COUPLED FIXED AND COINCIDENCE POINT THEOREMS FOR GENERALIZED CONTRACTIONS IN METRIC SPACES WITH A PARTIAL ORDER

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Abstract. In this paper, we establish results on the existence and uniqueness of coupled common fixed point theorems and coupled coincidence fixed point theorems for such non-linear contraction mappings having a mixed monotone property in partially ordered complete metric spaces with out using continuity.Our results generalize and extend the results of V. Lakshmikantham and L. Ciric [13], Sintunavarat and Poom Kumam [16].

Keywords: coupled fixed point, coupled coincidence point, mixed monotone property, partially ordered set.

1. Introduction

Recently V. Lakshmikantham and L. Ciric [13] generalized the concept of coupled fixed point theorems for non-linear contractions in partially ordered metric spaces. Subsequently Sintunavarat and Poom Kumam [16] studied unique coupled fixed point theorem in partially ordered metric spaces. The aim of this paper is to extend the results of T.G. Bhaskar and V. Lakshmikantham [5] and V. Lakshmikantham and L. Ciric [13] and Sintunavarat and Poom Kumam [16]

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for a mixed monotone non-linear contractive mapping and to generalize the notion of a mixed monotone mapping

We proved some coupled coincidence and coupled common fixed point theorems for a pair of mappings.Our results extend the recent fixed point theorems due to V. Lakshmikantham and L. Ciric [13], fixed point theorems due to Sintunavarat and Poom Kumam [16] and include several recent developments.

Suppose $(X, \leq)$ is a partially ordered set. Let $F : X \to X$ be such that for $x, y \in X$, $x \leq y \Rightarrow F(x) \leq F(y)$. Then the mapping F is said to be non-decreasing, similarly a non-increasing mapping is defined, Bhaskar and Lakshmikantham [5] introduced the following notions of a coupled fixed point theorems.

Before going to prove the main result, we need some basic definitions and results from the literature.

2. Preliminaries

Definition 2.1 (Bhaskar and Lakshmikantham [5]). Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$ .The mapping F is said to have the mixed monotone property,if F is monotone non-decreasing in its first argument and monotone non-increasing in its second argument,that is, for any $x, y \in X$,

$$x_1, x_2 \in X, x_1 \leq x_2 \Rightarrow F(x_1, y) \leq F(x_2, y)$$

and

$$y_1, y_2 \in X, y_1 \leq y_2 \Rightarrow F(x, y_1) \geq F(x, y_2).$$

Definition 2.2 (Bhaskar and Lakshmikantham [5]). Let X be a nonempty set. An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \to X$, if $F(x, y) = x$ and $F(y, x) = y$

Bhaskar and Lakshmikantham [5] proved the following two coupled fixed point theorems.

Theorem 2.3 (Bhaskar and Lakshmikantham [5], Theorem 2.1). *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X, such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a continuous mapping having the mixed monotone property on X. Assume there exists a $k \in [0, 1)$ with*

$$d(F(x, y), F(u, v)) \leq \frac{k}{2}(d(x, u) + d(y, v)) \tag{2.3.1}$$

for each $x \geq u$ and $y \leq v$.

If there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$.

Theorem 2.4 (Bhaskar and Lakshmikantham [5], Theorem 2.2). *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X, such that (X, d) is a complete metric space.Assume that X has the following property:*

i) if a non decreasing sequence $\{x_n\} \to x$*, then* $x_n \le x$*, for all* $n \in N$*.*

ii) if a non increasing sequence $\{y_n\} \to y$*,then* $y \le y_n$*, for all* $n \in N$*.*

Let $F : X \times X \to X$ *be a continuous mapping having the mixed monotone property on* X*. Assume that there exists a* $k \in [0, 1)$ *with*

$$d(F(x, y), F(u, v)) \le \frac{k}{2}(d(x, u) + d(y, v)) \ for\ all\ x, y, u, v \in X,$$

for which $x \ge u$ *and* $y \le v$*.*

If there exists $x_0, y_0 \in X$ *such that* $x_0 \le F(x_0, y_0)$ *and* $y_0 \ge F(y_0, x_0)$*,then there exist* $x, y \in X$ *such that* $x = F(x, y)$ *and* $y = F(y, x)$*.*

Definition 2.5. (V. Lakshmikantham and L. Ciric [13]). Let $(X, \le)$ be a partially ordered set and $F : X \times X \to X$ and $g : X \to X$. The mapping F is said to have the mixed g-monotone property, if F is monotone g-non decreasing in its first argument and is monotone g-non increasing in its second argument, that is, for any $x, y \in X$

$$x_1, x_2 \in X, g(x_1) \le g(x_2) \Rightarrow F(x_1, y) \le F(x_2, y)$$

and

$$y_1, y_2 \in X, g(y_1) \le g(y_2) \Rightarrow F(x, y_1) \ge F(x, y_2).$$

Definition 2.6 (Lakshmikantham and Ciric [12])**.** Let X be a nonempty set. An element $(x, y) \in X \times X$ is called a coupled coincidence point of a mapping $F : X \times X \to X$ and $g : X \to X$, if $x = g(x) = F(x, y)$ and $y = g(y) = F(y, x)$.

Definition 2.7 (Lakshmikantham and Ciric [12])**.** Let X be a nonempty set and $F : X \times X \to X$, $g : X \to X$. We say F and g are commutative, if $g(F(x, y)) = F(g(x), g(y))$, for all $x, y \in X$.

Theorem 2.8 (Lakshmikantham and Ciric [12], Theorem 2.1)**.** *Let* $(X, \le)$ *be a partially ordered set and suppose there is a metric* d *on* X*, such that* (X, d) *is a complete metric space. Assume there is a function* $\varphi : [0, \infty) \to [0, \infty)$ *with* $\varphi(t) < t$ *and* $\lim_{r \to t^+} \varphi(r) < t$ *for each* $t > 0$*,and also suppose that* $F : X \times X \to X$ *and* $g : X \to X$ *are such that* F *has the mixed* g*-monotone property and*

$$d(F(x, y), F(u, v)) \le \varphi\left(\frac{d(g(x), g(u)) + d(g(y), g(v))}{2}\right), \tag{2.8.1}$$

for all $x, y, u, v \in X$ *for which* $g(x) \le g(u)$ *and* $g(y) \ge g(v)$*.*

Suppose $F(X \times X) \subseteq g(X)$*,* g *is continuous and commutes with* F *and also suppose either*

(a) *F is continuous (or)*

(b) *X has the following property.*

i) if a non decreasing sequence $\{x_n\} \to x$*, then* $x_n \le x$*, for all* $n \in N$*.*

ii) if a non increasing sequence $\{y_n\} \to y$*, then* $y \le y_n$*, for all* $n \in N$*.*

If there exist $x_0, y_0 \in X$ *such that* $g(x_0) \le F(x_0, y_0)$ *and* $g(y_0) \ge F(y_0, x_0)$*, then there exist* $x, y \in X$ *such that* $g(x) = F(x, y)$ *and* $g(y) = F(y, x)$*, that is,* F *and* g *have a coupled coincidence point.*

Theorem 2.9 (Lakshmikantham and Ciric [12], Theorem 2.2). *In addition to the hypothesis of Theorem 2.8, suppose that F and g are commutative and for every $(x, y), (z, t) \in X \times X$, there exists a $(u, v) \in X \times X$ such that $(F(u, v), F(v, u))$ is comparable to $(F(x, y), F(y, x))$ and $(F(z, t), F(t, z))$. Then F and g have a unique coupled common fixed point, that is, there exists a unique $(x, y) \in X \times X$ such that*

$$x = g(x) = F(x, y) \text{ and } y = g(y) = F(y, x).$$

In 2013, Sintunavarat and Kumam [16] gave an extension of the result of Bhaskar and Lakshmikantham [5] , Lakshmikantham and Ciric [13]. They used this concept to establish the existence of coupled coincidence point and coupled common fixed point theorem.

Theorem 2.10 (Sintunavarat and Kumam [16], Theorem 2.1). *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X, such that (X, d) is a complete metric space. Assume there is a function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(t) < t$ and $\lim_{r \to t^+} \varphi(r) < t$, for each $t > 0$ and also suppose that $F : X \times X \to X$ is such that F has the mixed monotone property and*

$$d(F(x, y), F(u, v)) \leq \varphi\left(\frac{d(x, u) + d(y, v)}{2}\right), \tag{2.10.1}$$

for all $x, y, u, v \in X$ for which $x \leq u$ and $y \geq v$.

Suppose either

(a) F is continuous (or);

(b) X has the following property.

i) if a non decreasing sequence $\{x_n\} \to x$, then $x_n \leq x$ for all $n \in N$;

ii) if a non increasing sequence $\{y_n\} \to y$, then $y \leq y_n$ for all $n \in N$.

If there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$, that is, F has a coupled fixed point.

Theorem 2.11 (Sintunavarat and Kumam [16], Theorem 2.2). *In addition to the hypothesis of Theorem 2.10, suppose that for every $(x, y), (z, t) \in X \times X$, there exists a $(u, v) \in X \times X$, which is comparable to (x, y) and (z, t). Then F has a unique coupled fixed point.*

Corollary 2.12 (Sintunavarat and Kumam [16], Corollary 2.4). *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X, such that (X, d) is a complete metric space. Suppose that $F : X \times X \to X$ and $g : X \to X$ are such that F has the mixed g-monotone property and assume there is a $k \in [0, 1)$ such that*

$$d(F(x, y), F(u, v)) \leq \frac{k}{2}(d(g(x), g(u)) + d(g(y), g(v))), \tag{2.12.1}$$

for all $x, y, u, v \in X$ for which $g(x) \leq g(u)$ and $g(y) \geq g(v)$.

Suppose $F(X \times X) \subseteq g(X)$, g is continuous and suppose either
(a) *F is continuous (or)*
(b) *X has the following property.*
i) if a non decreasing sequence $\{x_n\} \to x$, then $x_n \leq x$ for all $n \in N$.
ii) if a non increasing sequence $\{y_n\} \to y$, then $y \leq y_n$ for all $n \in N$.
If there exist $x_0, y_0 \in X$ such that $g(x_0) \leq F(x_0, y_0)$ and $g(y_0) \geq F(y_0, x_0)$, then there exist $x, y \in X$ such that $g(x) = F(x, y)$ and $g(y) = F(y, x)$, that is, F and g have a coupled coincidence fixed point.

3. Main result

In this section, we improve the results in section 2, by replacing the conditions (i) $\varphi(t) < t$ and (ii). $\lim_{r\to t^+} \varphi(r) < t$ by the single condition: $\varphi(t+0) < t$, and the average in the argument of φ by maximum.

We introduce the class Φ of functions as follows:

$$\Phi = \{\varphi/\varphi : [0,\infty) \to [0,\infty), \varphi \text{ is increasing and } \varphi(t+0) < t \ \forall\ t > 0\}.$$

We observe that $\varphi \in \Phi \Rightarrow \varphi(t) < t$ and $\lim_{r\to t^+} \varphi(r) < t \ \forall\ t > 0$.

Now we prove our main result.

Theorem 3.1. *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X, such that (X, d) is a complete metric space. Assume there is an increasing function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(t+0) < t$ for each $t > 0$, and also suppose that $F : X \times X \to X$ is such that F has the mixed monotone property and*

$$d(F(x,y), F(u,v)) \leq \varphi\{\max(d(x,u), d(y,v))\}, \tag{3.1.1}$$

for all $x, y, u, v \in X$ for which $x \leq u$ and $y \geq v$. Suppose either
a) F is continuous (or);
b) X has the following property.
i) if a non decreasing sequence $\{x_n\} \to x$, then $x_n \leq x$, for all $n \in N$.
ii) if a non increasing sequence $\{y_n\} \to y$, then $y \leq y_n$, for all $n \in N$.
If there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$, that is, F has a coupled fixed point.

Proof. Suppose

$$u_0 \leq F(u_0, v_0) \text{ and } v_0 \geq F(v_0, u_0). \tag{3.1.2}$$

Define the sequences $\{u_n\}$ and $\{v_n\}$ by $u_1 = F(u_0, v_0)$ and $v_1 = F(v_0, u_0)$.

In general $u_{n+1} = F(u_n, v_n)$ and $v_{n+1} = F(v_n, u_n)$, for $n = 0, 1, 2, ...$

From (3.1.2), $u_0 \leq F(u_0, v_0) = u_1$. Therefore $u_0 \leq u_1$ and $v_0 \geq F(v_0, u_0) = v_1$. Therefore $v_0 \geq v_1$. Now $u_2 = F(u_1, v_1) \geq F(u_0, v_1) \geq F(u_0, v_0) = u_1$.

Therefore $u_2 \geq u_1$. And $v_2 = F(v_1, u_1) \leq F(v_0, u_1) \leq F(v_0, u_0) = v_1$. Therefore $v_2 \leq v_1$. Similarly $u_3 \geq u_2$ and $v_3 \leq v_2$. In general $u_{n+1} \geq u_n$ and $v_{n+1} \leq v_n$. Therefore

$$u_0 \leq u_1 \leq u_2 \leq \ldots \leq u_n \leq u_{n+1}$$

and

$$v_0 \geq v_1 \geq v_2 \ldots \geq v_n \geq v_{n+1}.$$

Therefore $\{u_n\}$ is $\uparrow$ and $\{v_n\}$ is $\downarrow$. First we show that $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences. If possible assume either $\{u_n\}$ or $\{v_n\}$ fails to be Cauchy. Then either

$$\lim_{m,n\to\infty} d(u_m, u_n) \neq 0$$

or

$$\lim_{m,n\to\infty} d(v_m, v_n) \neq 0.$$

Therefore,

$$\max\{\lim_{m,n\to\infty} d(u_m, u_n), \lim_{m,n\to\infty} d(v_m, v_n)\} \neq 0,$$

i.e

$$\lim_{m,n\to\infty} \max\{\lim_{m,n\to\infty} d(u_m, u_n), \lim_{m,n\to\infty} d(v_m, v_n)\} \neq 0,$$

i.e, there exist $\varepsilon > 0$, for which we can find sub sequences $\{m_k\}$ and $\{n_k\}$ of positive integers with $n_k > m_k > k$ such that

$$\max\{d(u_{m_k}, u_{n_k}), d(v_{m_k}, v_{n_k})\} \geq \varepsilon. \tag{3.1.3}$$

Further, we choose $\{n_k\}$ to be the smallest positive integer such that $n_k > m_k$ satisfying (3.1.1).

Hence, we have $\max\{d(u_{m_k}, u_{n_k}), d(v_{m_k}, v_{n_k})\} \geq \varepsilon$ and

$$\max\{d(u_{m_k}, u_{n_k-1}), d(v_{m_k}, v_{n_k-1})\} < \varepsilon. \tag{3.1.4}$$

Now, we prove that

I. $\lim_{k\to\infty} \max\{d(u_{n_k}, u_{m_k}), d(v_{n_k}, v_{m_k})\} = \varepsilon$;

II. $\lim_{k\to\infty} \max\{d(u_{n_k-1}, u_{m_k-1}), d(v_{n_k-1}, v_{m_k-1})\} = \varepsilon$;

III. $\lim_{k\to\infty} \max\{d(u_{m_k}, u_{n_k-1}), d(v_{m_k}, v_{n_k-1})\} = \varepsilon$.

First we prove I:

From the triangular inequality we have

$$d(u_{n_k}, u_{m_k}) \leq d(u_{n_k}, u_{n_k-1}) + d(u_{n_k-1}, u_{m_k}) < d(u_{n_k}, u_{n_k-1}) + \varepsilon \tag{3.1.5}$$

and

$$d(v_{n_k}, v_{m_k}) \leq d(v_{n_k}, v_{n_k-1}) + d(v_{n_k-1}, v_{m_k}) < d(v_{n_k}, v_{n_k-1}) + \varepsilon. \tag{3.1.6}$$

From (3.1.3),(3.1.5) and (3.1.6)

$$\varepsilon \leq \max\{d(u_{n_k}, u_{m_k}), d(v_{n_k}, v_{m_k})\} < \max\{d(u_{n_k}, u_{m_k}), d(v_{n_k}, v_{n_k-1})\}. \tag{3.1.7}$$

On letting $k \to \infty$,

$$\varepsilon \leq \lim_{k\to\infty} \max\{d(u_{n_k}, u_{m_k}), d(v_{n_k}, v_{m_k})\} \leq \varepsilon.$$

Therefore

$$\lim_{k\to\infty} \max\{d(u_{n_k}, u_{m_k}), d(v_{n_k}, v_{m_k})\} = \varepsilon$$

Therefore (I) holds.

Now we prove II:

$$\begin{aligned} d(u_{n_k-1}, u_{m_k-1}) &\leq d(u_{m_k-1}, u_{m_k}) + d(u_{m_k}, u_{n_k-1}) \\ &< d(u_{m_k-1}, u_{m_k}) + \varepsilon \text{ (} by \text{ (3.1.4))} \end{aligned}$$

and

$$\begin{aligned} d(v_{n_k-1}, v_{m_k-1}) &\leq d(v_{m_k-1}, v_{m_k}) + d(v_{m_k}, v_{n_k-1}) \\ &< d(v_{m_k-1}, v_{m_k}) + \varepsilon \text{ (} by \text{ (3.1.4))}. \end{aligned}$$

Therefore

$$\begin{aligned} &\max\{d(u_{n_k-1}, u_{m_k-1}), d(v_{n_k-1}, v_{m_k-1})\} \\ &\leq Max\{d(u_{m_k-1}, u_{m_k}), d(v_{m_k-1}, v_{m_k})\} + \varepsilon. \end{aligned}$$

Therefore

$$\limsup \max\{d(u_{n_k-1}, u_{m_k-1}), d(v_{n_k-1}, v_{m_k-1})\} \leq \varepsilon. \tag{3.1.8}$$

Now

$$d(u_{n_k}, u_{m_k}) \leq d(u_{n_k}, u_{n_k-1}) + d(u_{n_k-1}, u_{m_k-1}) + d(u_{m_k-1}, u_{m_k})$$

and

$$d(v_{n_k}, v_{m_k}) \leq d(v_{n_k}, v_{n_k-1}) + d(v_{n_k-1}, v_{m_k-1}) + d(v_{m_k-1}, v_{m_k}).$$

Therefore

$$\begin{aligned} \{d(u_{n_k}, u_{m_k}), d(v_{n_k}, v_{m_k})\} \leq & \max\{d(u_{n_k}, u_{n_k-1}), d(v_{n_k-1}, v_{n_k})\} \\ & + \max\{d(v_{n_k-1}, v_{m_k-1}), d(v_{m_k-1}, u_{m_k-1})\} \\ & + \max\{d(u_{m_k-1}, u_{m_k}), d(v_{m_k-1}, v_{m_k})\}. \end{aligned}$$

Letting $k \to \infty$, we get

$$\begin{aligned} 0 &\leq \liminf \max\{d(u_{n_k-1}, u_{m_k-1}), d(v_{n_k-1}, v_{m_k-1})\} \\ &\leq \limsup \max\{d(u_{n_k-1}, u_{m_k-1}), d(v_{n_k-1}, v_{m_k-1})\} \leq \varepsilon. \end{aligned}$$

Therefore

$$\lim \max\{d(u_{n_{k-1}}, u_{m_{k-1}}), d(v_{n_{k-1}}, v_{m_{k-1}})\} = \varepsilon.$$

Therefore (II) holds. Now

$$\begin{aligned} d(u_{n_k}, u_{m_k}) &\leq d(u_{n_k}, u_{n_k-1}) + d(u_{n_k-1}, u_{m_k}) \\ &\leq d(u_{n_k}, u_{n_k-1}) + d(u_{n_k-1}, u_{m_k-1}) + d(u_{m_k-1}, u_{m_k}) \end{aligned}$$

and

$$\begin{aligned} d(v_{n_k}, v_{m_k}) &\leq d(v_{n_k}, v_{n_k-1}) + d(v_{n_k-1}, v_{m_k}) \\ &\leq d(v_{n_k}, v_{n_k-1}) + d(v_{n_k-1}, v_{m_k-1}) + d(v_{m_k-1}, v_{m_k}). \end{aligned}$$

Therefore

$$\begin{aligned} \max\{d(u_{n_k}, u_{m_k}), d(v_{n_k}, v_{m_k})\} &\leq \max\{d(u_{n_k}, u_{n_k-1}), d(v_{n_k}, v_{n_k-1})\} \\ &\quad + \max\{d(u_{n_k-1}, u_{m_k}), d(v_{n_k-1}, v_{m_k})\} \\ &\leq \max\{d(u_{n_k}, u_{n_k-1}), d(v_{n_k}, v_{n_k-1})\} \\ &\quad + \max\{d(u_{n_k-1}, u_{m_k-1}), d(v_{n_k-1}, v_{m_k-1})\} \\ &\quad + \max\{d(u_{m_k-1}, u_{m_k}), d(v_{m_k-1}, v_{m_k})\}. \end{aligned}$$

On letting $k \to \infty$, from (I), (II), we get (III), since

$$\varepsilon \leq 0 + \lim \max\{d(u_{n_k-1}, u_{m_k}), d(v_{n_k-1}, v_{m_k})\} \leq \varepsilon.$$

Now we have

$$\begin{aligned} \varepsilon &\leq \liminf \max\{d(u_{n_k-1}, u_{m_k-1}), d(v_{n_k-1}, v_{m_k-1})\} \\ &\leq \limsup \max\{d(u_{n_k-1}, u_{m_k-1}), d(v_{n_k-1}, v_{m_k-1})\} = \varepsilon. \end{aligned}$$

Since $u_{n_k-1} \geq u_{m_k-1}$ and $v_{n_k-1} \leq v_{m_k-1}$. From (3.1.1),we get

$$\begin{aligned} d(u_{n_k}, u_{m_k}) &= d(F(u_{n_k-1}, v_{n_k-1}), F(u_{m_k-1}, v_{m_k-1})) \\ &\leq \varphi\{\max\{d(u_{n_k-1}, u_{m_k-1}), d(v_{n_k-1}, v_{m_k-1})\}\} \end{aligned} \tag{3.1.9}$$

similarly

$$\begin{aligned} d(v_{n_k}, v_{m_k}) &= d(F(v_{n_k-1}, u_{m_k-1}), F(v_{m_k-1}, u_{n_k-1})) \\ &\leq \varphi\{\max\{d(v_{n_k-1}, v_{m_k-1}), d(u_{m_k-1}, u_{n_k-1})\}\}. \end{aligned} \tag{3.1.10}$$

From (3.1.9) and (3.1.10),we have

$$\varepsilon \leq s_k \leq \varphi(p_k) < p_k. \tag{3.1.11}$$

Where $s_k = \max\{d(u_{n_k}, u_{m_k}), d(v_{n_k}, v_{m_k})\}$. Therefore $s_k \to \varepsilon$. and $p_k = \max\{d(u_{n_k-1}, u_{m_k-1}), d(v_{n_k-1}, v_{m_k-1})\}$. On letting $k \to \infty$, from (3.1.11), we have $\varepsilon \leq \lim \varphi(p_k) \leq \varepsilon$. Therefore $\varepsilon = \lim \varphi(p_k) < \varepsilon$, by (I), a Contradiction. Hence $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences. Suppose $u_n \to u$ and $v_n \to v$. Suppose (a) holds. Then F is continuous, hence $u_{n+1} = F(u_n, v_n) \to F(u, v)$. So $u = F(u, v)$, since $u_{n+1} \to u$. Similarly $v_{n+1} = F(v_n, u_n) \to F(v, u)$. So

$v = F(v, u)$, since $v_{n+1} \to v$. Therefore (u, v) is a coupled fixed point of F. Now suppose (b) holds. Then $u_n \leq u$ and $v_n \geq v$ for all n.

$$\begin{aligned} d(u_{n+1}, F(u, v)) &= d(F(u_n, v_n), F(u, v)) \\ &\leq \varphi\{\max\ (d(u_n, u), d(v_n, v))\} \\ &\leq \max\{d(u_n, u), d(v_n, v)\} \to 0 \text{ } as \text{ } n \to \infty \end{aligned}$$

i.e, $d(u_{n+1}, F(u, v)) \to 0$ as $n \to \infty$. i.e, $u_{n+1} \to F(u, v)$. So $u = F(u, v)$. Similarly $v = F(v, u)$. Therefore (u, v) is a coupled fixed point of F.

Lemma 3.2. *Under the hypothesis of Theorem* 3.1*, suppose* (x, y) *is a coupled fixed point of* F *and* (u, v) *is comparable to* (x, y)*. Write* $u_0 = u$ *and* $v_0 = v$ *and construct the sequences* $\{u_n\}$ *and* $\{v_n\}$ *by* $u_{n+1} = F(u_n, v_n)$ *and* $v_{n+1} = F(v_n, u_n)$ *for* $n = 0, 1, 2, 3, \ldots$ *Then* $\{u_n\}$ *and* $\{v_n\}$ *are Cauchy sequences and* $\{u_n\} \to x$ *and* $\{v_n\} \to y$*.*

Proof. Case (i): Suppose $(u, v) \geq (x, y)$, so that

$$u \geq x \text{ and } v \leq y. \tag{3.2.1}$$

Write

$$u_0 = u \text{ and } v_0 = v, \tag{3.2.2}$$

and construct the sequences $\{u_n\}$ and $\{v_n\}$ by

$$u_{n+1} = F(u_n, v_n) \text{ and } v_{n+1} = F(v_n, u_n). \tag{3.2.3}$$

From (3.2.1) and (3.2.2) $u_0 \geq x$ and $v_0 \leq y$ for every n. Now we have to show that $u_n \geq x$ and $v_n \leq y$ for every n. From (3.2.3),$u_1 = F(u_0, v_0) \geq F(x, v_0) \geq F(x, y) = x$. Therefore $u_1 \geq x$. And $v_1 = F(v_0, u_0) \leq F(y, u_0) \leq F(y, x) = y$. Therefore $v_1 \leq y$. Similarly $u_2 \geq x$ and $v_2 \leq y$. Hence by induction $u_n \geq x$ and $v_n \leq y$ for every n.

As in theorem 3.1, we can show that $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences.

Suppose $\{u_n\} \to a$ and $\{v_n\} \to b$. Now

$$\begin{aligned} d(u_{n+1}, x) &= d(F(u_n, v_n), F(x, y)) \\ &\leq \varphi\{\max\ (d(u_n, x), d(v_n, y))\} \\ &< \max\{d(u_n, x), d(v_n, y)\} \text{ (} since \text{ } u_n \geq x \text{ } and \text{ } v_n \leq y) \end{aligned} \tag{3.2.4}$$

and

$$\begin{aligned} d(v_{n+1}, y) &= d(F(v_n, u_n), F(y, x)) \\ &\leq \varphi\{\max\ (d(v_n, y), d(u_n, x))\} \\ &< \max\{d(v_n, y), d(u_n, x)\} \text{ (} since \text{ } u_n \geq x \text{ } and \text{ } v_n \leq y). \end{aligned} \tag{3.2.5}$$

Write $A_n = \max\{d(u_n, x), d(v_n, y)\}$. Then by (3.2.4) and (3.2.5)

$$A_{n+1} \leq \varphi(A_n) < A_n. \tag{3.2.6}$$

Write $\lim_{n\to\infty} A_n = \alpha$. Then by (3.2.6), $\lim \varphi(A_n) = \alpha$. But by hypothesis $\lim \varphi(A_n) = \varphi(\alpha + 0) < \alpha$, if $\alpha > 0$, which is a contradiction. Therefore $\alpha = 0$. Therefore $\lim_{n\to\infty} A_n = 0$. Therefore

$$\max\{d(u_{n+1}, x), d(v_{n+1}, y)\} \to 0 \text{ as } n \to \infty. \tag{3.2.7}$$

Therefore $d(u_{n+1}, x) = 0$ and $d(v_{n+1}, y) = 0$ i.e,$d(a, x) = 0$ since $u_{n+1} \to a$. Therefore $a = x$ and $d(b, y) = 0$ since $v_{n+1} \to b$. Therefore $b = y$. Therefore $\{u_n\} \to x$ and $\{v_n\} \to y$.

Case (ii): Suppose $(u, v) \leq (x, y)$.

Then $u \leq x$ and $v \geq y$. We assume that $(u_0, v_0) = (u, v) \leq (x, y)$. Now $u_1 = F(u_0, v_0) \leq F(x, v_0) \leq F(x, y) = x$. Therefore $u_1 \leq x$ and $v_1 = F(v_0, u_0) \geq F(y, u_0) \geq F(y, x) = y$. Therefore $v_1 \geq y$. Similarly $u_2 = F(u_1, v_1) \leq F(x, v_1) \leq F(x, y) = x$. Therefore $u_2 \leq x$ and $v_2 = F(v_1, u_1) \geq F(y, u_1) \geq F(y, x) = y$. Therefore $v_2 \geq y$. Thus by induction follows that $u_n \leq x$ and $v_n \geq y$ for every n. As in Theorem (3.1), we can show that $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences. Suppose $\{u_n\} \to a$ and $\{v_n\} \to b$. Now

$$\begin{aligned} d(u_{n+1}, x) &= d(F(u_n, v_n), F(x, y)) \\ &\leq \varphi\{\max\ (d(u_n, x), d(v_n, y))\} \\ &< \max\{d(u_n, x), d(v_n, y)\} \text{ (since } u_n \geq x \text{ and } v_n \leq y) \end{aligned}$$

and

$$\begin{aligned} d(v_{n+1}, y) &= d(F(v_n, u_n), F(y, x)) \\ &\leq \varphi\{\max\ (d(v_n, y), d(u_n, x))\} \\ &< \max\{d(v_n, y), d(u_n, x)\} \text{ (since } u_n \geq x \text{ and } v_n \leq y). \end{aligned}$$

Therefore $\max\{d(u_{n+1}, x), d(v_{n+1}, y)\} \to 0$ as $n \to \infty$ (as in (3.2.7)) i.e, $d(u_{n+1}, x) = 0$ and $d(v_{n+1}, y) = 0$ i.e, $d(a, x) = 0$ since $u_{n+1} \to a$. Therefore $a = x$, and $d(b, y) = 0$ since $v_{n+1} \to b$. Therefore $b = y, \{u_n\} \to x$ and $\{v_n\} \to y$. □

Theorem 3.3. *In addition to the hypothesis of Theorem 3.1, suppose that for every* $(x, y), (z, t) \in X \times X$, *there exists a* $(u, v) \in X \times X$, *which is comparable to* (x, y) *and* (z, t). *Then* F *has a unique coupled fixed point.*

Proof. Suppose (x, y) and (z, t) are coupled fixed points of F.

Suppose (u, v) is comparable with (x, y) and (z, t). Let $\{u_n\}$ and $\{v_n\}$ be as in Lemma 3.2, since (u, v) is comparable with (x, y), $\{u_n\} \to x$ and $\{v_n\} \to y$. Similarly, since (u, v) is comparable with (z, t) , again by Lemma 3.2, $\{u_n\} \to z$ and $\{v_n\} \to t$. Therefore $x = z$ and $y = t$. Therefore F has a unique coupled fixed point. □

Theorem 3.4. *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X, such that (X, d) is a complete metric space. Assume there is an increasing function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(t + 0) < t$ for each $t > 0$, and also suppose that $F : X \times X \to X$ and $g : X \to X$ are such that F has the mixed g-monotone property and*

$$d(F(x, y), F(u, v)) \leq \varphi\{\max(d(g(x), g(u)), d(g(y), g(v)))\}, \tag{3.4.1}$$

for all $x, y, u, v \in X$ for which $g(x) \leq g(u)$ and $g(y) \geq g(v)$.

Suppose that $F(X \times X) \subseteq g(X)$, g is continuous and commutes with F and also Suppose either

a) F is continuous (or);

b) X has the following property.

i) if a non decreasing sequence $\{x_n\} \to x$, then $x_n \leq x$ for all $n \in N$.

ii) if a non increasing sequence $\{y_n\} \to y$, then $y \leq y_n$ for all $n \in N$.

If there exist $x_0, y_0 \in X$ such that $g(x_0) \leq F(x_0, y_0)$ and $g(y_0) \geq F(y_0, x_0)$, then there exist $x, y \in X$ such that $g(x) = F(x, y)$ and $g(y) = F(y, x)$, that is, F and g have a coupled coincidence point.

Proof. Suppose that $x_0, y_0 \in X$ such that

$$g(x_0) \leq F(x_0, y_0) \text{ and } g(y_0) \geq F(y_0, x_0). \tag{3.4.2}$$

Since

$$F(X \times X) \subseteq g(X) \tag{3.4.3}$$

we can choose $x_1, y_1 \in X$ such that $g(x_1) = F(x_0, y_0)$ and $g(y_1) = F(y_0, x_0)$. Again from (3.4.3), we can choose $x_2, y_2 \in X$ such that $g(x_2) = F(x_1, y_1)$ and $g(y_2) = F(y_1, x_1)$. Continuing this process, inductively we construct the sequences $\{g(x_n)\}$ and $\{g(y_n)\}$ in X such that

$$g(x_{n+1}) = F(x_n, y_n) \text{ and } g(y_{n+1}) = F(y_n, x_n) \; for \; n = 0, 1, 2..... \tag{3.4.4}$$

We show that

$$g(x_n) \leq g(x_{n+1}) \text{ for } n = 0, 1, 2, ... \tag{3.4.5}$$

and

$$g(y_n) \geq g(y_{n+1}) \text{ for } n = 0, 1, 2, ... \tag{3.4.6}$$

From (3.4.2), $g(x_0) \leq F(x_0, y_0)$ and $g(y_0) \geq F(y_0, x_0)$. Therefore from (3.4.3), $g(x_1) = F(x_0, y_0)$ and $g(y_1) = F(y_0, x_0)$. Therefore $g(x_0) \leq g(x_1)$ and $g(y_0) \geq g(y_1)$. Thus (3.4.5) and (3.4.6) hold for $n = 0$. Suppose that (3.4.5) and (3.4.6) hold for some $n \geq 0$. Then, since $g(x_n) \leq g(x_{n+1})$ and $g(y_n) \geq g(y_{n+1})$, and as F has the mixed g-monotone property, from (3.4.3),

$$g(x_{n+1}) = F(x_n, y_n) \leq F(x_{n+1}, y_n), F(y_{n+1}, x_n) \leq F(y_n, x_n) = g(y_{n+1}). \tag{3.4.7}$$

Again from (3.4.3),

$$g(x_{n+2}) = F(x_{n+1}, y_{n+1}) \geq F(x_{n+1}, y_n),$$
$$F(y_{n+1}, x_n) \geq F(y_{n+1}, x_{n+1}) = g(y_{n+2}). \tag{3.4.8}$$

From (3.4.7) and (3.4.8), we get $g(x_{n+1}) \leq g(x_{n+2})$ and $g(y_{n+1}) \geq g(y_{n+2})$. Thus by mathematical induction we conclude that (3.4.5) and (3.4.6) hold for all $n \geq 0$. Therefore

$$g(x_0) \leq g(x_1) \leq g(x_2) \leq ....g(x_n) \leq g(x_{n+1}) \leq ..... \tag{3.4.9}$$

and

$$g(y_0) \geq g(y_1) \geq g(y_2) \geq ....g(y_n) \geq g(y_{n+1}) \geq .... \tag{3.4.10}$$

Therefore $\{g(x_n)\} is \uparrow$ and $\{g(y_n)\} is \downarrow$.

Denote $\delta_n = \max\{d((g(x_n),\ g(x_{n+1})), d((g(y_n), g(y_{n+1}))\}$. We show that

$$\delta_n \leq \varphi(\delta_{n-1}). \tag{3.4.11}$$

Since $g(x_{n-1}) \leq g(x_n)$ and $g(y_{n-1}) \geq g(y_n)$ from (3.4.3) and (3.4.1), we have

$$\begin{aligned} d((g(x_n), g(x_{n+1})) &= d((F(x_{n-1}, y_{n-1}), (F(x_n, y_n)) \\ &\leq \varphi\{\max((d(g(x_{n-1}), g(x_n)), (d(g(y_{n-1}), g(y_n)))\} \\ &= \varphi(\delta_{n-1}). \end{aligned} \tag{3.4.12}$$

Similarly

$$\begin{aligned} d((g(y_n), g(y_{n+1})) &= d((F(y_{n-1}, x_{n-1}), (F(y_n, x_n)) \\ &\leq \varphi\{\max((d(g(y_{n-1}), g(y_n)), (d(g(x_{n-1}), g(x_n)))\} \\ &= \varphi(\delta_{n-1}). \end{aligned} \tag{3.4.13}$$

From (3.4.12) and (3.4.13) we obtain (3.4.11). From (3.4.11),since $\varphi(t) < t$ for $t > 0$, it follows that sequence $\{\delta_n\}$ is monotone decreasing. Therefore there is some $\delta \geq 0$ such that $\lim_{n\to\infty} \delta_n = \delta$. We show that $\delta = 0$. Suppose to the contrary that $\delta > 0$. Then taking the limit as $n \to \infty$ on both sides of (3.4.11), we have $\delta = \lim_{n\to\infty} \delta_n \leq \lim_{n\to\infty} \varphi(\delta_{n-1}) = \varphi(\delta + 0) < \delta$, a contradiction. Thus $\delta = 0$,

$$\lim_{n\to\infty}\{\max((d(g(x_n), g(x_{n+1})), d(g(y_n), g(y_{n+1})))\} = 0. \tag{3.4.14}$$

Now we prove that $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy sequences.

Suppose to the contrary that at least one of $\{g(x_n)\}$ or $\{g(y_n)\}$ is not a Cauchy sequence.

Then there exist an $\epsilon > 0$, and two sub sequences of integers $\{l_k\}$ and $\{m_k\}$, $m_k > l_k \geq k$ with

$$r_k = \max(d(g(x_{l_k}), g(x_{m_k})), d(g(y_{l_k}), g(y_{m_k})) > \epsilon \text{ for } k = 1, 2, 3... \tag{3.4.15}$$

We may also assume

$$\max\left(d(g(x_{l_k}), g(x_{m_k})), d(g(y_{l_k}), g(y_{m_k})\right) \leq \epsilon \tag{3.4.16}$$

by choosing m_k to be the smallest number exceeding l_k for which(3.4.15) holds. From (3.4.15), (3.4.16) and by the triangular inequality

$$\begin{aligned}\epsilon < \max\{&(d(g(x_{l_k}), g(x_{m_k-1})) + d(g(x_{m_k-1}), g(x_{m_k}))),\\ &(d(g(y_{l_k}), g(y_{m_k-1})) + d(g(y_{m_k-1}), g(y_{m_k}))\}.\end{aligned}$$

Taking the limit as $k \to \infty$ we get by (3.4.14)

$$\lim_{k\to\infty} r_k = \epsilon^+ \tag{3.4.17}$$

Since from (3.4.3) and (3.4.1), $g(x_{l_k}) \leq g(x_{m_k})$ and $g(y_{l_k}) \geq g(y_{m_k})$ we have

$$\begin{aligned}(d(g(x_{l_{k+1}}), g(x_{m_{k+1}})) &= d(F(x_{l_k}, y_{l_k}), F(x_{m_k}, y_{m_k})\\ &\leq \varphi\{\max((d(g(x_{l_k}), g(x_{m_k})), d(g(y_{l_k}), g(y_{m_k}))\}\\ &= \varphi(r_k).\end{aligned} \tag{3.4.18}$$

Similarly

$$\begin{aligned}(d(g(y_{l_{k+1}}), g(y_{m_{k+1}})) &= d(F(y_{l_k}, x_{l_k}), F(y_{m_k}, x_{m_k})\\ &\leq \varphi\{\max((d(g(y_{l_k}), g(y_{m_k})), d(g(x_{l_k}), g(x_{m_k}))\}\\ &= \varphi(r_k).\end{aligned} \tag{3.4.19}$$

From (3.4.18) and (3.4.19), $\epsilon < r_{k+1} \leq \varphi(r_k)$. Taking $k \to \infty$, using (3.4.14) and (3.4.17) we get

$$\epsilon \leq \lim_{k\to\infty}(\varphi(r_k)) = \varphi(\varepsilon + 0) < \epsilon, \text{ a contradiction.}$$

Thus our supposition is wrong. Therefore $\{g(x_n)\}$and $\{g(y_n)\}$ are Cauchy sequences. Since X is complete,there exist $x, y \in X$such that

$$\lim_{n\to\infty} g(x_n) = x \text{ and } \lim_{n\to\infty} g(y_n) = y. \tag{3.4.20}$$

From (3.4.20) and continuity of g,

$$\lim_{n\to\infty} g(g(x_n)) = g(x) \text{ and } \lim_{n\to\infty} g(g(y_n)) = g(y). \tag{3.4.21}$$

Since F and g commute, from (3.4.21)

$$g(g(x_{n+1})) = g(F(x_n, y_n)) = F(g((x_n), g(y_n)) \tag{3.4.22}$$

and

$$g(g(y_{n+1})) = g(F(y_n, x_n)) = F(g((y_n), g(x_n)). \tag{3.4.23}$$

We show that $g(x) = F(x, y)$ and $g(y) = F(y, x)$.

Suppose (a) holds. Taking the limit as $n \to \infty$ in (3.4.22) and (3.4.23) by (3.4.20),(3.4.21) and continuity of F, we get

$$\begin{aligned} g(x) = \lim_{n\to\infty} g(g(x_{n+1})) &= \lim_{n\to\infty} F(g((x_n), g(y_n)) \\ &= F(\lim_{n\to\infty} g((x_n), \lim_{n\to\infty} g(y_n)) \\ &= F(x, y) \end{aligned}$$

and

$$\begin{aligned} g(y) = \lim_{n\to\infty} g(g(y_{n+1})) &= \lim_{n\to\infty} F(g((y_n), g(x_n)) \\ &= F(\lim_{n\to\infty} g((y_n), \lim_{n\to\infty} g(x_n)) \\ &= F(y, x). \end{aligned}$$

Therefore $g(x) = F(x, y)$ and $g(y) = F(y, x)$.

Suppose (b) holds. Since $\{g(x_n)\}$ is non decreasing and $g(x_n) \to x$ and $\{g(y_n)\}$ is non increasing and $g(y_n) \to y$ from hypothesis we have $g(x_n) \leq x$ and $g(y_n) \geq y$, for all n. Then by the triangular inequality (3.4.22),(3.4.23) and (3.4.1),we get

$$\begin{aligned} d(g(x), F(x, y))) &\leq d(g(x), g(g(x_{n+1}))) + d(g(g(x_{n+1})), F(x, y)) \\ &= \{d(g(x), g(g(x_{n+1}))) + d(F(g(x_n), g(y_n))), F(x, y))\} \\ &\leq \{d(g(x), g(g(x_{n+1})))+ \\ &\quad \varphi(\max(d(g(g(x_n), g(x))), d(g(g(y_n), g(y)))))\} \to 0 \text{ as } n \to \infty. \end{aligned}$$

Therefore $d(g(x), F(x, y))) \leq 0$. Therefore $g(x) = F(x, y)$. Similarly $g(y) = F(y, x)$. Therefore F and g have a coupled coincidence point. □

Lemma 3.5. *Suppose (x, y) is coupled coincidence point of F and g and $(g(u), g(v))$ is comparable with $(g(x), g(y))$. Write $u_0 = u$ and $v_0 = v$ and construct the sequences $\{g(u_n)\}$ and $\{g(v_n)\}$ by $g(u_{n+1}) = F(u_n, v_n)$ and $g(v_{n+1}) = F(v_n, u_n)$ for $n = 0, 1, 2, ...$ Then $g(u_n) \to g(x)$ and $g(v_n) \to g(y)$.*

Proof. Case (i): $(g(u), g(v)) \leq (g(x), g(y))$. Then $g(u) \leq g(x)$ and $g(v) \geq g(y)$. Write $u_0 = u$ and $v_0 = v$. Then $(g(u_0), g(v_0)) = (g(u), g(v)) \leq (g(x), g(y))$. Therefore $g(u_0) \leq g(x)$ and $g(v_0) \geq g(y)$. Choose u_1,v_1 such that $g(u_1) = F(u_0, v_0)$ and $g(v_1) = F(v_0, u_0)$. Therefore $g(u_1) = F(u_0, v_0) \leq F(x, v_0) \leq F(x, y) = g(x)$. Therefore $g(u_1) \leq g(x)$ and $g(v_1) = F(v_0, u_0) \geq F(y, u_0) \leq F(y, x) = g(y)$. Therefore $g(v_1) \geq g(y)$. In general $(g(u_n), g(v_n)) \leq (g(x), g(y)$. Now

$$\begin{aligned} d(g(u_n), g(x)) &= d(F(u_{n-1}, v_{n-1}), F(x, y)) \\ &\leq \varphi\{\max(d(g(u_{n-1}), g(x)), d(g(v_{n-1}), g(y)))\} \end{aligned}$$

and

$$\begin{aligned} d(g(v_n), g(y)) &= d(F(v_{n-1}, u_{n-1}), F(y, x)) \\ &\leq \varphi\{\max(d(g(v_{n-1}), g(y)), d(g(u_{n-1}), g(x)))\}. \end{aligned}$$

Let $r_k = \max\{d(g(u_n), g(x)), d(g(v_n), g(y))\} \leq \varphi(r_{k-1}) < r_{k-1}$. Therefore $\{r_k\}$ is a decreasing sequence. Suppose $r_k \downarrow \alpha$. Hence $r_k \leq \varphi(r_{k-1}) < r_{k-1}$. Letting $k \to \infty$,we get $\lim_{k\to\infty} \varphi(r_k) = \alpha$. But by hypothesis $\lim \varphi(r_k) = \varphi(\alpha + 0) < \alpha$, if $\alpha > 0$. Therefore $\alpha = 0$. Therefore $\lim_{k\to\infty} r_k = 0$. Therefore $\max\{d(g(u_n), g(x)), d(g(v_n), g(y))\} \to 0$ as $n \to \infty$. Therefore $d(g(u_n), g(x)) \to 0$ as $n \to \infty$ and $d(g(v_n), g(y)) \to 0$ as $n \to \infty$. Therefore $g(u_n) \to g(x)$ and $g(v_n) \to g(y)$.

Case(ii): Suppose $(g(u), g(v)) \geq (g(x), g(y))$. In this case the proof is similar to case (i). □

Theorem 3.6. *In addition to the hypothesis of Theorem 3.4, suppose* $(x, y), (z, t) \in X \times X \Rightarrow$ *there exist* $u, v \in X \times X$ *such that* $(g(u), g(v))$ *is comparable with* $(g(x), g(y))$ *and* $(g(z), g(t))$*.Then F and g have unique coupled coincidence point,in the sense that* $g(x) = g(z)$ *and* $g(y) = g(t)$.

Proof. Suppose $(x, y) and (z, t)$ are coupled coincidence points of F and g.

Suppose $(g(u), g(v))$ is comparable with $(g(x), g(y))$ and $(g(z), g(t))$.

Let $\{u_n\}$ and $\{v_n\}$ be in Lemma 3.5,$g(u_n) \to g(x)$ and $g(v_n) \to g(y)$.

Similarly, since $(g(u), g(v))$ is comparable with $(g(z), g(t))$, again by Lemma 3.5, $g(u_n) \to g(z)$ and $g(v_n) \to g(t)$. Therefore $g(x) = g(z)$ and $g(y) = g(t)$. Now $F(x, y) = g(x) = g(z) = F(z, t)$ and $F(y, x) = g(y) = g(t) = F(t, z)$. □

Note: In view of the comment made at the beginning of this section, Theorem 2.3, 2.4, 2.8, 2.10 and 2.11 follow as corollaries to Theorems 3.1, 3.3, 3.4 and 3.6 of this section.

Corollary 3.7. *Let* $(X, \leq)$ *be a partially ordered set and suppose there is a metric d on X, such that* (X, d) *is a complete metric space. Suppose that* $F : X \times X \to X$ *and* $g : X \to X$ *are such that F has the mixed g-monotone property and assume there is a* $k \in [0, 1)$ *such that*

$$d(F(x, y), F(u, v)) \leq k\{\max(d(g(x), g(u)), d(g(y), g(v)))\},$$

for all $x, y, u, v \in X$ *for which* $g(x) \leq g(u)$ *and* $g(y) \geq g(v)$.

Suppose that $F(X \times X) \subseteq g(X)$, *g is continuous and also Suppose either*

(a) *F is Continuous (or);*

(b) *X has the following property.*

i) if a non decreasing sequence $\{x_n\} \to x$*,then* $x_n \leq x$ *for all* $n \in N$.

ii) if a non increasing sequence $\{y_n\} \to y$*,then* $y \leq y_n$ *for all* $n \in N$.

If there exist $x_0, y_0 \in X$ *such that* $g(x_0) \leq F(x_0, y_0)$ *and* $g(y_0) \geq F(y_0, x_0)$, *then there exist* $x, y \in X$ *such that* $g(x) = F(x, y)$ *and* $g(y) = F(y, x)$, *that is, F and g have a coincidence point.*

Proof. Taking $\varphi(t) = kt$ where $k \in [0, 1)$ in Theorem 3.4, we obtain Corollary 3.7. □

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