SOME CLASSES OF INVARIANT SUBMANIFOLDS OF $(LCS)_n$-MANIFOLDS

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Abstract. The object of the present paper is to study the pseudoparallel, Ricci generalized pseudoparallel and η-parallel invariant submanifolds of $(LCS)_n$-manifolds and we obtained some equivalent conditions of invariant submanifolds of $(LCS)_n$-manifolds under which the submanifolds are totally geodesic. Among others we found the necessary

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and sufficient condition of the second fundamental form to be η-parallel in an invariant submanifold of a $(LCS)_n$-manifold. Finally an example of invariant submanifold of $(LCS)_5$-manifold is constructed.

Keywords: $(LCS)_n$-manifold, invariant submanifold, pseudo parallel submanifold, Ricci generalized pseudo parallel submanifold, η-parallel submanifold, totally geodesic.

1. Introduction

In 2003 Shaikh [29] introduced the notion of Lorentzian concircular structure manifolds (briefly, $(LCS)_n$-manifolds), with an example, which generalizes the notion of LP-Sasakian manifolds introduced by Matsumoto [18] and also by Mihai and Rosca [20]. Then Shaikh and Baishya ([32], [33]) investigated the applications of $(LCS)_n$-manifolds to the general theory of relativity and cosmology. It is to be noted that the most interesting fact is $(LCS)_n$-manifold remains invariant under a D-homothetic transformation, which does not hold for an LP-Sasakian manifold [31]. The $(LCS)_n$-manifolds have been also studied by Atceken [3], Narain and Yadav [21], Prakasha [26], Hui et. al ([4], [9],[13], [14]), Shaikh [30], Shaikh, Basu and Eyasmin ([35], [36]), Shaikh and Binh [34], Shaikh and Hui [37], Sreenivasa, Venkatesha and Bagewadi [39], Yadav, Dwivedi and Suthar [41] and others.

In modern analysis the geometry of submanifolds have become a subject of growing interest for its significant application in applied mathematics and theoretical physics. For instance, the notion of invariant submanifold is used to discuss properties of non-linear autonomous system [12]. Also the notion of geodesics play an important role in the theory of relativity [19]. For totally geodesic submanifolds, the geodesics of the ambient manifolds remain geodesics in the submanifolds. Hence, totally geodesic submanifolds are also very much important in physical sciences. The study of geometry of invariant submanifolds was initiated by Bejancu and Papaghuic [7]. In general the geometry of an invariant submanifold inherits almost all properties of the ambient manifold. The invariant submanifolds have been studied by many geometers to different extent such as [1], [2], [5], [6], [11], [16], [17], [22], [23], [25], [27], [28], [40], [43] and many others. Recently Shaikh et. al [38] studied invariant submanifolds of $(LCS)_n$-manifolds.

In the present paper we study some classes of invariant submanifolds of $(LCS)_n$-manifolds. The paper is organized as follows. Section 2 is concerned with preliminaries of $(LCS)_n$-manifolds. Section 3 deals with the study of some classes of invariant submanifolds of $(LCS)_n$-manifolds. We obtain the necessary and sufficient conditions for some classes of invariant submanifolds of $(LCS)_n$-manifolds to be totally geodesic. It is shown that pseudoparallelism and Ricci generalized pseudoparallelism of an invariant submanifold of a $(LCS)_n$-manifold are equivalent with a certain condition. We also find the necessary and sufficient condition of the second fundamental form h to be η-parallel in an invariant submanifold of a $(LCS)_n$-manifold. Finally, we construct an example of invariant submanifold of $(LCS)_5$-manifold.

2. Preliminaries

An n-dimensional Lorentzian manifold M is a smooth connected paracompact Hausdorff manifold with a Lorentzian metric g, that is, M admits a smooth symmetric tensor field g of type (0,2) such that for each point $p \in M$, the tensor $g_p : T_pM \times T_pM \to \mathbb{R}$ is a non-degenerate inner product of signature $(-, +, \cdots, +)$, where T_pM denotes the tangent vector space of M at p and $\mathbb{R}$ is the real number space. A non-zero vector $v \in T_pM$ is said to be timelike (resp., non-spacelike, null, spacelike) if it satisfies $g_p(v, v) < 0$ (resp., ≤ 0, $= 0$, > 0) [24].

Definition 2.1 ([42]). In a Lorentzian manifold (M, g) a vector field P defined by

$$g(X, P) = A(X),$$

for any $X \in \Gamma(TM)$, is said to be a concircular vector field if

$$(\overline{\nabla}_X A)(Y) = \alpha\{g(X, Y) + \omega(X)A(Y)\}$$

where α is a non-zero scalar and ω is a closed 1-form and $\overline{\nabla}$ denotes the operator of covariant differentiation of M with respect to the Lorentzian metric g.

Let M be an n-dimensional Lorentzian manifold admitting a unit timelike concircular vector field ξ, called the characteristic vector field of the manifold. Then we have

$$g(\xi, \xi) = -1. \tag{2.1}$$

Since ξ is a unit concircular vector field, it follows that there exists a non-zero 1-form η such that for

$$g(X, \xi) = \eta(X), \tag{2.2}$$

the equation of the following form holds

$$(\overline{\nabla}_X \eta)(Y) = \alpha\{g(X, Y) + \eta(X)\eta(Y)\}, \quad \alpha \neq 0 \tag{2.3}$$

for all vector fields X, Y and α is a non-zero scalar function satisfies

$$\overline{\nabla}_X \alpha = (X\alpha) = d\alpha(X) = \rho\eta(X), \tag{2.4}$$

ρ being a certain scalar function given by $\rho = -(\xi\alpha)$.

Let us take

$$\phi X = \frac{1}{\alpha}\overline{\nabla}_X \xi, \tag{2.5}$$

then from (2.3) and (2.5) we have

$$\phi X = X + \eta(X)\xi, \tag{2.6}$$

from which it follows that ϕ is a symmetric (1,1) tensor and called the structure tensor of the manifold. Thus the Lorentzian manifold M together with the unit timelike concircular vector field ξ, its associated 1-form η and an (1,1) tensor field ϕ is said to be a Lorentzian concircular structure manifold (briefly, $(LCS)_n$-manifold), [29]. Especially, if we take $\alpha = 1$, then we can obtain the LP-Sasakian structure of Matsumoto [18]. In a $(LCS)_n$-manifold ($n > 2$), the following relations hold ([29], [30]):

$$\eta(\xi) = -1, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.7}$$

$$\phi^2 X = X + \eta(X)\xi, \tag{2.8}$$

$$\overline{S}(X, \xi) = (n-1)(\alpha^2 - \rho)\eta(X), \tag{2.9}$$

$$\overline{R}(X, Y)\xi = (\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y], \tag{2.10}$$

$$\overline{R}(\xi, Y)Z = (\alpha^2 - \rho)[g(Y, Z)\xi - \eta(Z)Y], \tag{2.11}$$

$$\overline{R}(\xi, X)\xi = (\alpha^2 - \rho)[\eta(X)\xi + X], \tag{2.12}$$

$$(\overline{\nabla}_X \phi)(Y) = \alpha\{g(X, Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\}, \tag{2.13}$$

$$(X\rho) = d\rho(X) = \beta\eta(X), \tag{2.14}$$

$$\overline{R}(X, Y)Z = \phi\overline{R}(X, Y)Z + (\alpha^2 - \rho)\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\}\xi \tag{2.15}$$

for all X, Y, $Z \in \Gamma(TM)$ and $\beta = -(\xi\rho)$ is a scalar function, where $\overline{R}$ is the curvature tensor and $\overline{S}$ is the Ricci tensor of the manifold.

For a $(0, l)$ tensor field T, $l \geq 1$, and a symmetric $(0, 2)$ tensor field B, we have

$$\begin{aligned} &Q(B, T)(X_1, \cdots, X_l; X, Y) \\ &= -T((X \wedge_B Y)X_1, X_2, \cdots, X_l) - \cdots - T(X_1, \cdots, X_{l-1}, (X \wedge_B Y)X_K), \end{aligned} \tag{2.16}$$

where

$$(X \wedge_B Y)Z = B(Y, Z)X - B(X, Z)Y. \tag{2.17}$$

Putting $T = h$ and $B = g$ or $B = S$, we obtain $Q(g, h)$ and $Q(S, h)$ respectively.

Let N be a submanifold of a $(LCS)_n$-manifold M with induced metric g and let ∇ and $\overline{\nabla}$ be the Levi-Civita connection of N and M respectively. Also let ∇ and $\nabla^{\perp}$ be the induced connection on the tangent bundle TN and the normal bundle $T^{\perp}N$ of N respectively. Then the Gauss and Weingarten formulae are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.18}$$

and

$$\overline{\nabla}_X V = -A_V X + \nabla^{\perp}_X V \tag{2.19}$$

for all $X,\ Y \in \Gamma(TN)$ and $V \in \Gamma(T^{\perp}N)$, where h and A_V are second fundamental form and the shape operator (corresponding to the normal vector field V) respectively for the immersion of N into M. The second fundamental form h and the shape operator A_V are related by [44]

$$g(h(X, Y), V) = g(A_V X, Y) \tag{2.20}$$

for any $X,\ Y \in \Gamma(TN)$ and $V \in \Gamma(T^{\perp}N)$. We note that $h(X, Y)$ is bilinear and since $\nabla_{fX} Y = f \nabla_X Y$ for any smooth function f on a manifold, we have

$$h(fX, Y) = fh(X, Y). \tag{2.21}$$

For the second fundamental form h, the covariant derivative of h is defined by

$$(\overline{\nabla}_X h)(Y, Z) = \nabla^{\perp}_X (h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) \tag{2.22}$$

for any vector fields X, Y, Z tangent to N. Then $\overline{\nabla} h$ is a normal bundle valued tensor of type (0,3) and is called the third fundamental form of N, $\overline{\nabla}$ is called the Vander-Waerden-Bortolotti connection of M, i.e. $\overline{\nabla}$ is the connection in $TN \oplus T^{\perp}N$ built with ∇ and $\nabla^{\perp}$. If $\overline{\nabla} h = 0$, then N is said to have parallel second fundamental form [44].

An immersion is said to be pseudo parallel if

$$\overline{R}(X, Y) \cdot h = (\overline{\nabla}_X \overline{\nabla}_Y - \overline{\nabla}_Y \overline{\nabla}_X - \overline{\nabla}_{[X,Y]}) h = L_1 Q(g, h) \tag{2.23}$$

for all vector fields X, Y tangent to N [10]. If in particular, $L_1 = 0$ then the manifold is said to be semiparallel. Again the submanifold N of a $(LCS)_n$-manifold M is said to be Ricci generalized pseudoparallel [10] if its second fundamental form h satisfies

$$\overline{R}(X, Y) \cdot h = L_2 Q(S, h), \text{where } L_2 \text{ is constant.} \tag{2.24}$$

Also the second fundamental form h of submanifold N of a $(LCS)_n$-manifold M is said to be η-parallel [44] if

$$(\nabla_X h)(\phi Y, \phi Z) = 0 \tag{2.25}$$

for all vector fields X, Y and Z tangent to N. A submanifold N of a $(LCS)_n$-manifold M is said to be totally umbilical if

$$h(X,Y) = g(X,Y)H \tag{2.26}$$

for any vector fields X, $Y \in \Gamma(TN)$, where H is the mean curvature of N. Moreover if $h(X,Y) = 0$ for all X, $Y \in \Gamma(TN)$, then N is said to be totally geodesic and if $H = 0$ then N is minimal in M.

Definition 2.2 ([7]). A submanifold N of a $(LCS)_n$-manifold M is said to be invariant if the structure vector field ξ is tangent to N at every point of N and ϕX is tangent to N for any vector field X tangent to N at every point of N, that is $\phi(TN) \subset TN$ at every point of N.

From the Gauss and Weingarten formulae we obtain

$$\overline{R}(X,Y)Z = R(X,Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X, \tag{2.27}$$

where $\overline{R}(X,Y)Z$ denotes the tangential part of the curvature tensor of the submanifold.

Now we have

$$\begin{aligned}(\overline{R}(X,Y)\cdot h)(Z,U) &= R^{\perp}(X,Y)h(Z,U)\\ &\quad - h(R(X,Y)Z,U) - h(Z,R(X,Y)U)\end{aligned} \tag{2.28}$$

for all vector fields X, Y, Z and U, where

$$R^{\perp}(X,Y) = [\nabla^{\perp}_X, \nabla^{\perp}_Y] - \nabla^{\perp}_{[X,Y]}.$$

In an invariant submanifold N of a $(LCS)_n$-manifold M, the following relations hold [38]:

$$\nabla_X \xi = \alpha\phi X, \tag{2.29}$$

$$h(X,\xi) = 0, \tag{2.30}$$

$$R(X,Y)\xi = (\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y], \tag{2.31}$$

$$S(X,\xi) = (n-1)(\alpha^2 - \rho)\eta(X), \quad \text{i.e., } Q\xi = (n-1)(\alpha^2 - \rho)\xi, \tag{2.32}$$

$$(\nabla_X \phi)(Y) = \alpha\{g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\}, \tag{2.33}$$

$$h(X,\phi Y) = \phi h(X,Y) = h(\phi X,Y) = h(X,Y). \tag{2.34}$$

3. Totally geodesic submanifolds of $(LCS)_n$-manifolds

In this section we prove the following:

Theorem 3.1. *Let N be an invariant submanifold of a $(LCS)_n$-manifold M with $L_1 \neq -(\alpha^2 - \rho)$, then N is totally geodesic if and only if N is pseudoparallel.*

Proof. Let N be an invariant submanifold of a $(LCS)_n$-manifold M with $L_1 \neq -(\alpha^2 - \rho)$.

Since N is pseudoparallel, we have

$$\overline{R}(X,Y) \cdot h = L_1 Q(g,h),$$

i.e.,

$$\begin{aligned} (3.1)\quad & R^{\perp}(X,Y)h(Z,U) - h(R(X,Y)Z,U) - h(Z,R(X,Y)U) \\ = \quad & L_1[g(Y,Z)h(X,U) - g(X,Z)h(Y,U) + g(Y,U)h(X,Z) - g(X,U)h(Y,Z)] \end{aligned}$$

for all vector fields $X,\ Y,\ Z,\ U$ on N.

Putting $X = U = \xi$ in (3.1) and using (2.30), we obtain

$$h(Z, R(\xi,Y)\xi) = -L_1 h(Y,Z). \tag{3.2}$$

Feeding (2.30) and (2.31) in (3.2) we get

$$(\alpha^2 - \rho + L_1)h(Y,Z) = 0,$$

which implies that $h(Y,Z) = 0$ for all $Y,\ Z$ on N, i.e., N is totally geodesic, since $L_1 \neq -(\alpha^2 - \rho)$. The converse part is trivial.

Hence the theorem is proved.

Corollary 3.1. Let N be an invariant submanifold of a $(LCS)_n$-manifold M with $\alpha^2 - \rho \neq 0$. Then N is totally geodesic if and only if N is semiparallel.

Theorem 3.2. *Let N be an invariant submanifold of a $(LCS)_n$-manifold M with $\alpha^2 - \rho \neq 0$ and $L_2 \neq \frac{1}{n-1}$, then N is totally geodesic if and only if N is Ricci generalized pseudoparallel.*

Proof. Let N be a Ricci generalized pseudoparallel invariant submanifold of a $(LCS)_n$-manifold M with $\alpha^2 - \rho \neq 0$ and $L_2 \neq \frac{1}{n-1}$. Then

$$R(X,Y) \cdot h = L_2 Q(S,h),$$

i.e.,

$$\begin{aligned} & R^{\perp}(X,Y)h(Z,U) - h(R(X,Y)Z,U) - h(Z,R(X,Y)U) \\ & = -L_2[S(Y,Z)h(X,U) - S(X,Z)h(Y,U) \\ & \quad + S(Y,U)h(X,Z) - S(X,U)h(Y,Z)] \end{aligned} \tag{3.3}$$

for all vector fields X, Y, Z, U on N.

Putting $X = U = \xi$ in (3.3) and using (2.30) we get

$$h(Z, R(\rho, \xi)\xi) = -L_2 S(\xi, \xi)h(Y, Z). \tag{3.4}$$

Using (2.31) and (2.32) in (3.4) we get

$$[1 - (n-1)L_2](\alpha^2 - \rho)h(Y, Z) = 0, \tag{3.5}$$

which implies that $h(Y, Z) = 0$, since $\alpha^2 - \rho \neq 0$ and $L_2 \neq \frac{1}{n-1}$.

Thus N is totally geodesic. The converse part is trivial.

By virtue of Theorem 3.1 and Theorem 3.2, we can state that

Theorem 3.3. *Let N be an invariant submanifold of a $(LCS)_n$-manifold M, then the following statements are equivalent:*

(i) N is totally geodesic.

(ii) N is pseudoparallel with $L_1 \neq -(\alpha^2 - \rho)$.

(iii) N is Ricci generalized pseudoparallel with $(\alpha^2 - \rho) \neq 0$ and $L_2 \neq \frac{1}{n-1}$.

Suppose that N be an invariant submanifold of a $(LCS)_n$-manifold M whose second fundamental tensor h is pseudo parallel [15], i.e. h satisfies

$$(\nabla_X h)(Y, Z) = 2A(X)h(Y, Z) + A(Y)h(X, Z) + A(Z)h(X, Y). \tag{3.6}$$

Setting $Z = \xi$ in (3.6) and using (2.30) we get,

$$(\nabla_X h)(Y, \xi) = A(\xi)h(X, Y). \tag{3.7}$$

Now, $(\nabla_X h)(Y, \xi) = \nabla_X h(Y, \xi) - h(\nabla_X Y, \xi) - h(Y, \nabla_X \xi) = -h(Y, \alpha\phi X)$, i.e.,

$$(\nabla_X h)(Y, \xi) = -\alpha h(Y, \phi X). \tag{3.8}$$

Using (3.8) in (3.7) we get,

$$-\alpha h(Y, \phi X) = A(\xi)h(X, Y). \tag{3.9}$$

In view of (2.6) and (2.30), (3.9) yields,

$$[\alpha + A(\xi)]h(X, Y) = 0, \tag{3.10}$$

which implies that $h(X, Y) = 0$, provided $\alpha \neq -A(\xi)$.

This leads to the following theorem:

Theorem 3.4. *Let N be an invariant submanifold of $(LCS)_n$-manifold M with pseudo parallel second fundamental tensor, then N is totally geodesic if $\alpha \neq -A(\xi)$.*

We now consider that N be an invariant submanifold of a $(LCS)_n$-manifold M such that h is η-parallel. Since h is η-parallel we have

$$\nabla_X h(\phi Y, \phi Z) = h(\nabla_X \phi Y, \phi Z) + h(\phi Y, \nabla_X \phi Z). \tag{3.11}$$

From (2.34) we get

$$h(\phi Y, \phi Z) = h(Y, Z). \tag{3.12}$$

Using (3.12) in (3.11) we get

$$\begin{aligned} \nabla_X h(Y, Z) &= h(\{(\nabla_X \phi)Y + \phi(\nabla_X Y)\}, \phi Z) \\ &\quad + h(\phi Y, \{(\nabla_X \phi)Z + \phi(\nabla_X Z)\}) \\ &= h((\nabla_X \phi)Y, \phi Z) + h(\nabla_X Y, Z) \\ &\quad + h(\phi Y, (\nabla_X \phi)Z) + h(Y, \nabla_X Z). \end{aligned} \tag{3.13}$$

Feeding (2.30) and (2.33) in (3.13), we obtain

$$\begin{aligned} (\nabla_X h)(Y, Z) &= h(\alpha\{g(X, Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\}, \phi Z) \\ &\quad + h(\phi Y, \alpha\{g(X, Z)\xi + 2\eta(X)\eta(Z)\xi + \eta(Z)X\}) \\ &= \alpha[\eta(Y)h(X, \phi Z) + \eta(Z)h(\phi Y, X)] \\ &= \alpha[\eta(Y)h(X, Z) + \eta(Z)h(Y, X)]. \end{aligned} \tag{3.14}$$

Conversely let N be an invariant submanifold of a $(LCS)_n$-manifold M such that the relation (3.14) holds. Then from (3.14) it follows that $(\nabla_X h)(\phi Y, \phi Z) = 0$, i.e., the second fundamental form h is η-parallel. Thus we can obtain the following:

Theorem 3.5. *let N be an invariant submanifold of a $(LCS)_n$-manifold M, then the second fundamental form h is η-parallel if and only if* (3.14) *holds.*

We now prove the following:

Theorem 3.6. *let N be an invariant submanifold of a $(LCS)_n$-manifold M, then*

$$\phi(A_V X) = A_V X = A_{\phi V} X \tag{3.15}$$

for all $X \in \Gamma(TN)$ and $V \in \Gamma(T^{\perp}N)$.

Proof. By virtue of (2.6) we have

$$g(\phi(A_V X), Y) = g(A_V X, \phi Y). \tag{3.16}$$

In view of (2.20) and (2.34), (3.16) yields,

$$g(\phi(A_V X), Y) = g(h(X, \phi Y), V) = g(h(X, Y), V) = g(A_V X, Y).$$

Thus we have

$$\phi(A_V X) = A_V X. \tag{3.17}$$

Again using (2.20) and (2.34), we obtain

$$\begin{aligned} g(A_{\phi V}X, Y) &= g(h(X,Y), \phi V) = g(\phi(h(X,Y)), V) \\ &= g(h(X,Y), V) = g(A_V X, Y). \end{aligned}$$

Hence we get

$$A_{\phi V}X = A_V X. \tag{3.18}$$

From (3.17) and (3.18) we get the desired result.

We now provide an example of invariant submanifold of $(LCS)_n$-manifold.

Example 3.1. Let us consider the 5-dimensional manifold $\overline{M} = \{(x,y,z,u,v) \in \mathbb{R}^5 : (x,y,z,u,v) \neq (0,0,0,0,0)\}$, where (x,y,z,u,v) are the standard coordinates in $\mathbb{R}^5$. The vector fields

$$e_1 = e^z \frac{\partial}{\partial x}, \quad e_2 = e^{z-ax} \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}, \quad e_4 = e^z \frac{\partial}{\partial u}, \quad e_5 = e^{z-u} \frac{\partial}{\partial v}$$

are linearly independent at each point of M where 'a' is a scalar.

For $\alpha \neq 0$, let $\overline{g}$ be the metric defined by

$$\begin{aligned} \overline{g}(e_i, e_j) &= \frac{1}{\alpha}, \quad for \;\; i = j \neq 3, \\ &= 0, \quad for \;\; i = j, \\ &= -\frac{1}{\alpha}, \quad for \;\; i = j = 3. \end{aligned}$$

Here i and j runs over 1 to 5.

Let η be the 1-form defined by $\eta(X) = \overline{g}(X, e_3)$ for any vector field X tangent to $\overline{M}$.

Let ϕ be the (1,1) tensor field defined by

$$\phi e_1 = -e_1, \quad \phi e_2 = -e_2, \quad \phi e_3 = 0, \quad \phi e_4 = -e_4, \quad \phi e_5 = -e_5.$$

Then using the linearity property of ϕ and $\overline{g}$ we have $\eta(e_3) = -\frac{1}{\alpha}$. Thus for $\xi = \alpha e_3$ we get $\eta(\xi) = -1$ and $\phi^2 X = X + \eta(X)\xi$.

Let $\overline{\nabla}$ be the Levi-Civita connection on $\overline{M}$ with respect to the metric $\overline{g}$. Then we have

$$[e_1, e_2] = -ae^z e_2, \quad [e_1, e_3] = -e_1, \quad [e_1, e_4] = 0, \quad [e_1, e_5] = 0, \quad [e_2, e_3] = -e_2,$$

$$[e_2, e_4] = 0, \quad [e_2, e_5] = 0, \quad [e_3, e_4] = e_4, \quad [e_4, e_5] = -e^z e_5.$$

Now, using Koszul's formula for $\overline{g}$, it can be calculated that
$\overline{\nabla}_{e_1}e_1 = e_3$, $\overline{\nabla}_{e_1}e_2 = 0$, $\overline{\nabla}_{e_1}e_3 = -e_1$, $\overline{\nabla}_{e_1}e_4 = 0$, $\overline{\nabla}_{e_1}e_5 = 0$,
$\overline{\nabla}_{e_2}e_1 = ae^z e_2$, $\overline{\nabla}_{e_2}e_2 = -ae^z + e_3$, $\overline{\nabla}_{e_2}e_3 = -e_2$, $\overline{\nabla}_{e_2}e_4 = 0$, $\overline{\nabla}_{e_2}e_5 = 0$,
$\overline{\nabla}_{e_3}e_1 = 0$, $\overline{\nabla}_{e_3}e_2 = 0$, $\overline{\nabla}_{e_3}e_3 = 0$, $\overline{\nabla}_{e_3}e_4 = 0$, $\overline{\nabla}_{e_3}e_5 = 0$, $\overline{\nabla}_{e_4}e_1 = 0$,
$\overline{\nabla}_{e_4}e_2 = 0$, $\overline{\nabla}_{e_4}e_3 = -e_4$, $\overline{\nabla}_{e_4}e_4 = 0$ $\overline{\nabla}_{e_4}e_5 = 0$, $\overline{\nabla}_{e_5}e_1 = 0$, $\overline{\nabla}_{e_5}e_2 = 0$,
$\overline{\nabla}_{e_5}e_3 = -e_5$, $\overline{\nabla}_{e_5}e_4 = e^z e_5$, $\overline{\nabla}_{e_5}e_5 = e_3 - e^z e_5$.

From the above calcultions we see that the (ϕ, ξ, η, g) structure satisfies $\eta(\xi) = -1$ and $\overline{\nabla}_X \xi = \alpha\phi X$. Hence $\overline{M}$ is an $(LCS)_5$-manifold. Let f be an isometric immersion from M to $\overline{M}$ defined by $f(x, y, z) = (x, y, z, 0, 0)$.

Let $M = \{(x, y, z) \in \mathbb{R}^3 : (x, y, z) \neq (0, 0, 0)\}$, where (x, y, z) are the standard coordinates in $\mathbb{R}^3$. The vector fields $e_1 = e^z \frac{\partial}{\partial x}$, $e_2 = e^{z-ax} \frac{\partial}{\partial y}$, $e_3 = \frac{\partial}{\partial z}$ are linearly independent at each point of M where 'a' is a scalar.

For $\alpha \neq 0$, let g be the metric defined by

$$g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0, \;\; g(e_1, e_1) = g(e_2, e_2) = \frac{1}{\alpha}, \;\; g(e_3, e_3) = -\frac{1}{\alpha}.$$

Let η be the 1-form defined by $\eta(X) = g(X, e_3)$ for any vector field X tangent to M.

Let ϕ be the (1,1) tensor field defined by $\phi e_1 = -e_1$, $\phi e_2 = -e_2$, $\phi e_3 = 0$. Thus for $\xi = \alpha e_3$ we get $\eta(\xi) = -1$, $\phi^2 X = X + \eta(X)e_3$ for any vector field X tangent to M.

Let ∇ be the Levi-Civita connection on M with respect to the metric g. Then we have $[e_1, e_2] = -ae^z e_2, [e_1, e_3] = -e_1, [e_2, e_3] = -e_2$.

Now, using Koszul's formula for g, it can be calculated that

$$\nabla_{e_1}e_1 = e_3, \;\; \nabla_{e_1}e_2 = 0, \;\; \nabla_{e_1}e_3 = -e_1, \;\; \nabla_{e_2}e_1 = ae^z e_2, \; \nabla_{e_2}e_2 = -ae^z + e_3,$$

$$\nabla_{e_2}e_3 = -e_2 \;\; \nabla_{e_3}e_1 = 0, \;\; \nabla_{e_3}e_2 = 0, \;\; \nabla_{e_3}e_3 = 0.$$

Therefore (ϕ, ξ, η, g) structure satisfies $\eta(\xi) = -1$, $\nabla_X \xi = \alpha\phi X$. Hence M is a $(LCS)_3$-manifold. It is obvious that M is a submanifold of $\overline{M}$. Also $\phi X \in TM$ for $X \in TM$. Hence M is invariant.

Let $U = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 \;\in TM$ and $V = \mu_1 e_1 + \mu_2 e_2 + \mu_3 e_3 \;\in TM$, where λ_i, μ_i are scalars such that $i = 1, 2, 3$. Then

$$h(U, V) = \sum_{i=1}^{3} \lambda_i \mu_i \; h(e_i, e_j) = \sum_{i=1}^{3} \lambda_i \mu_i \; h(\overline{\nabla}_{e_i} e_j - \nabla_{e_i} e_j) = 0.$$

Hence the submanifold is totally geodesic.

Acknowledgements

The authors wishes to express their sincere thanks and gratitude to the referee for her/his valuable suggestions towards the improvement of the paper. The first author (S. K. Hui) gratefully acknowledges to the SERB (Project No.: EMR/2015/002302), Govt. of India for the financial assistance of this work.

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Accepted: 5.04.2017