ON SUBCLASS OF MEROMORPHIC UNIVALENT FUNCTIONS DEFINED BY A LINEAR OPERATOR ASSOCIATED WITH λ-GENERALIZED HURWITZ-LERCH ZETA FUNCTION AND q-HYPERGEOMETRIC FUNCTION

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Abstract. In this article, a linear operator associated with the λ-generalized Hurwitz-Lerch zeta function and q-hypergeometric function by using the Hadamard product (or convolution) is defined by the authors, a different interesting properties of certain subclass of meromorphic univalent functions related to a linear operator in the punctured unit disk are introduced and investigated. The authors also consider some closely related (known or new) corollaries and consequences of the main results presented in this paper.
Keywords: analytic functions, meromorphic functions, univalent functions, Hadamard product or (convolution), λ-generalized Hurwitz-Lerch zeta function, q-hypergeometric function, Srivastava-Attiya operator.

1. Introduction, definitions and preliminaries

Normally, we are considering the class of meromorphic function f of the form

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n, \tag{1.1}$$

in the punctured open unit disc $\mathbb{U}^* = \{z : z \in \mathbb{C}, \text{ and } 0 < |z| < 1\} = \mathbb{U} \setminus \{0\}$ and denoted by Σ.

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The set of complex numbers is, as usual $\mathbb{C}$. By using $\Sigma S^*(\beta)$ and $\kappa(\beta)$ $(\beta \geq 0)$, we denote the subclasses of Σ that encompass all of the meromorphic functions, which are, starlike of the order β and convex of order β in $\mathbb{U}^*$, respectively (see also the recent studies [33], [34]).

In the case of the functions $f_j(z)$ $(j = 1, 2)$ which have been defined by

$$f_j(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,j} z^n, \quad (j = 1, 2),$$

the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ can be dented by using

$$(f_1 * f_2)(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,1} a_{n,2} z^n.$$

For complex parameters $\beta_1, ..., \beta_m$ $(\beta_j \neq 0, -1, ...; j = 1, 2, ..., m)$ and $\alpha_1, ..., \alpha_l$ the q-hypergeometric function ${}_l\Psi_m(z)$ can be defined as

$$\begin{aligned} {}_l\Psi_m(\alpha_1, ..., \alpha_l; \beta_1, ..., \beta_m; q, z) := \sum_{n=0}^{\infty} & \frac{(\alpha_1, q)_n ... (\alpha_l, q)_n}{(q, q)_n (\beta_1, q)_n ... (\beta_m, q)_n} \\ & \times \left[(-1)^n q^{\binom{n}{2}}\right]^{1+m-l} z^n, \end{aligned} \tag{1.2}$$

with $\binom{n}{2} = n(n-1)/2$ where $q \neq 0$ when $l > m+1$ $(l, m \in \mathbb{N}_0 = \mathbb{N} \bigcup \{0\}; z \in \mathbb{U})$.

Also the q-shifted factorial can be defined for $\alpha, q \in \mathbb{C}$ as a product of n factors by using

$$(\alpha; q)_n = \begin{cases} (1-\alpha)(1-\alpha q) ... \left(1 - \alpha q^{n-1}\right), & (n \in \mathbb{N}) \\ 1, & (n = 0), \end{cases} \tag{1.3}$$

and in terms of basic analogue of the gamma function

$$(q^\alpha; q)_n = \frac{\Gamma_q(\alpha + n)(1-q)^n}{\Gamma_q(\alpha)}, \qquad n > 0. \tag{1.4}$$

Interesting to note is that, $\lim_{q \to -1}((q^\alpha; q)_n/(1-q)^n) = (\alpha)_n = \alpha(\alpha+1)...(\alpha+n-1)$ is the common Pochhammer symbol, and

$${}_l\Psi_m(\alpha_1, ..., \alpha_l; \beta_1, ..., \beta_m; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n ... (\alpha_l)_n}{(\beta_1)_n ... (\beta_m)_n} \frac{z^n}{n!}. \tag{1.5}$$

Now for $z \in \mathbb{U}$, $0 < |q| < 1$, and $l = m + 1$, the form of

$${}_l\Psi_m(\alpha_1, ..., \alpha_l; \beta_1, ..., \beta_m; q, z) = \sum_{n=0}^{\infty} \frac{(\alpha_1, q)_n ... (\alpha_l, q)_n}{(q, q)_n (\beta_1, q)_n ... (\beta_m, q)_n} z^n, \tag{1.6}$$

is taken by the basic hypergeometric function which is defined in (1.2), and converges absolutely in the open unit disk $\mathbb{U}$ see [1].

Huda and Darus [1] and K.A Challab et al. [[2], [5]] introduced and studied a q-analogue of the Liu-Srivastava operator in correspondence with the function ${}_l\Psi_m(\alpha_1,...,\alpha_l;\beta_1,...,\beta_m;q,z)$ for the meromorphic functions of $f\in\Sigma$, which consist of the functions in the form of (1.1), and as presented below:

$$\begin{aligned}&{}_l\Upsilon_m(\alpha_1,...,\alpha_l;\beta_1,...,\beta_m;q,z)*f(z)\\ &=\frac{1}{z}{}_l\Psi_m(\alpha_1,...,\alpha_l;\beta_1,...,\beta_m;q,z)*f(z)\\ &=\frac{1}{z}+\sum_{n=1}^{\infty}\frac{\prod_{i=1}^{l}(\alpha_i,q)_{n+1}}{(q,q)_{n+1}\prod_{i=1}^{m}(\beta_i,q)_{n+1}}a_nz^n,\end{aligned} \tag{1.7}$$

where $\prod_{k=1}^{s}(\alpha_k,q)_{n+1}=(\alpha_1,q)_{n+1}(\alpha_2,q)_{n+1}...(\alpha_s,q)_{n+1}$, where $z\in\mathbb{U}^*:=\{z\in\mathbb{C}:0<|z|<1\}$.

Recently, Ghanim ([8]; see also [9]) introduced the function $G_{s,a}$ which defined by

$$\begin{aligned}&G_{s,a}:=(a+1)^s\left[\Phi(z,s,a)-a^s+\frac{1}{z(a+1)^s}\right],\\ &G_{s,a}=\frac{1}{z}+\sum_{n=1}^{\infty}\left(\frac{a+1}{a+n}\right)^s z^n,\quad(z\in\mathbb{U}^*).\end{aligned} \tag{1.8}$$

Also, the function $\Phi(z,s,a)$ be the well-known Hurwitz-Lerch zeta function as was defined by (see, e.g. [[28], p. 121 et seq.]; see also [[24], [23], p. 194 et seq.])

$$\Phi(z,s,a):=\sum_{n=0}^{\infty}\frac{z^n}{(n+a)^s}(a\in\mathbb{C}\setminus\mathbb{Z}_0^-;s\in\mathbb{C}\text{ when }|z|<1;\Re(s)>1\text{ when }|z|=1).$$

We recollect here that, Srivastava introduced and systematically investigated the following new group of λ-generalised Hurwitz-Lerch zeta functions (see for example, [3], [4], [17], [18], [19], [21], [22], [26], [27], [29], and [32]):

$$\Phi^{(\rho_1,...,\rho_p,\sigma_1,...,\sigma_r)}_{\lambda_1,...,\lambda_p;\mu_1,...,\mu_r}(z,s,a;b,\lambda)$$

$$\frac{1}{\lambda\Gamma(s)}.\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(\lambda_j)_{n\rho_j}}{(a+n)^s.\prod_{j=1}^{r}(\mu_j)_{n\sigma_j}}H^{2,0}_{0,2}\left[(a+n)b^{\frac{1}{\lambda}}\mid\overline{(s,1),\left(0,\frac{1}{\lambda}\right)}\right]\frac{z^n}{n!} \tag{1.9}$$

$$(\min\{\Re(a),\Re(s)\}>0;\Re(b)>0;\lambda>0),$$

where

$$\Big(\lambda_j \in \mathbb{C}(j=1,...,p) \;\; and \;\; \mu_j \in \mathbb{C}\setminus \mathbb{Z}_0^-(j=1,...,r); \rho_j > 0(j=1,...,p);$$
$$\sigma_j > 0(j=1,...,r); 1+\sum_{j=1}^{r}\sigma_J - \sum_{j=1}^{p}\rho_j \geq 0\Big)$$

it was found that the equality in the convergence condition remained true for the suitably bounded values of $|z|$, which

$$|z| < \nabla := \left(\prod_{j=1}^{p}\rho_j^{-\rho_j}\right).\left(\prod_{j=1}^{r}\sigma_j^{\sigma_j}\right)$$

had given.

Definition 1.1. *The H-function which was involved on the right-hand side of* (1.9) *was the well-known Fox's H-function* [[12], *Definition* 1.1] (*see also* [16], [20]) *that*

$$H_{p,r}^{m,n}(z) = H_{p,r}^{m,n}\left[z\;|_{(b_1,B_1),...,(b_r,B_r)}^{(a_1,A_1),...,(a_p,A_p)}\right]$$
$$= \frac{1}{2\pi i}\int_{\ell}\Xi(s)z^{-s}ds \qquad (z\in\mathbb{C}\setminus\{0\}; |\arg(z)| < \pi)\,,$$

had defined, where

$$\Xi(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j+B_js).\prod_{j=1}^{n}\Gamma(1-a_j-A_js)}{\prod_{j=n+1}^{p}\Gamma(a_j+A_js).\prod_{j=m+1}^{r}\Gamma(1-b_j-B_js)},$$

a hollow product is also depicted as $1, m, n, p$ *and* r *are integers such that*

$$1 \leq m \leq r \;\; and \;\; 0 \leq n \leq p,$$
$$A_j > 0 \;\; (j=1,...,p) \;\; and \;\; B_j > 0 \;\; (j=1,...,r),$$
$$a_j \in \mathbb{C} \;\; (j=1,...,p) \;\; and \;\; b_j \in \mathbb{C} \;\; (j=1,...,r)$$

and ℓ *is a suitable Mellin-Barnes type contour separating the poles of the gamma functions*

$$\{\Gamma(b_j+B_js)\}_{j=1}^{m}$$

from the poles of the gamma functions

$$\{\Gamma(1-a_j+A_js)\}_{j=1}^{n}.$$

It is worth mentioning that, if we use the fact that [[32], *p.* 1496, *Remark* 7]

$$\lim_{b\to 0} H_{0,2}^{2,0}\left[(a+n)b^{\frac{1}{\lambda}} \;|\; \overline{(s,1),\left(0,\frac{1}{\lambda}\right)}\right] = \lambda\Gamma(s) \quad (\lambda > 0),$$

equation (1.8) is reduced to the form of the following:

$$\Phi^{(\rho_1,...,\rho_p,\sigma_1,...,\sigma_r)}_{\lambda_1,...,\lambda_p;\mu_1,...,\mu_r}(z,s,a;b,\lambda) = \Phi^{(\rho_1,...,\rho_p,\sigma_1,...,\sigma_r)}_{\lambda_1,...,\lambda_p;\mu_1,...,\mu_r}(z,s,a)$$

$$\sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(\lambda_j)_{n\rho_j}}{(a+n)^s.\prod_{j=1}^{r}(\mu_j)_{n\sigma_j}} \frac{z^n}{n!}. \tag{1.10}$$

Definition 1.2. *The function* $\Phi^{(\rho_1,...,\rho_p,\sigma_1,...,\sigma_r)}_{\lambda_1,...,\lambda_p;\mu_1,...,\mu_r}(z,s,a)$ *which was involved in* (1.10) *was a multiparameter extension and generalisation of the Hurwitz-Lerch zeta function* $\Phi(z,s,a)$ *that Srivastava et al. introduced* [17, *p*.503*, Equation* (6.2)] *and was defined by*

$$\Phi^{(\rho_1,...,\rho_p,\sigma_1,...,\sigma_r)}_{\lambda_1,...,\lambda_p;\mu_1,...,\mu_r}(z,s,a) =: \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(\lambda_j)_{n\rho_j}}{(a+n)^s.\prod_{j=1}^{r}(\mu_j)_{n\sigma_j}} \frac{z^n}{n!}$$

$$\Big(p,q \in \mathbb{N}_0; \lambda_j \in \mathbb{C}(j=1,...,p); a,\mu_j \in \mathbb{C}\setminus \mathbb{Z}_0^-(j=1,...,r); \rho_j,\sigma_k \in \mathbb{R}^+$$
$$(j=1,...,p;k=1,...,r); \triangle > -1 \quad when \quad s,z\in\mathbb{C}; \triangle = -1 \quad and \quad s \in \mathbb{C}$$
$$when \quad |z| < \nabla^*; \triangle = -1 \quad and \quad \Re(\Xi) > \frac{1}{2} \quad when \quad |z| = \nabla^*\Big)$$

with

$$\nabla^* := \left(\prod_{j=1}^{p} \rho_j^{-\rho_j}\right) \cdot \left(\prod_{j=1}^{r} \sigma_j^{\sigma_j}\right),$$
$$\triangle := \sum_{j=1}^{r}\sigma_j - \sum_{j=1}^{p}\rho_j \quad and \quad \Xi := s + \sum_{j=1}^{r}\mu_j - \sum_{j=1}^{p}\lambda_j + \frac{p-r}{2}.$$

Srivastava et al. [30, 31] presented and developed a new linear operator by applying new family of meromorphic λ-generalised Hurwitz-Lerch zeta functions. The convolution operator which was studied by Dziok-Srivastava [6], [7] is a generalisation of two other operators: they are the Ruscheweyh [15] operator and the Hohlov [10] operator. As a matter of fact, the Dziok-Srivastava convolution operator is in and of itself, a special case of the Srivastava-Wright operator (see, for more details[11], [25]).

We have considered the new linear operator $K^{\alpha_l}f(z)$ in this study such that,

$$K^{\alpha_l}f(z) \equiv K^{s,a,\lambda,\alpha_l,\beta_m}_{(\lambda_p),(\mu_r),b}f(z) : \Sigma \to \Sigma,$$

this has been defined by

$$K^{\alpha_l}f(z) = G^{s,a,\lambda}_{(\lambda_p),(\mu_r),b}(z) *_l \Upsilon_m(\alpha_1,...,\alpha_l;\beta_1,...,\beta_m;q,z), \tag{1.11}$$

where the Hadamard product (or convolution) of the analytical functions has been denoted by $*$, and

$$\begin{aligned} &G^{s,a,\lambda}_{(\lambda_p),(\mu_r),b}(z) \\ (1.12)\quad &:=(a+1)^s.\left[\Phi^{(1,\dots,1,1,\dots,1)}_{\lambda_1,\dots,\lambda_p;\mu_1,\dots,\mu_q}(z,s,a;b,\lambda)-\frac{a^{-s}}{\lambda\Gamma(s)}\Lambda(a,b,s,\lambda)+\frac{(a+1)^{-s}}{z}\right] \\ &=\frac{1}{z}+\sum_{n=1}^{\infty}\frac{\prod_{j=1}^{p}(\lambda_j)_n}{\prod_{j=1}^{r}(\mu_j)_n}\left(\frac{a+1}{a+n}\right)^s\frac{\Lambda(a+n,b,s,\lambda)}{\lambda\Gamma(s)}\frac{z^n}{n!} \end{aligned}$$

gave the function $G^{s,a,\lambda}_{(\lambda_p),(\mu_r),b}(z)$, with

$$\Lambda(a,b,s,\lambda):=H^{2,0}_{0,2}\left[ab^{\frac{1}{\lambda}}\mid\overline{(s,1),\left(0,\frac{1}{\lambda}\right)}\right].$$

Now

$$\begin{aligned} K^{\alpha_l}f(z)&=\frac{1}{z}+\sum_{n=1}^{\infty}\frac{\prod_{i=1}^{l}(\alpha_i,q)_{n+1}}{(q,q)_{n+1}\prod_{i=1}^{m}(\beta_i,q)_{n+1}}\frac{\prod_{j=1}^{p}(\lambda_j)_n}{\prod_{j=1}^{r}(\mu_j)_n}\left(\frac{a+1}{a+n}\right)^s \\ &\quad\cdot\frac{\Lambda(a+n,b,s,\lambda)}{\lambda\Gamma(s)}a_n\frac{z^n}{n!} \\ K^{\alpha_l}f(z)&=\frac{1}{z}+\sum_{n=1}^{\infty}\Omega^{\alpha_l,\beta_m,a,s}_{\lambda_p,\mu_r,q,b}a_n\frac{z^n}{n!} \end{aligned}$$

$$(1.13)\quad (z\in\mathbb{U}^*;\alpha,\lambda_j\in\mathbb{C}\quad(j=1,\dots,p);\beta,\mu_j\in\mathbb{C}\backslash\mathbb{Z}_0^-\quad(j=1,\dots,r);p\le r+1)$$

is obtained if (1.11) and (1.12) are combined, with

$$\min\{\Re(a),\Re(s)\}>0;\quad \lambda>0\ \ if\ \ \Re(b)>0$$

and

$$s\in\mathbb{C}\quad a\in\mathbb{C}\setminus\mathbb{Z}_0^-\quad if\quad b=0$$

where

$$\Omega^{\alpha_l,\beta_m,a,s}_{\lambda_p,\mu_r,q,b}=\frac{\prod_{i=1}^{l}(\alpha_i,q)_{n+1}}{(q,q)_{n+1}\prod_{i=1}^{m}(\beta_i,q)_{n+1}}\frac{\prod_{j=1}^{p}(\lambda_j)_n}{\prod_{j=1}^{r}(\mu_j)_n}\left(\frac{a+1}{a+n}\right)^s\frac{\Lambda(a+n,b,s,\lambda)}{\lambda\Gamma(s)}.$$

Let the class of all functions $f(z)\in\Sigma$ such that

$$\Re\left((1-\rho)\left(\frac{K^{\alpha_l}f(z)}{K^{\alpha_l}g(z)}\right)^{\mu}+\rho\left(\frac{K^{\alpha_l+1}f(z)}{K^{\alpha_l+1}g(z)}\right)\cdot\left(\frac{K^{\alpha_l}f(z)}{K^{\alpha_l}g(z)}\right)^{\mu-1}\right)>\gamma,$$
$$(z\in\mathbb{U}^*;0\le\gamma<1),$$

be denoted by $\Sigma^{s,a,\lambda,\alpha_l,\beta_m}_{(\lambda_p),(\mu_r),b}(\gamma,\delta,\mu,\rho)$, where the following inequality of:

$$\Re\left(\frac{K^{\alpha_l}g(z)}{K^{\alpha_l+1}g(z)}\right) > \delta \quad (0 \le \delta < 1; z \in \mathbb{U}^*) \tag{1.14}$$

is satisfied by $g(z) \in \Sigma$.

At this point and as follows, γ and μ are real numbers such that $0 \le \gamma < 1$ and $\mu > 0$ and $\rho \in \mathbb{C}$ with $\Re(\rho) > 0$.

Different properties of certain subclass of the meromorphically analytical function class Σ in the punctured unit disk $\mathbb{U}^*$ have been investigated. One of these function class was introduced, first, and then the properties of the linear operator

$$K^{\alpha_l+1} \equiv K^{s,a,\lambda,\alpha_l+1,\beta_m}_{(\lambda_p),(\mu_r),b} f(z)$$

were investigated.

2. Main results

The following Lemmas were needed so that the main results could be proved:

Lemma 2.1. (*see[13]*). *Let Ω be a set in the complex plane $\mathbb{C}$ and let the function $\Psi : \mathbb{C}^2 \to \mathbb{C}$ satisfy the following condition:*

$$\Psi(ir_2, a_1) \notin \Omega \quad for\ \ all\ \ real \quad r_2, a_1 \le \frac{1}{2}(1 + {r_2}^2).$$

If $q(z)$ is analytic in $\mathbb{U}^*$ with $q(0) = 1$ and $\Psi(q(z), zq'(z)) \in \Omega$ $(z \in \mathbb{U}^*)$, then $\Re\{q(z)\} > 0$.

Our first main result is now stated and proved as Theorem 2.1, which is presented below:

Theorem 2.1. *Let $f(z) \in \Sigma^{s,a,\lambda,\alpha_l,\beta_m}_{(\lambda_p),(\mu_r),b}(\gamma,\delta,\mu,\rho)$, $\alpha \in \mathbb{R} \setminus \{0\}$ and $\rho \ge 0$. Then*

$$\Re\left(\frac{K^{\alpha_l}f(z)}{K^{\alpha_l}g(z)}\right) > \frac{2\alpha\gamma\mu + \delta\rho}{2\alpha\mu + \delta\rho} \quad (0 \le \gamma < 1; \mu > 0; z \in \mathbb{U}^*), \tag{2.1}$$

where the condition (1.14) is satisfied by the function $g(z) \in \Sigma$.

Proof. Let

$$\xi = \frac{2\alpha\gamma\mu + \delta\rho}{2\alpha\mu + \delta\rho}$$

Now, let us suppose that

$$q(z) = \frac{1}{1-\xi}\left[\left(\frac{K^{\alpha_l}f(z)}{K^{\alpha_l}g(z)}\right)^{\mu} - \xi\right] \tag{2.2}$$

defines the function $q(z)$.

In that case, the function $q(z)$ is analytical in $\mathbb{U}^*$ and $q(0) = 1$. If

$$h(z) = \frac{K^{\alpha_l} g(z)}{K^{\alpha_l+1} g(z)}, \tag{2.3}$$

is put in, then, according to the hypothesis of Theorem 2.1, we get $\Re\{h(z)\} > \delta$. Then, since $f(z) \in \Sigma^{s,a,\lambda,\alpha_l,\beta_m}_{(\lambda_p),(\mu_r),b}(\gamma, \delta, \mu, \rho)$

$$\begin{aligned}&(1-\rho)\left(\frac{K^{\alpha_l} f(z)}{K^{\alpha_l} g(z)}\right)^{\mu} + \rho\left(\frac{K^{\alpha_l+1} f(z)}{K^{\alpha_l+1} g(z)}\right) \cdot \left(\frac{K^{\alpha_l} f(z)}{K^{\alpha_l} g(z)}\right)^{\mu-1}\\ &= [\xi + (1-\xi) q(z)] + \frac{\rho(1-\xi)}{\alpha\mu} h(z) z q'(z),\end{aligned}$$

is obtained by differentiating (2.2) with respect to z.

Let us use

$$\Psi(r, s) = \xi + (1-\xi) r + \left(\frac{\rho(1-\xi)}{\alpha\mu}\right) h(z) s$$

to define the function $\Psi(r, s)$. Then, if we use (2.3) and the fact that

$$\begin{aligned}&f(z) \in \Sigma^{s,a,\lambda,\alpha_l,\beta_m}_{(\lambda_p),(\mu_r),b}(\gamma, \delta, \mu, \rho),\\ &\{\Psi(q(z), zq'(z)); z \in \mathbb{U}^*\} \subset \Omega = \{w \in \mathbb{C} : \Re(w) > \gamma\}\end{aligned}$$

is obtained.

At this point, for all real numbers $r_2, a_1 \leq \frac{1}{2}(1 + {r_2}^2)$, we get

$$\begin{aligned}\Re\{\Psi(ir_2, a_1)\} &= \xi + \left(\frac{\rho(1-\xi)}{\alpha\mu}\right) \Re\left(\frac{K^{\alpha_l} g(z)}{K^{\alpha_l+1} g(z)}\right)\\ &\leq \xi - \frac{\rho\delta(1-\xi)(1+{r_2}^2)}{2\alpha\mu} \leq \xi - \frac{\rho\delta(1-\xi)}{\alpha\mu} =: \gamma.\end{aligned}$$

Hence, for every $z \in \mathbb{U}^*$, we have $\Psi(ir_2, a_1) \notin \Omega$. Therefore, by using Lemma 2.1, we get that $\Re\{q(z)\} > 0$, which is,

$$\Re\left(\frac{K^{\alpha_l} f(z)}{K^{\alpha_l} g(z)}\right)^{\mu} > \xi \quad (z \in \mathbb{U}^*).$$

The proof of Theorem 2.1 is obviously completed at this point. □

Corollary 2.1. *If the functions of $f(z)$ and $g(z)$ are in the class Σ and also suppose that condition (1.14) is satisfied by the function of $g(z)$, if the $\alpha \in \mathbb{R} \setminus \{0\}$, $\rho \geq 0$ and*

$$\Re\left((1-\rho)\frac{K^{\alpha_l} f(z)}{K^{\alpha_l} g(z)} + \rho\frac{K^{\alpha_l+1} f(z)}{K^{\alpha_l+1} g(z)}\right) > \gamma \quad (0 \leq \gamma < 1; z \in \mathbb{U}^*). \tag{2.4}$$

Then,

$$\Re\left(\frac{K^{\alpha_l+1} f(z)}{K^{\alpha_l+1} g(z)}\right) > \eta := \frac{\gamma(2\alpha + \delta) + \delta(\rho - 1)}{2\alpha + \rho\delta}.$$

Proof. We have

$$\lambda\frac{K^{\alpha_l+1}f(z)}{K^{\alpha_l+1}g(z)} = \left((1-\rho)\frac{K^{\alpha_l}f(z)}{K^{\alpha_l}g(z)} + \rho\frac{K^{\alpha_l+1}f(z)}{K^{\alpha_l+1}g(z)}\right) + (\rho-1)\frac{K^{\alpha_l}f(z)}{K^{\alpha_l}g(z)}.$$

We can deduce the following desired inequality

$$\Re\left(\frac{K^{\alpha_l+1}f(z)}{K^{\alpha_l+1}g(z)}\right) > \eta := \frac{\gamma(2\alpha+\delta)+\delta(\rho-1)}{2\alpha+\rho\delta},$$

if we use of (2.4) and (2.1) ($for \quad \mu = 1$), and since $\rho > 1$. □

Corollary 2.2. *Let* $\alpha \in \mathbb{R}\setminus\{0\}$ *and* $\Re\{\rho\} > 0$. *If the following condition:*

$$\Re\{(1-\rho)(zK^{\alpha_l}f(z))^{\mu} + \rho(zK^{\alpha_l+1}f(z))(zK^{\alpha_l}f(z))^{\mu-1}\} > \gamma$$

$(0 \le \gamma < 1; \mu > 0; z \in \mathbb{U}^*)$, *is satisfied by* $f(z) \in \Sigma$, *then*

$$\Re\{(zK^{\alpha_l}f(z))^{\mu}\} > \frac{2\alpha\mu\gamma + \Re\{\rho\}}{2\mu\alpha + \Re\{\rho\}}. \tag{2.5}$$

In addition, if $\rho \ge 1$ *and* $\alpha \in \mathbb{R}\setminus\{0\}$, *and if* $f(z) \in \Sigma$ *satisfies the following condition* $\Re((1-\rho)zK^{\alpha_l}f(z) + \rho(zK^{\alpha_l+1}f(z))) > \gamma$, *then*

$$\Re(zK^{\alpha_l+1}f(z)) > \frac{(2\alpha+1)\gamma+\rho-1}{2\alpha+\rho} \quad (0 \le \gamma < 1; z \in \mathbb{U}^*). \tag{2.6}$$

Proof. The results of (2.5) and (2.6) are achieved by putting $g(z) = \frac{1}{z}$ in Theorem 2.1 and Corollary 2.1, respectively. □

Remark 1. (i) By putting $\rho = 1$ and for $\alpha_i, \beta_j > 0 \quad (i = 1, 2, ..., l)$ and $(j = 1, 2, ..., m)$ in Corollary 2.2, we get $\Re(zK^{\alpha_l+1}f(z).(zK^{\alpha_l}f(z))^{\mu-1}) > \gamma$, this implies that

$$\Re\{(zK^{\alpha_l}f(z))^{\mu}\} > \frac{2\alpha\gamma + \Re\{\rho\}}{2\alpha + \Re\{\rho\}} \quad (z \in \mathbb{U}^*).$$

(ii) For $\rho \in \mathbb{C}\setminus\{0\}$ with $\Re\{\rho\} > 0$, $\mu = 1$ and $\alpha_i, \beta_j > 0 \quad (i = 1, 2, ..., l)$ and $(j = 1, 2, ..., m)$ in Corollary 2.2, we get

$$\Re\{(1-\rho)zK^{\alpha_l}f(z) + \rho(zK^{\alpha_l+1}f(z))\} > \gamma \quad (0 \le \gamma < 1; z \in \mathbb{U}^*),$$

this implies that

$$\Re(zK^{\alpha_l}f(z)) > \frac{2\alpha\gamma + \Re\{\rho\}}{2\alpha + \Re\{\rho\}} \quad (0 \le \gamma < 1; z \in \mathbb{U}^*).$$

(iii)For $\rho = 1$, $s = 0$, $\alpha_i = \beta_j = 1 \quad (i = 1, 2, ..., l)$ and $(j = 1, 2, ..., m)$, $p - 1 = r = 0$ and $\lambda_1 = 1$, if we proceed to the limit as $b \to 0$ in Corollary 2.2, we have

$$\Re\left(\frac{zf'(z)}{f(z)}(zf(z))^{\mu}\right) > \gamma \quad (0 \le \gamma < 1; \mu > 0; z \in \mathbb{U}^*),$$

which implies that

$$\Re([zf(z)]^{\mu}) > \frac{2\gamma\mu+1}{2\mu+1} \quad (0 \leq \gamma < 1; \mu > 0; z \in \mathbb{U}^*).$$

(iv)For $\rho \in \mathbb{C} \setminus \{0\}$ with $\Re\{\rho\} > 0$, $\alpha_i = \beta_j = 1 \quad (i = 1, 2, ..., l)$ and $(j = 1, 2, ..., m)$, $\mu = 1$, $s = 0$, $p - 1 = r = 0$, and $\lambda_1 = 1$, if we take the limit as $b \to 0$ in Corollary 2.2, we get

$$\Re([(1-\rho)zf(z) + \rho(z^2 f'(z))]) > \gamma \quad (0 \leq \gamma < 1; \mu > 0; z \in \mathbb{U}^*),$$

which implies that

$$\Re\{zf(z)\} > \frac{2\gamma + \Re\{\rho\}}{2 + \Re\{\rho\}} \quad (0 \leq \gamma < 1; \mu > 0; z \in \mathbb{U}^*).$$

(v) Replacing $f(z)$ by $-zf'(z)$ in Remark 1 (ii) above, we have

$$-\Re\{(1-\rho)z^2 f(z) + \rho(z^3 f''(z))\} > \gamma,$$

which implies that

$$-\Re\{z^2 f'(z)\} > \frac{2\gamma + \Re\{\rho\}}{2 + \Re\{\rho\}} \quad (z \in \mathbb{U}^*).$$

(vi) For $\rho \in \mathbb{R}$ with $\rho \geq 1$, $\mu = 1$, $s = 0$, $\alpha_i = \beta_j = 1 \quad (i = 1, 2, ..., l)$ and $(j = 1, 2, ..., m)$, $p - 1 = r = 0$, and $\lambda_1 = 1$, if we take the limit as $b \to 0$ in Corollary 2.2, we obtain $\Re\{(1-\rho)zf(z) + \rho(z^2 f'(z))\} > \gamma$, which implies that

$$\Re\{zf(z)\} > \frac{3\gamma + \rho - 1}{2 + \rho}.$$

The following theorem gives a further extension of the previous result.

Theorem 2.2. *Let the functions $f(z)$ and $g(z)$ be in the class Σ. Let us also suppose that, if*

$$\Re\left(\frac{K^{\alpha_l+1} f(z)}{K^{\alpha_l+1} g(z)} - \frac{K^{\alpha_l} f(z)}{K^{\alpha_l} g(z)}\right) > \frac{(1-\gamma)\delta}{2\alpha}, \tag{2.7}$$
$$(0 \leq \gamma < 1; \alpha \in \mathbb{R} \setminus \{0\}; 0 \leq \delta < 1; z \in \mathbb{U}^*),$$

then the condition (1.14) is satisfied by the function $g(z)$, hence

$$\Re\left(\frac{K^{\alpha_l} f(z)}{K^{\alpha_l+1} g(z)}\right) > \gamma, \quad (0 \leq \gamma < 1; \alpha \in \mathbb{R} \setminus \{0\}; 0 \leq \delta < 1; z \in \mathbb{U}^*) \tag{2.8}$$

and

$$\Re\left\{\frac{K^{\alpha_l+1} f(z)}{K^{\alpha_l+1} g(z)}\right\} > \frac{(2\alpha + 1 + \delta)\gamma - \delta}{2\alpha}, \tag{2.9}$$
$$(0 \leq \gamma < 1; \alpha \in \mathbb{R} \setminus \{0\}; 0 \leq \delta < 1; z \in \mathbb{U}^*).$$

Proof. Let

$$q(z) = \frac{1}{1-\gamma}\left(\frac{K^{\alpha_l} f(z)}{K^{\alpha_l} g(z)} - \gamma\right).$$

Then, the function $q(z)$ is analytic in $\mathbb{U}^*$ with $q(0) = 1$. At this point, by setting

$$\Phi(z) = \frac{K^{\alpha_l} g(z)}{K^{\alpha_l+1} g(z)}$$

then, by using the hypothesis, we observe that $\Re\{\Phi(z)\} > \delta, (z \in \mathbb{U}^*)$. Now,

$$\frac{(1-\gamma)zq'(z)\Re\{\Phi\}}{\alpha} = \frac{K^{\alpha_l+1} f(z)}{K^{\alpha_l+1} g(z)} - \frac{K^{\alpha_l} f(z)}{K^{\alpha_l} g(z)} = \Psi(q(z), zq'(z)),$$

is shown by a simple computation, where

$$\Psi(r, a) = \frac{(1-\gamma)\Phi(z)a}{\alpha} \quad (\alpha \in \mathbb{R} \setminus \{0\}).$$

Therefore, we get

$$\Psi(q(z), zq'(z))_{(z\in\mathbb{U}^*)} \subset \Omega := \left\{w : w \in \mathbb{C} \;\; and \;\; \Re(w) > -\frac{\delta(1-\gamma)}{2\alpha}\right\}$$

if we use hypothesis (2.7). Now, for all real $r_2, a_1 \leq -(1 + {r_2}^2)/2$, or $r_2, a_1 \leq (1 + {r_2}^2)/2$, we obtain

$$\Re\{\Psi(ir_2, a_1)\} = \frac{a_1(1-\gamma)\Re\{\Phi\}}{\alpha} \leq -\frac{\delta(1-\gamma)(1-{r_2}^2)}{2\alpha} \leq -\frac{\delta(1-\gamma)}{2\alpha}.$$

This shows that $\Re\{\Psi(ir_2, a_1)\} \notin \Omega, (z \in \mathbb{U}^*)$. Then, we get $\Re\{q(z)\} > 0, (z \in \mathbb{U}^*)$, with Lemma 2.1. The assertion of (2.8) is thus proved.

The assertion of (2.9) is proved if (2.8) and (2.9) are used in the following identity,

$$\frac{K^{\alpha_l+1} f(z)}{K^{\alpha_l+1} g(z)} = \left(\frac{K^{\alpha_l+1} f(z)}{K^{\alpha_l+1} g(z)} - \frac{K^{\alpha_l} f(z)}{K^{\alpha_l} g(z)}\right) + \frac{K^{\alpha_l} f(z)}{K^{\alpha_l} g(z)}.$$

The proof of Theorem 2.2 is now obviously completed. □

Remark 2. Upon putting $\alpha_i = \beta_j = 1 \quad (i = 1, 2, ..., l)$ and $(j = 1, 2, ..., m)$, $s = 0$, $g(z) = \frac{1}{z}$, $p - 1 = r = 0$, and $\lambda_1 = 1$, if we take the limit as $b \to 0$ in Theorem 2.2, we obtain

$$\Re\{zf(z) + z^2 f'(z)\} > -\frac{\delta(1-\gamma)}{2} \quad (0 \leq \gamma < 1; 0 \leq \delta < 1; z \in \mathbb{U}^*),$$

which implies that $\Re\{zf(z)\} > \gamma \;\; (0 \leq \gamma < 1; z \in \mathbb{U}^*)$ and

$$\Re\{2zf(z) + z^2 f'(z)\} > \frac{\gamma(2+\delta) - \delta}{2} \quad (0 \leq \gamma < 1; 0 \leq \delta < 1; z \in \mathbb{U}^*).$$

3. Concluding remarks and observations

A remarkably general group of linear operators related to the λ generalised Hurwitz-Lerch zeta functions has been successfully applied in our present investigation. By using this general linear operator, we have a variety of properties of some new subclass of meromorphically univalent functions in the punctured unit disk $\mathbb{U}^*$ which were introduced and investigated. Several closely related (new or known) consequences and corollaries of the main results (Theorem 2.1 and 2.2) presented in this paper have also been considered.

Acknowledgements

The work is supported by MOHE: FRGS/1/2016/STG06/UKM/01/1.

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Accepted: 4.07.2017