in & x \circ y^{-1} & = & x \circ y & = & \{x, y\} \\
y & \in & x \circ y & \Longrightarrow & x & \in & y \circ y^{-1} & = & y \circ y & = & \{e, x, y\} \\
x & = & y \circ e & \Longrightarrow & y & = & x \circ e^{-1} & = & x \circ e & = & y \\
x & \in & y \circ x & \Longrightarrow & y & \in & x \circ x^{-1} & = & x \circ x & = & \{e, x, y\} \\
y & \in & y \circ x & \Longrightarrow & y & \in & y \circ x^{-1} & = & y \circ x & = & \{x, y\} \\
e & \in & y \circ y & \Longrightarrow & y & = & e \circ y^{-1} & = & e \circ y & = & y \\
x & \in & y \circ y & \Longrightarrow & y & \in & x \circ y^{-1} & = & x \circ y & = & \{x, y\} \\
y & \in & y \circ y & \Longrightarrow & y & \in & y \circ y^{-1} & = & y \circ y & = & \{e, x, y\}
\end{array}$$

Hence $\langle P, \circ, e, ^{-1} \rangle$ is an LA-polygroup.

Example 3. Consider a finite set P with at least 3 elements. Define a hyperoperation "$*$" on P as follows:

$$x_i * x_j = \begin{cases} x_j, & \text{for } i = 1, \\ x_k, & \text{for } j = 1 \text{and } k \equiv 2 - i mod\,|P|\,, \\ P, & \text{for } i = j, i \neq 1, j \neq 1, \\ P \backslash \{x_1\}, & \text{for } i \neq j, i \neq 1, j \neq 1. \end{cases}$$

Then P under the hyperoperation "$*$" forms an LA-polygroup with the left identity x_1 and $^{-1}$ is a unitary operation on P taken as

	x_1	x_2	x_3	.	.	.	$x_{\|P\|}$
-1	x_1	x_2	x_3	.	.	.	$x_{\|P\|}$

Besides "$*$" is non-associative because $\{x_2\} = (x_2 * x_1) * x_1 \neq x_2 * (x_1 * x_1) = \{x_{|P|}\}$. Consider $P = \{x_1, x_2, x_3, x_4, x_5\}$. Then under the binary hyperoperation $*$ defined above, P is an LA-polygroup. See the following Cayley's table:

$*$	x_1	x_2	x_3	x_4	x_5
x_1	x_1	x_2	x_3	x_4	x_5
x_2	x_5	P	$\{x_2, x_3, x_4, x_5\}$	$\{x_2, x_3, x_4, x_5\}$	$\{x_2, x_3, x_4, x_5\}$
x_3	x_4	$\{x_2, x_3, x_4, x_5\}$	P	$\{x_2, x_3, x_4, x_5\}$	$\{x_2, x_3, x_4, x_5\}$
x_4	x_3	$\{x_2, x_3, x_4, x_5\}$	$\{x_2, x_3, x_4, x_5\}$	P	$\{x_2, x_3, x_4, x_5\}$
x_5	x_2	$\{x_2, x_3, x_4, x_5\}$	$\{x_2, x_3, x_4, x_5\}$	$\{x_2, x_3, x_4, x_5\}$	P

One can see that "$*$" satisfies the left invertive law, x_1 is the left identity, $^{-1}$ is a unitary operation on P taken as

	x_1	x_2	x_3	x_4	x_5
-1	x_1	x_2	x_3	x_4	x_5

but "$*$" is non-associative because $\{x_2\} = (x_2 * x_1) * x_1 \neq x_2 * (x_1 * x_1) = \{x_5\}$.

Theorem 1. *Let $\langle P, \circ, e, ^{-1} \rangle$ be an LA-polygroup satisfying the condition $e = a \circ a^{-1}$, for any $a \in P$. Then P is a polygroup if and only if $a \circ (b \circ c) = (c \circ b) \circ a$ holds, for all $a, b, c \in P$.*

Proof. Let $\langle P, \circ, e, ^{-1} \rangle$ be a polygroup. Then, by associativity, we have $(a \circ b) \circ c = a \circ (b \circ c)$ for all $a, b, c \in P$, and since P is an LA-polygroup, it follows that $(a \circ b) \circ c = (c \circ b) \circ a$, therefore $a \circ (b \circ c) = (c \circ b) \circ a$, for all $a, b, c \in P$. Conversely, suppose $a \circ (b \circ c) = (c \circ b) \circ a$ holds, for all $a, b, c \in P$.

(i) Since $\langle P, \circ, e, ^{-1} \rangle$ is an LA-polygroup, we have $a \circ (b \circ c) = (c \circ b) \circ a = (a \circ b) \circ c$.

(ii) Based on the following identities $a\circ e = a\circ(e\circ e) = (e\circ e)\circ a = e\circ a = a$, it results that e is also the right identity.

(iii) Let $a, b, c \in P$ such that $a \in b\circ c \Rightarrow b \in a\circ c^{-1}$. Now,

$$\begin{aligned}
c &= e\circ c = (b^{-1}\circ b)\circ c \\
&= (c\circ b)\circ b^{-1} \quad \text{(left invertive law)} \\
&= b^{-1}\circ(b\circ c) \quad (\text{by } a\circ(b\circ c) = (c\circ b)\circ a) \\
&\subseteq b^{-1}\circ((a\circ c^{-1})\circ c) \\
&= b^{-1}\circ((c\circ c^{-1})\circ a) \quad \text{(left invertive law)} \\
&= b^{-1}\circ(e\circ a) \\
&= b^{-1}\circ a.
\end{aligned}$$

We can conclude that $\langle P, \circ, e, ^{-1}\rangle$ is a polygroup. □

Definition 4. *A non-empty subset K of an LA-polygroup $\langle P, \circ, e, ^{-1}\rangle$ is called LA-subpolygroup of P if, under the hyperoperation in P, K itself forms an LA-polygroup.*

Example 4. Let $P = \{e, x, y, z\}$ and the hyperoperation on P be defined in the following table.

$\circ$	e	x	y	z
e	e	x	y	z
x	y	$\{x, y\}$	$\{e, x\}$	z
y	x	$\{e, y\}$	$\{x, y\}$	z
z	z	z	z	$\{e, x, y\}$

Here all the elements of P satisfy the left invertive law and e is the left identity. $^{-1}$ is a unitary operation on P taken as

	e	x	y	z
$^{-1}$	e	y	x	z

Besides P is not a polygroup since $\{x, y\} = (x\circ e)\circ y \neq x\circ(e\circ y) = \{e, x\}$. So $\langle P, \circ, e, ^{-1}\rangle$ is an LA-polygroup and $K = \{e, x, y\}$ is an LA-subpolygroup of P.

Lemma 1. *A non-empty subset K of the LA-polygroup $\langle P, \circ, e, ^{-1}\rangle$ is an LA-subpolygroup of it, if and only if the following relations are satisfied.*

(i) For all $a, b \in K \Rightarrow a\circ b \subseteq K$.

(ii) For all $a \in K \Rightarrow a^{-1} \in K$.

Proof. Let K be an LA-subpolygroup of P. Then relations (i) and (ii) are obvious.

Conversely, suppose that relations (i) and (ii) are true. Since K is a non-empty subset of P, it follows that the left invertive law holds in K. Now from (ii) we have the implication $a \in K \Rightarrow a^{-1} \in K$, and from ($i$) it results that $e \in a\circ a^{-1} \subseteq K$, for $a, a^{-1} \in K$. Moreover, let $c \in a\circ b$. It implies that $a \in c\circ b^{-1}$ by (ii). Hence, K is an LA-subpolygroup of P. □

Lemma 2. *In an LA-polygroup $\langle P, \circ, e, ^{-1} \rangle$ the following laws holds:*
(i) *Medial law:* $(a \circ b) \circ (c \circ d) = (a \circ c) \circ (b \circ d)$,
(ii) $a \circ (b \circ c) = b \circ (a \circ c)$,
(ii) *Paramedial law:* $(a \circ b) \circ (c \circ d) = (d \circ c) \circ (b \circ a)$, *for all* $a, b, c, d \in P$.

Proof. Straightforward. □

Lemma 3. *If K is an LA-subpolygroup of the LA-polygroup $\langle P, \circ, e, ^{-1} \rangle$, then, for every $a, b \in P$, we have:*
(i) $K = K \circ K$.
(ii) $e \circ K = K \circ e = K$.
(ii) $a \circ K = (K \circ a) \circ e$,
(iv) $(a \circ b) \circ K = K \circ (b \circ a)$.

Proof. Straightforward. □

It is important to note that there is no concept of polygroup theoretic normality in LA-polygroups, meaning that we can factor an LA-polygroup by any of its LA-subpolygroups. We know that if $\langle P, \circ, e, ^{-1} \rangle$ is a polygroup and K is its subpolygroup, then $(K \circ a) \circ (K \circ b) \neq K \circ (a \circ b)$, unless K is normal in P. But for LA-polygroups we have no such condition because of the medial property, that is, if $K \circ a$, $K \circ b$ belong to P/K, then

$$(K \circ a) \circ (K \circ b) = (K \circ K) \circ (a \circ b) \quad \text{(by the medial law)} = K \circ (a \circ b),$$

without having an extra condition on P.

Remark 1. An LA-polygroup can be partitioned only into right cosets (or left cosets) and we do not require the two side decomposition.

Theorem 2. *If $\langle P, \circ, e, ^{-1} \rangle$ is an LA-polygroup and K is an LA-subpolygroup of P, then $P/K = \{K \circ a \mid a \in P\}$ is an LA-polygroup, too.*

Proof. Let us define the hyperoperation in P/K as $K \circ a \boxtimes K \circ b = K \circ (a \circ b)$, which is obviously closed.

(i) Let $K \circ a$, $K \circ b$ and $K \circ c \in P/K$. Then one obtains

$$\begin{aligned}(K \circ a \boxtimes K \circ b) \boxtimes K \circ c &= K \circ (a \circ b) \boxtimes K \circ c \\ &= K \circ ((a \circ b) \circ c) = K \circ ((c \circ b) \circ a) \\ &= K \circ (c \circ b) \boxtimes K \circ a \\ &= (K \circ c \boxtimes K \circ b) \boxtimes K \circ a.\end{aligned}$$

(ii) There exists $K \circ e \in P/K$ such that $K \circ e \boxtimes K \circ b = K \circ (e \circ b) = K \circ b$, so $K \circ e = K$ is the left identity in P/K.

(iii) Let $K \circ a$, $K \circ b$ and $K \circ c \in P/K$ such that $K \circ a \subseteq K \circ b \boxtimes K \circ c = K \circ (b \circ c)$ i.e. $K \circ a \subseteq K \circ (b \circ c)$, so $a \in b \circ c$. This implies that $b \in a \circ c^{-1}$ and therefore $K \circ b \subseteq K \circ (a \circ c^{-1})$. It results that $K \circ b \subseteq K \circ a \boxtimes K \circ c^{-1}$. Hence $P/K = \{K \circ a : a \in P\}$ is an LA-polygroup. We conclude this section with some constructions of LA-polygroups. □

Definition 5. *Let* $\langle P_1, \circ_1, e_1, ^{-1} \rangle$ *and* $\langle P_2, \circ_2, e_2, ^{-1} \rangle$ *be two LA-polygroups. Then on* $P_1 \times P_2$ *we can define the hyperproduct as follows:*

$$(a_1, b_1) \circ (a_2, b_2) = \{(c, d) \mid c \in a_1 \circ_1 a_2,\ d \in b_1 \circ_2 b_2\},$$

for all $(a_1, b_1), (a_2, b_2) \in P_1 \times P_2$.

Proposition 1. *The direct product of two LA-polygroups is an LA-polygroup, too.*

Proof. Straightforward. □

Corollary 1. *If* K_1, K_2 *are LA-subpolygroups of the LA-polygroups* P_1, P_2 *respectively, then* $K_1 \times K_2$ *is an LA-subpolygroup of* $P_1 \times P_2$ *and* $(P_1 \times P_2)/(K_1 \times K_2) \cong P_1/K_1 \times P_2/K_2$.

Definition 6. *Let* $\langle P, \circ, e, ^{-1} \rangle$ *be an LA-polygroup, and let* $a, b \in P$. *We write* $a \blacktriangleright b$ *if* $a \circ c \subseteq b \circ c$, *for all* $c \in P$, *and we call* $\blacktriangleright$ *a hyperorder on* P.

If $a \blacktriangleright b$ and $b \blacktriangleright a$, then we say a is hyperequal to b, and we write $a \sim b$. It is clear that the relation " $\sim$ " is an equivalence relation on P.

Proposition 2. *Let* $\langle P, \circ, e, ^{-1} \rangle$ *be an LA-polygroup. We define the class* $[a] = \{b \in P \mid a \sim b\}$ *represented by* a, *and let* $C(P) = \{[a] \mid a \in P\}$ *denote the set of all classes of the elements in* P. *If we define the hyperoperation on* $C(P)$ *as* $[a] \bullet [b] = \{[n] \mid n \in a \circ b\}$, *then* $\langle C(P), \bullet, e, ^{-1} \rangle$ *is an LA-polygroup.*

Proof. (i) Let $[a], [b], [c] \in C(P)$. One finds that

$$\begin{aligned} ([a] \bullet [b]) \bullet [c] &= (\{[n] \mid n \in a \circ b\}) \bullet [c] = \{[m] \mid m \in n \circ c\} \\ &= \{[m] \mid m \in (a \circ b) \circ c\} = \{[m] \mid m \in (c \circ b) \circ a\} \\ &= \{[m] \mid m \in n \circ a\} = (\{[n] \mid n \in c \circ b\}) \bullet [a] \\ &= ([c] \bullet [b]) \bullet [a]. \end{aligned}$$

(ii) Let $[e], [a] \in C(P)$. Then $([e] \bullet [a]) = (\{[n] \mid n \in e \circ a\}) = (\{[n] \mid n = a\}) = [a]$. (iii) Let $[a], [b], [c] \in C(P)$ and consider $[a] \in [b] \bullet [c] = \{[x] \mid x \in b \circ c\}$. Therefore, there exist $y \in [b]$ and $z \in [c]$ such that $x \in y \circ z$, so $y \in x \circ z^{-1}$. This implies that $[y] \in [x] \bullet [z^{-1}]$, thus $[b] \in [a] \bullet [c^{-1}]$. Hence $\langle C(P), \bullet, e, ^{-1} \rangle$ is an LA-polygroup. □

4. Homomorphisms of LA-polygroups

This section is devoted to the study of some homomorphism problems related to LA-polygroups.

Definition 7. *Let* $\langle P_1, \circ_1, e_1, ^{-1} \rangle$ *and* $\langle P_2, \circ_2, e_2, ^{-1} \rangle$ *be two LA-polygroups. Let* f *be a mapping from* P_1 *into* P_2 *such that* $f(e_1) = e_2$. *Then,* f *is called*

(i) *an inclusion homomorphism if* $f(x \circ_1 y) \subseteq f(x) \circ_2 f(y)$.

(ii) *a good homomorphism if* $f(x \circ_1 y) = f(x) \circ_2 f(y)$.

Example 5. (i) Let $P_1 = \{e, x, y\}$ and $P_2 = \{g, a, b\}$ be two LA-polygroups with the hyperoperations defined in the following tables:

$\circ_1$	e	x	y
e	e	x	y
x	y	$\{x, y\}$	$\{e, x\}$
y	x	$\{e, y\}$	$\{x, y\}$

$\circ_2$	g	a	b
g	g	a	b
a	b	$\{a, b\}$	P_2
b	a	P_2	$\{a, b\}$

and let $f : P_1 \longrightarrow P_2$ be defined by $f(e) = g$, $f(x) = a$, $f(y) = b$. Then, clearly, f is an inclusion homomorphism.

(ii) Let $P_1 = \{e, x, y, z\}$ and $P_2 = \{g, a, b, c\}$ be two LA-polygroups with the hyperoperations defined in the following tables:

$\circ_1$	e	x	y	z
e	e	x	y	z
x	y	$\{x, y, z\}$	$\{e, x, z\}$	$\{x, y\}$
y	x	$\{e, y, z\}$	$\{x, y, z\}$	$\{x, y\}$
z	z	$\{x, y\}$	$\{x, y\}$	$\{e, z\}$

$\circ_2$	g	a	b	c
g	g	a	b	c
a	c	$\{a, b, c\}$	$\{a, c\}$	$\{g, a, b\}$
b	b	$\{a, c\}$	$\{g, b\}$	$\{a, c\}$
c	a	$\{g, b, c\}$	$\{a, c\}$	$\{a, b, c\}$

and let $f : P_1 \longrightarrow P_2$ be defined by $f(e) = g$, $f(x) = a$, $f(y) = c$, $f(z) = b$. Then, clearly f is a good homomorphism.

Lemma 4. *Let f be a good homomorphism from $\langle P_1, \circ_1, e_1, ^{-1} \rangle$ into $\langle P_2, \circ_2, e_2, ^{-1} \rangle$. Let K_1 and K_2 be LA-subpolygroups of P_1 and P_2, respectively. Then the following statements are valid.*

(i) The image $f(K_1)$ of K_1 under f is an LA-subpolygroup of P_2.

(ii) The inverse image $f^{-1}(K_2)$ of K_2 under f is an LA-subpolygroup of P_1.

Proof. Straightforward. □

Lemma 5. *Let f be a good homomorphism from $\langle P_1, \circ_1, e_1, ^{-1} \rangle$ into $\langle P_2, \circ_2, e_2, ^{-1} \rangle$. Then:*

(i) $f(e_1) = e_2$,

(ii) $f(a^{-1}) \subseteq f(a)^{-1}$.

Proof. Straightforward. □

Lemma 6. *Let f be a good homomorphism from $\langle P_1, \circ_1, e_1, ^{-1} \rangle$ into $\langle P_2, \circ_2, e_2, ^{-1} \rangle$. Then f is injective if and only if $ker f = \{e_1\}$.*

Proof. Let f be injective and assume that $x \in ker f$. By Lemma 5, we have $f(e_1) = e_2$. Therefore $f(x) = e_2 = f(e_1) \Rightarrow x = e_1$ and hence $\ker f = \{e_1\}$.

Conversely, let $\ker f = \{e_1\}$ and assume that $f(x) = f(y)$ for $x, y \in P_1$. Now considering $f(x) = f(y)$, we have $f(x) \circ_2 f(x^{-1}) = f(y) \circ_2 f(x^{-1})$. It follows that $f(e_1) \in f(x \circ_1 x^{-1}) = f(y \circ_1 x^{-1})$. So there exists $t \in y \circ_1 x^{-1}$ such that $e_2 = f(e_1) = f(t)$. Thus $e_1 = t \in y \circ_1 x^{-1}$, whence $x = y$. □

Theorem 3. *Let f be a good homomorphism from $\langle P_1, \circ_1, e_1, ^{-1} \rangle$ into $\langle P_2, \circ_2, e_2, ^{-1} \rangle$ with the kernel κ, such that κ is an LA-subpolygroup of $\langle P_1, \circ_1, e_1, ^{-1} \rangle$. Then $P_1/\kappa \cong P_2$.*

Proof. Let f be a good homomorphism, i.e $f(e_1) = e_2$ and $f(a \circ_1 b) = f(a) \circ_2 f(b)$, for all $a, b \in P_1$. Define a mapping $\lambda : P_1/\kappa \to P_2$ by $\lambda(\kappa x) = f(x)$, for all $x \in P_1$. We first show that λ is well-defined. For $x, y \in P_1$, $\kappa x = \kappa y \Longrightarrow x \circ_1 y^{-1} \subseteq \kappa$. Let $t \in x \circ_1 y^{-1}$. Consequently, $f(t) = e_2$ and $f(t) \subseteq f(x) \circ_2 f(y^{-1}) = f(x) \circ_2 f(y)^{-1}$. Thus $f(x) = f(y)$. Clearly λ is onto. Now we have to show that λ is one to one. Suppose $f(x) = f(y)$. Then $e_2 \in f(x \circ_1 y^{-1})$ and so there exists $t \in x \circ_1 y^{-1}$ with $t \in \ker f$. Therefore $x \circ_1 y^{-1} \subseteq \kappa$, which implies that $\kappa x = \kappa y$, and so λ is one to one. Now it remains only to prove that λ is a good homomorphism. Let $\kappa x, \kappa y \in P_1/\kappa$. It results that $\lambda(\kappa x \circ_1 \kappa y) = \lambda(\kappa(x \circ_1 y)) = f(x \circ_1 y) = f(x) \circ_2 f(y) = \lambda(\kappa x) \circ_2 \lambda(\kappa y)$ and also $\lambda(\kappa e_1) = f(e_1) = e_2$. Hence $P_1/\kappa \cong P_2$. □

Theorem 4. *If K and N are LA-subpolygroups of the LA-polygroup $\langle P, \circ, e, ^{-1} \rangle$, then $K/(N \cap K) \cong KN/N$.*

Proof. Let us define $f : K \to KN/N$ by $f(k) = Nk$, for all $k \in K$. It is easy to show that f is a good homomorphism. Since each element of KN/N has the form of knN, where $k \in K$ and $n \in N$, it follows that $nN = N$. Therefore each element of KN/N is of the form KN, which is the image of K under f. Hence f is onto. Therefore, by Theorem 3, $K/\ker f \cong KN/N$. Now we need to show that $\ker f = N \cap K$.

$$\begin{aligned} \ker f &= \{k \in K : f(k) = \text{identity of } KN/N\} \\ &= \{k \in K : kN = N\} \\ &= \{k \in K : k \in N\} = N \cap K. \end{aligned}$$

Hence $K/(N \cap K) \cong KN/N$. □

Theorem 5. *If K and N are LA-subpolygroups of the LA-polygroup $\langle P, \circ, e, ^{-1} \rangle$ such that $N \subseteq K$, then $(P/N)/(K/N) \cong P/K$.*

Proof. Let us define a mapping $f : P/N \to P/K$ by $f(Na) = Ka$, for any $a \in P$. It is easy to show that f is a good homomorphism. Since, for each $Na \in P/N$ there exist $Ka \in P/K$ such that $f(Na) = Ka$, it results that the mapping f is onto. So, by Theorem 3, $(P/N)/\ker f \cong P/K$. Now we show that $\ker f = K/N$. This follows from the following identities:

$$\begin{aligned} \ker f &= \{Na \in P/N : a \in P \text{ and } f(Na) = \text{identity of } P/K\} \\ &= \{Na \in P/N : a \in P \text{ and } Ka = K\} \\ &= \{Na \in P/N : a \in K\} \quad = K/N. \end{aligned}$$

Hence $(P/N)/(K/N) \cong P/K$. □

5. Conclusions

Substituting in a polygroup the associativity with the left invertive law, one obtains the notion of left almost polygroup, by short LA-polygroup. Most of the properties of polygroups are valid also for LA-polygroups, but the normality concept is different here. This means that an LA-polygroup may be factorize by any of its LA-subpolygroup K, without asking K to be a normal LA-subpolygroup, as it happens in polygroups framework. This property is assured by the medial law, that holds in an LA-polygroup. Consequently, the three isomorphism theorems are simplified for LA-polygroups, with respect to polygroups.

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