# ON SOME GENERATING FUNCTIONS FOR THE TWO-PARAMETERS ONE-VARIABLE SRIVASTAVA POLYNOMIALS

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**Abstract.** In the present paper we prove a general theorems on generating functions involving the two-parameter one-variable Srivastava polynomials, Hermite and Laguerre polynomials of two variables. It is also shown how these theorems can be used to derive several bilateral generating functions (known or new) involving Hermite and Laguerre polynomials of two variables and other classical polynomials of one variable which are contained by the two-parameter one-variable Srivastava polynomials.

**Keywords:** Generating functions, Srivastava polynomials, Hermite polynomials, Laguerre polynomials.

## 1. Introduction

In 1972, Srivastava [8] introduced the following family of polynomials:

$$S_n^N(x) = \sum_{k=0}^{[\frac{n}{N}]} \frac{(-n)_{Nk}}{k!} A_{n,k} x^k \qquad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; N \in \mathbb{N}), \tag{1.1}$$

where $\mathbb{N}$ is the set of positive integers, $\{A_{n,k}\}_{n,k=0}^{\infty}$ is a bounded double sequence of real or complex numbers, $[a]$denotes the greatest integer in $a \in \mathbb{R}$ and $(\lambda)_n$ denotes the well-known Pochhammers symbol.
In [4],Gonzalez *at al.* extended the Srivastava polynomials $S_n^N(x)$ as follows:

$$S_{n,m}^N(x) = \sum_{k=0}^{[\frac{n}{N}]} \frac{(-n)_{Nk}}{k!} A_{n+m,k} x^k \qquad (m, n \in \mathbb{N}_0; N \in \mathbb{N}). \tag{1.2}$$

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In 2013, Kaanoglu and Ozarslan [5] introduced the following family of two-parameter one-variable Srivastava polynomials:

$$S_n^{p,q}(x) = \sum_{k=0}^{n} \frac{(-n)_k}{k!} A_{p+q+n,q+k} x^k \qquad (p,q,n,k \in \mathbb{N}_0), \tag{1.3}$$

where $\{A_{n,k}\}$ is a bounded double sequence of real or complex numbers.
Also the following remarks are given in [5]:

**Remark 1.1.** Choosing $A_{m,n} = (-\alpha - m)_n, (m,n \in \mathbb{N}_0)$ in (1.3), we get

$$S_n^{p,q}\left(\frac{-1}{x}\right) = (-1)^q(\alpha+p+n+1)_q \frac{n!}{(-x)^n} L_n^{(\alpha+p)}(x), \tag{1.4}$$

where $L_n^{(\alpha)}(x)$ are the classical Laguerre polynomials [9]

$$L_n^{(\alpha)}(x) = \frac{(-x)^n}{n!}\, {}_2F_0\left[-n, -\alpha-n; -; \frac{-1}{x}\right]. \tag{1.5}$$

**Remark 1.2.** Choosing $A_{m,n} = \frac{(\alpha+\beta+1)_{2m}(-\beta-m)_n}{(\alpha+\beta+1)_m(-\alpha-\beta-2m)_m}, (m,n \in \mathbb{N}_0)$ in (1.3), we get

$$S_n^{p,q}\left(\frac{2}{1+x}\right) = \frac{(\alpha+\beta+1)_{2p+2q+2n}(-\beta-p-q-n)_q(1+\alpha+\beta+2p+q)_n}{(\alpha+\beta+1)_{p+q+n}(-\alpha-\beta-2p-2q-2n)_q(1+\alpha+\beta+2p+q)_{2n}}$$

$$\times n!\left(\frac{2}{1+x}\right)^n P_n^{(\alpha+p+q,\beta+p)}(x), \tag{1.6}$$

where $P_n^{(\alpha,\beta)}(x)$ are the classical Jacobi polynomials [7]

$$P_n^{(\alpha,\beta)}(x) = \binom{\alpha+\beta+2n}{n} \times$$

$$\times \left(\frac{1+x}{2}\right)^n {}_2F_1\left[-n, -\beta-n; -\alpha-\beta-2n; \frac{2}{1+x}\right]. \tag{1.7}$$

Further, we add the following remark:

**Remark 1.3.** Choosing $A_{m,n} = \frac{(\alpha)_{2m}}{(\alpha)_{m+n}}, (m,n \in \mathbb{N}_0)$ in (1.3), we get

$$S_n^{p,q}(x) = n!(\alpha+p+2q+2n)_p R_n(\alpha+p+2q;x), \tag{1.8}$$

where $R_n(\alpha,x)$ are the Shivelys pseudo Laguerre polynomials [7]

$$R_n(\alpha,x) = \frac{(\alpha)_{2n}}{n!(\alpha)_n}\, {}_1F_1[-n, \alpha+n; x]. \tag{1.9}$$

The Hermite polynomials of two variables are defined by [6]

$$H_n(x,y) = \sum_{r=0}^{[\frac{n}{2}]} \frac{(-1)^r n! H_{n-2r}(x) x^{2r} y^{n-2r}}{r!(n-2r)!}, \tag{1.10}$$

where $H_n(x)$ are the well-known Hermite polynomials [7].
The Laguerre polynomials of two variables are defined by (see [2],[3])

$$L_n(x,y) = n! \sum_{k=0}^{n} \frac{(-1)^k x^k y^{n-k}}{(k!)^2 (n-k)!}. \tag{1.11}$$

Also, we note that the Hermite and Laguerre polynomials of two variables (1.10) and (1.11) are satisfy the following generating functions respectively :

$$\sum_{n=0}^{\infty} \frac{H_n(x,y) t^n}{n!} = \exp[2xyt - (x^2+y^2)t^2], \tag{1.12}$$

$$\sum_{n=0}^{\infty} \frac{(c)_n H_n(x,y) t^n}{n!} = [1-2xyt]^{-c} \tag{1.13}$$

$$\times F \begin{matrix} 2:0;0 \\ 0:0;0 \end{matrix} \left[ \begin{matrix} \frac{c}{2}, \frac{c}{2}+\frac{1}{2} & : & - & ; & - & ; \\ - & : & - & ; & - & ; \end{matrix} \frac{-4x^2t^2}{(1-2xyt)^2}, \frac{-4y^2t^2}{(1-2xyt)^2} \right],$$

where $F \begin{matrix} A:B;D \\ E:G;H \end{matrix} [x,y]$ is the Kamp de Friet function [9]

$$\sum_{n=0}^{\infty} \frac{(a)_n L_n(x,y) t^n}{n!} = (1-yt)^{-a} {}_1F_1 \left[ a; 1; \frac{-xt}{(1-yt)} \right] \quad (|yt| < 1), \tag{1.14}$$

$$\sum_{n=0}^{\infty} \frac{L_n(x,y) t^n}{n!} = \exp(yt) C_0(xt), \tag{1.15}$$

where $C_n(x)$ denotes the $n^{th}$ order Tricomi function [9]

$$C_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!(n+k)!}. \tag{1.16}$$

## 2. Main results

In this section ,we have proved the following theorems :

**Theorem 2.1.** *The following family of bilateral generating functions involving the two-parameter one-variable Srivastava polynomials and Hermite polynomials of two variables holds true:*

$$\sum_{p,q,n=0}^{\infty} H_n(x,y) S_n^{p,q}(z) \frac{u^p}{p!} \frac{v^q}{q!} \frac{t^n}{n!} \tag{2.1}$$

$$= \sum_{p,q=0}^{\infty} H_{p+q}(x,y) A_{p+q,q} \frac{(u+t)^p}{p!} \frac{(v-zt)^q}{q!}.$$

**Theorem 2.2.** *The following family of bilateral generating functions involving the two-parameter one-variable Srivastava polynomials and Laguerre polynomials of two variables holds true:*

$$\sum_{p,q,n=0}^{\infty} L_n(x,y) S_n^{p,q}(z) \frac{u^p}{p!} \frac{v^q}{q!} \frac{t^n}{n!}$$

$$= \sum_{p,q=0}^{\infty} L_{p+q}(x,y) A_{p+q,q} \frac{(u+t)^p}{p!} \frac{(v-zt)^q}{q!}. \tag{2.2}$$

**Proof of 2.1.** Denoting the left hand side of (2.1) by $S$, expressing $S_n^{p,q}(z)$ as in (1.3) and using the result[9]

$$(-n)_k = \frac{(-1)^k n!}{(n-k)!}, 0 \leq k \leq n, \tag{2.3}$$

we obtain

$$S = \sum_{p,q,n=0}^{\infty} H_{p+q+n}(x,y) \sum_{k=0}^{n} \frac{(-z)_k}{k!} A_{p+q+n,q+k} \frac{u^p}{p!} \frac{v^q}{q!} \frac{t^n}{(n-k)!}. \tag{2.4}$$

Using the following result [9]:

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} A(k,n) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k,n+k), \tag{2.5}$$

we get

$$S = \sum_{p,q,n,k=0}^{\infty} H_{p+q+n+k}(x,y) A_{p+q+n+k,q+k} \frac{u^p}{p!} \frac{v^q}{q!} \frac{t^n}{n!} \frac{(-zt)^k}{k!}. \tag{2.6}$$

Now, using the following results [9]:

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} A(k,n-k), \tag{2.7}$$

$$\sum_{n=0}^{\infty} (\lambda)_n \frac{x^n}{n!} = (1-x)^- \lambda, \tag{2.8}$$

we get

$$S = \sum_{p,q,k=0}^{\infty} H_{p+q+k}(x,y) A_{p+q+k,q+k} \frac{(u+t)^p}{p!} \frac{v^q}{q!} \frac{(-zt)^k}{k!}. \tag{2.9}$$

Finally using the results (2.7) and (2.8), after a little simplification, we arrive at the right-hand side of (2.1). This completes the proof of Theorem 2.1.The Theorem 2.2. can be established similarly .

**Remark 2.1.** On taking $u = -t$ in Theorems 2.1. and 2.2., we obtain the following family of bilateral generating functions:

**Corollary 2.1.**

$$\sum_{p,q,n=0}^{\infty} H_{p+q+n}(x,y) S_n^{p,q}(z) \frac{(-t)^p}{p!} \frac{v^q}{q!} \frac{t^n}{n!} = \sum_{q=0}^{\infty} H_q(x,y) A_{q,q} \frac{(v-zt)^q}{q!}. \tag{2.10}$$

**Corollary 2.2.**

$$\sum_{p,q,n=0}^{\infty} L_{p+q+n}(x,y) S_n^{p,q}(z) \frac{(-t)^p}{p!} \frac{v^q}{q!} \frac{t^n}{n!} = \sum_{q=0}^{\infty} L_q(x,y) A_{q,q} \frac{(v-zt)^q}{q!}. \tag{2.11}$$

**Remark 2.2.** On taking $v = 0$ in Theorems 2.1. and 2.2. and using the relation $S_n^{p,0}(z) = S_{n,p}^1(z)$ , we obtain the following family of bilateral generating functions :

**Corollary 2.3.**

$$\sum_{p,n=0}^{\infty} H_{p+n}(x,y) S_{n,p}^1(z) \frac{u^p}{p!} \frac{t^n}{n!} = \sum_{p,q=0}^{\infty} H_{p+q}(x,y) A_{p+q,q} \frac{(u+t)^p}{p!} \frac{(-zt)^q}{q!}. \tag{2.12}$$

**Corollary 2.4.**

$$\sum_{p,n=0}^{\infty} L_{p+n}(x,y) S_{n,p}^1(z) \frac{u^p}{p!} \frac{t^n}{n!} = \sum_{p,q=0}^{\infty} L_{p+q}(x,y) A_{p+q,q} \frac{(u+t)^p}{p!} \frac{(-zt)^q}{q!}, \tag{2.13}$$

*where $S_{n,m}^N(z)$ is the extended Srivastava polynomials (1.2).*

## 3. Applications

I. In (2.10) and (2.11) Choosing $A_{m,n} = (-\alpha - m)_n$ and using (1.4), we get

$$\begin{aligned} &\sum_{p,q,n=0}^{\infty} (1+\alpha+p+n)_q H_{p+q+n}(x,y) L_n^{(\alpha+p)}(z) \frac{t^p}{p!} \frac{v^q}{q!} \left(\frac{t}{z}\right)^n \\ &= \sum_{q=0}^{\infty} (\alpha+1)_q H_q(x,y) \frac{(v+z/t)^q}{q!} \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} &\sum_{p,q,n=0}^{\infty} (1+\alpha+p+n)_q L_{p+q+n}(x,y) L_n^{(\alpha+p)}(z) \frac{t^p}{p!} \frac{v^q}{q!} \left(\frac{t}{z}\right)^n \\ &= \sum_{q=0}^{\infty} (\alpha+1)_q L_q(x,y) \frac{(v+z/t)^q}{q!}. \end{aligned} \tag{3.2}$$

Now, by using (1.13) and (1.14) in (3.1) and (3.2) respectively , we get

$$(3.3)\quad \sum_{p,q,n=0}^{\infty} (1+\alpha+p+n)_q H_{p+q+n}(x,y) L_n^{(\alpha+p)}(z) \frac{t^p}{p!}\frac{v^q}{q!}\left(\frac{t}{z}\right)^n = [1-2xyw]^{-\alpha-1}$$

$$\times F\, \begin{matrix} 2:0;0 \\ 0:0;0 \end{matrix} \left[ \begin{matrix} \frac{\alpha}{2}, \frac{\alpha}{2}+\frac{1}{2} & : & - & ; & - & ; \\ - & : & - & ; & - & ; \end{matrix} \frac{-4x^2w^2}{(1-2xyw)^2}, \frac{-4y^2w^2}{(1-2xyw)^2} \right],$$

where $w = v + \frac{t}{z}$ and

$$(3.4)\quad \sum_{p,q,n=0}^{\infty} (1+\alpha+p+n)_q L_{p+q+n}(x,y) L_n^{(\alpha+p)}(z) \frac{t^p}{p!}\frac{v^q}{q!}\left(\frac{t}{z}\right)^n$$

$$= \left(\frac{z}{z-yt-yvz}\right)^{1+\alpha} {}_1F_1\left[\alpha+1;1;\frac{-x(vz+t)}{(z-yt-yvz)}\right].$$

Further, if we take $v = 0$ in (3.3) and (3.4) respectively we obtain

$$(3.5)\quad \sum_{p,n=0}^{\infty} H_{p+n}(x,y) L_n^{(\alpha+p)}(z) \frac{t^p}{p!}\left(\frac{t}{z}\right)^n = [1-2xyt/z]^{-\alpha-1}$$

$$\times F\, \begin{matrix} 2:0;0 \\ 0:0;0 \end{matrix} \left[ \begin{matrix} \frac{\alpha}{2}, \frac{\alpha}{2}+\frac{1}{2} & : & - & ; & - & ; \\ - & : & - & ; & - & ; \end{matrix} \frac{-4x^2t^2}{(z-2xyt)^2}, \frac{-4y^2t^2}{(z-2xyt)^2} \right]$$

and

$$(3.6)\quad \sum_{p,n=0}^{\infty} L_{p+n}(x,y) L_n^{(\alpha+p)}(z) \frac{t^p}{p!}\left(\frac{t}{z}\right)^n = \left(\frac{z}{z-yt}\right)^{1+\alpha} {}_1F_1\left[\alpha+1;1;\frac{-xt}{z-yt}\right].$$

II. In (2.10) and (2.11) Choosing $A_{m,n} = \frac{(\alpha+\beta+1)_{2m}(-\beta-m)_n}{(\alpha+\beta+1)_m(-\alpha-\beta-2m)_m}$ and using (1.6), we get

$$(3.7)\quad \sum_{p,q,n=0}^{\infty} \frac{(1+\alpha+\beta+p+q+n)_{p+q+n}(-\beta-p-q-n)_q}{(1+\alpha+\beta+2p+q+n)_n(-\alpha-\beta-2p-2q-2n)_q} H_{p+q+n}(x,y)$$

$$\times P_n^{(\alpha+p+q,\beta+p)}(z) \frac{(-t)^p}{p!}\frac{v^q}{q!}\left(\frac{2t}{1+z}\right)^n = \sum_{q=0}^{\infty} (\beta+1)_q H_q(x,y) \frac{(v-2t/(1+z))^q}{q!}$$

and

$$(3.8)\quad \sum_{p,q,n=0}^{\infty} \frac{(1+\alpha+\beta+p+q+n)_{p+q+n}(-\beta-p-q-n)_q}{(1+\alpha+\beta+2p+q+n)_n(-\alpha-\beta-2p-2q-2n)_q} L_{p+q+n}(x,y)$$

$$\times P_n^{(\alpha+p+q,\beta+p)}(z) \frac{(-t)^p}{p!}\frac{v^q}{q!}\left(\frac{2t}{1+z}\right)^n = \sum_{q=0}^{\infty} (\beta+1)_q L_q(x,y) \frac{(v-2t/(1+z))^q}{q!}.$$

Now, using (1.13) and (1.14) in (3.7) and (3.8) respectively , we get

$$\sum_{p,q,n=0}^{\infty}\frac{(1+\alpha+\beta+p+q+n)_{p+q+n}(-\beta-p-q-n)_q}{(1+\alpha+\beta+2p+q+n)_n(-\alpha-\beta-2p-2q-2n)_q}H_{p+q+n}(x,y) \tag{3.9}$$

$$\times P_n^{(\alpha+p+q,\beta+p)}(z)\frac{(-t)^p}{p!}\frac{v^q}{q!}\left(\frac{2t}{1+z}\right)^n=[1-2xyw]^{-\beta-1}$$

$$F\begin{matrix}2:0;0\\0:0;0\end{matrix}\left[\begin{matrix}\frac{\beta+1}{2},\frac{\beta+2}{2} & : & - & ; & - & ; \\ - & : & - & ; & - & ;\end{matrix}\ \frac{-4x^2w^2}{(1-2xyw)^2},\frac{-4y^2w^2}{(1-2xyw)^2}\right]$$

and

$$\sum_{p,q,n=0}^{\infty}\frac{(1+\alpha+\beta+p+q+n)_{p+q+n}(-\beta-p-q-n)_q}{(1+\alpha+\beta+2p+q+n)_n(-\alpha-\beta-2p-2q-2n)_q}L_{p+q+n}(x,y) \tag{3.10}$$

$$\times P_n^{(\alpha+p+q,\beta+p)}(z)\frac{(-t)^p}{p!}\frac{v^q}{q!}\left(\frac{2t}{1+z}\right)^n=(1-yw)^{-1-\beta}\ {}_1F_1\left[\beta+1;1;\frac{-xw}{1-yw}\right]$$

where $w=v-\frac{2t}{1+z}$.

Further, if we take $v=0$ in (3.9) and (3.10) respectively we obtain

$$\sum_{p,n=0}^{\infty}(1+\alpha+\beta+p+n)_pH_{p+n}(x,y)P_n^{(\alpha+p,\beta+p)}(z)\frac{(-t)^p}{p!}\left(\frac{2t}{1+z}\right)^n \tag{3.11}$$

$$=\left(\frac{1+z+4xyt}{1+z}\right)^{-\beta-1}$$

$$\times F\begin{matrix}2:0;0\\0:0;0\end{matrix}\left[\begin{matrix}\frac{\beta+1}{2},\frac{\beta+2}{2} & : & - & ; & - & ; \\ - & : & - & ; & - & ;\end{matrix}\ \frac{-(4xt)^2}{(1+z+4xyt)^2},\frac{-(4yt)^2}{(1+z+4xyt)^2}\right]$$

and

$$\sum_{p,n=0}^{\infty}(1+\alpha+\beta+p+n)_pL_{p+n}(x,y)P_n^{(\alpha+p,\beta+p)}(z)\frac{(-t)^p}{p!}\left(\frac{2t}{1+z}\right)^n \tag{3.12}$$

$$=\left(\frac{1+z}{1+z+2yt}\right)^{1+\beta}\ {}_1F_1\left[\beta+1;1;\frac{2xt}{1+z+2yt}\right].$$

III. In (2.10) and (2.11) choosing $A_{m,n}=\frac{(\alpha)_{2m}}{(\alpha)_{m+n}}$ and using (1.8), we get

$$\sum_{p,q,n=0}^{\infty}(\alpha+p+2q+2n)_pH_{p+q+n}(x,y)R_n(\alpha+p+2q;x)\frac{(-t)^p}{p!}\frac{v^q}{q!}t^n \tag{3.13}$$

$$=\sum_{q=0}^{\infty}H_q(x,y)\frac{(v-zt)^q}{q!},$$

$$\sum_{p,q,n=0}^{\infty}(\alpha+p+2q+2n)_p L_{p+q+n}(x,y)R_n(\alpha+p+2q;x)\frac{(-t)^p}{p!}\frac{v^q}{q!}t^n = \sum_{q=0}^{\infty} L_q(x,y)\frac{(v-zt)^q}{q!}. \tag{3.14}$$

Now, by using (1.12) and (1.15) in (3.13) and (3.14) respectively , we get

$$\sum_{p,q,n=0}^{\infty}(\alpha+p+2q+2n)_p H_{p+q+n}(x,y)R_n(\alpha+p+2q;x)\frac{(-t)^p}{p!}\frac{v^q}{q!}t^n = \exp[2xyt(v-zt)-(x^2+y^2)(v-zt)^2], \tag{3.15}$$

and

$$\sum_{p,q,n=0}^{\infty}(\alpha+p+2q+2n)_p L_{p+q+n}(x,y)R_n(\alpha+p+2q;x)\frac{(-t)^p}{p!}\frac{v^q}{q!}t^n = \exp(y(v-zt))C_0(x(v-zt)). \tag{3.16}$$

Further, if we take $v=0$ in (3.15) and (3.16) respectively we obtain

$$\sum_{p,n=0}^{\infty}(\alpha+p+2n)_p H_{p+n}(x,y)R_n(\alpha+p;x)\frac{(-t)^p}{p!}t^n = \exp[t^2(-2xyz-z^2(x^2+y^2))] \tag{3.17}$$

and

$$\sum_{p,q,n=0}^{\infty}(\alpha+p+2n)_p L_{p+n}(x,y)R_n(\alpha+p;x)\frac{(-t)^p}{p!}t^n = \exp(-yzt)C_0(-xzt). \tag{3.18}$$

**Remark 3.1.** The results (3.6), (3.12) and (3.18) are a known results of Al-Gonah [1].

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