ON TOPOLOGICAL EFFECT ALGEBRAS

M.R. Rakhshani
Department of Mathematics
Sistan and Balouchestan University
Zahedan
Iran
rezarakhshani@yahoo.com

R.A. Borzooei*
Department of Mathematics
Shahid Beheshti University
Tehran
Iran
borzooei@sbu.ac.ir

G.R. Rezaei
Department of Mathematics
Sistan and Balouchestan University
Zahedan
Iran
grezaei@hamoon.usb.ac.ir

Abstract. In this paper the notions of topological and paratopological effect algebras are defined and their properties are investigated. Then by considering the notion of uniformity space and using of Riesz ideals in effect algebras, a topology and a uniformity space on any effect algebra is obtained. Finally, by definition of an equivalence relation on the set of all Cauchy nets of an effect algebra, and geting a quotient structure by that set, a completion for above uniformity space is constructed.

Keywords: effect algebra, topology, paratopology, uniformity, completion.

1. Introduction

Topology and algebra, the two fundamental domains of mathematics, play complementary roles. Topology studies continuity and convergence and provides a general framework to study the concept of a limit. Algebra considers all kinds of operations and provides a basis for algorithms and calculations. Many of the most important objects of mathematics represent a blend of algebraic and topological structures. For example, in applications, in higher level domains of mathematics, such as functional analysis, dynamical systems, representation theory, topology and algebra come in contact naturally. Several mathematicians have endowed a number of algebraic structures associated with logical systems

$*$. Corresponding author

with a topology and have found some of their properties. For example, R. A. Borzooei et al. [1, 2, 14, 15] who defined semitopological and topological BL-algebras and MV-algebras and M. Haveshki et al. [11] introduced the topology induced by uniformity on BL-algebras. Effect algebras have been introduced by D. J. Foulis and M. K. Bennet in 1994 [6] for modelling unsharp measurement in a quantum mechanical system. They are a generalization of many structures which arises in quantum physics [3] and in mathematical economics. In the last few years, the theory of effect algebras has enjoyed a rapid development. As an important tool of studying, the topological structures of effect algebras not only can help us to describe the convergence properties, but also can help us to characterize some algebra properties of effect algebras. In this paper we present the main properties of effect algebras, and introduce topologies that cause the operation $\oplus$ and $'$ to be continuous, and study their properties. We define (para) topological effect algebra and some examples are presented. We apply the notion of ideals, especially Riesz ideals in an effect algebra and we produce a binary relation which is an congruence relation. Then we show that an effect algebra with the topology that is defined by Riesz ideals, form a uniform space. Finally, we introduce a completion on an effect algebra.

2. Preliminary

Recall that a set A with a family $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ of its subsets is called a *topological space*, denoted by $(A,\mathcal{U})$, if $A, \emptyset \in \mathcal{U}$, the intersection of any finite numbers of members of $\mathcal{U}$ is in $\mathcal{U}$ and the arbitrary union of members of $\mathcal{U}$ is in $\mathcal{U}$. The members of $\mathcal{U}$ are called *open sets* of A and the complement of $U \in \mathcal{U}$, that is $A\setminus U$, is said to be a *closed set.* If B is a subset of A, the smallest closed set containing B is called the *closure* of B and denoted by $\overline{B}$ (or $cl_u B$). A subset P of A is said to be a *neighborhood* of $x \in A$, if there exists an open set U such that $x \in U \subseteq P$. A subfamily $\{U_\alpha\}$ of $\mathcal{U}$ is said to be a *base* of $\mathcal{U}$ if for each $x \in U \in \mathcal{U}$ there exists an $\alpha \in I$ such that $x \in U_\alpha \subseteq U$, or equivalently, each U in $\mathcal{U}$ is the union of members of $\{U_\alpha\}$. A subfamily $\{U_\beta\}$ of $\mathcal{U}$ is a *subbase* for $\mathcal{U}$ if the family of finite intersections of members of $\{U_\beta\}$ forms a base of $\mathcal{U}$. Let $\mathcal{U}_x$ denote the totality of all neighborhoods of x in A. Then a subfamily $\mathcal{V}_x$ of $\mathcal{U}_x$ is said to form a *fundamental system* of neighborhoods of x, if for each U_x in $\mathcal{U}_x$, there exists V_x in $\mathcal{V}_x$ such that $V_x \subseteq U_x$ (See [3]). Let $(A, *)$ be an algebra of type 2 and $\mathcal{U}$ be a topology on A. Then $\mathcal{A} = (A, *, \mathcal{U})$ is called a *left (right) topological algebra*, if for all $a \in A$ the map $* : A \to A$ is defined by $x \to a * x$ ($x \to x * a$) is continuous, or equivalently, for any x in A and any open set U of $a * x$ $(x * a)$, there exists an open set V of x such that $a * V \subseteq U$ $(V * a \subseteq U)$, *topological algebra*, if the operation $*$ is continuous, or equivalently, if for any x, y in A and any open set (neighborhood) W of $x * y$ there exist two open sets (neighborhoods) U and V of x and y, respectively, such that $U * V \subseteq W$ (See [2]). An *effect algebra* is algebraic structure $(E, \oplus, 0, 1)$, where 0,1 are distinct elements of E and $\oplus$ is a partial binary operation on E that satisfies the following conditions:

($E1$) (Commutative law) If $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$.

($E2$) (Associative law) If $a \oplus b$ and $(a \oplus b) \oplus c$ are defined, then $b \oplus c$ and $a \oplus (b \oplus c)$ are defined and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.

($E3$) (Orthosupplementation law) For each $a \in E$ there exist a unique $b \in E$ such that $a \oplus b$ is defined and $a \oplus b = 1$.

($E4$) (Zero-Unit law) If $a \oplus 1$ is defined, then $a = 0$.

For each $a \in E$, we denote the unique b in condition ($E3$) by a' and call it the orthosupplement of a. The sense is that if a presents a proposition, then a' corresponds to the negation. We define $a \leq b$, if there exists $c \in E$ such that $a \oplus c = b$. Such an element c is unique, and therefore we can introduce a dual operation $\ominus$ in E by $a \ominus b = c$ if and only if $a = c \oplus b$. Partial order $\leq$ on E which is defined as above, is a totally order (See [6]). Let A and B be subsets of a effect algebra E. Then $A \oplus B$ denotes the set $\{a \oplus b : a \in A, b \in B\}$, and A' denotes the set $\{a' : a \in A\}$. If $a \oplus b$ is defined, then we say that a and b are *orthogonal* and write $a \perp b$.

Example 2.1 ([4]). (i) Let $E = [0,1] \subseteq R$, where $a \oplus b$ is defined if and only if $a + b \leq 1$, in which case we define, $a \oplus b = a + b$. Then $(E, \oplus, 0, 1)$ is an effect algebra.

(ii) Let $C_n = \{0, a, 2a.....na = 1\}$, where $ia \oplus ja$ on C_n is defined if and only if $i + j \leq n$ for any $i, j = 0, 1, 2, ..., n$, and in this case $ia \oplus ja = (i + j)a$. Then n-chain $(C_n, \oplus, 0, 1)$ is an effect algebra.

Example 2.2. Let $E = \{0, a, b, 1\}$, where $0 < a, b < 1$. Consider the following table:

$\oplus$	0	a	b	1
0	0	a	b	1
a	a	a	1	-
b	b	1	b	-
1	1	-	-	-

Then, $(E, \oplus, 0, 1)$ is an effcet algebra.

Proposition 2.3 ([6]). The following properties hold for any effect algebra:

(i) $a'' = a$,
(ii) $1' = 0$ and $0' = 1$,
(iii) $0 \leq a \leq 1$,
(iv) $a \oplus 0 = a$,
(v) $a \oplus b = 0 \Rightarrow a = b = 0$,
(vi) $a \leq a \oplus b$,
(vii) $a \leq b \Rightarrow b' \leq a'$,
($viii$) $b \ominus a = (a \oplus b')'$,
(ix) $a \oplus b' = (b \ominus a)'$,

(x) $a = a \ominus 0$,
(xi) $a \ominus a = 0$,
(xii) $a' = 1 \ominus a$ and $a = 1 \ominus a'$.

Let $(E, \oplus, 0, 1)$ be an effect algebra. A nonempty subset I of E is said to be an ideal of E, if for all $a, b \in E$, $a \in I$ and $b \leq a$ implies $b \in I$ and $a \ominus b \in I$ and $b \in I$ implies $a \in I$.

Equivalently, if $a \oplus b$ is defined, then $a \oplus b \in I \Leftrightarrow a, b \in I$.(See [13])

A binary relation $\sim$ on effect algebra E is called a *congruence relation* if:

$(C1)$ $\sim$ is an equivalence relation,

$(C2)$ $a \sim a_1, b \sim b_1, a \perp b$ and $a_1 \perp b_1$ then $a \oplus b \sim a_1 \oplus b_1$.

$(C3)$ if $a \sim b$ and $b \perp c$, then there exists $d \in E$ such that $d \sim c$ and $a \perp d$.

Note: Condition $(C3)$ is equivalent to the conditions:

$(C4)$ If $a \sim b$ and $a \oplus a_1 \sim b \oplus b_1$, then $a_1 \sim b_1$.

$(C5)$ If $a \sim b \oplus c$, then there are a_1, a_2 such that $a = a_1 \oplus a_2$ and $a_1 \sim b, a_2 \sim c$. and condition $(C4)$ is equivalent to condition

$(C6)$ if $a \sim b$, then $a' \sim b'$.(See [8])

If $\sim$ is a congruence relation on effect algebra E, then quotient $E/\sim$ is an effect algebra ([5]). If I is an ideal on effect algebra E, we define a binary relation $\sim_I$ on E by $a \sim_I b$ if and only if there are $i, j \in I$, such that $i \leq a, j \leq b$ and $a \ominus i = b \ominus j$ or equivalently, there exists $k \in E$, such that $k \leq a, b$, $a \ominus k$ and $b \ominus k \in I$. Let $(E, \oplus, 0, 1)$ be an effect algebra and I be an ideal of E. We say that I is a *Riesz ideal* of E if for any $i \in I$ and $a, b \in E$ if $a \perp b$ and $i \leq a \oplus b$, then there exist $a_1, b_1 \in I$ such that $a_1 \leq a, b_1 \leq b$ and $i \leq a_1 \oplus b_1$.

Theorem 2.4 ([7])**.** *Let I be an ideal in an effect algebra E. Then $\sim_I$ is a congruence relation on E if and only if I is a Riesz ideal of E.*

Note: From now one, in this paper we let $(E, \oplus, 0, 1)$ or E is an effect algebra.

3. On topological effect algebras

In this section we construct a topology and paratopology on effect algebras.

From now on, we consider the operation $'$ as a single operation from E into E by $'(a) = a'$.

Definition 3.1. Let $\mathcal{T}$ be a topology on E. Then we say that $(E, \mathcal{T})$ is a:

(i) *paratopological effect algebra* if the operation $\oplus$ is continuous, or equivalently, for any $x, y \in E$ and any open neighborhood W of $x \oplus y$, there exist two open neighborhoods U and V of x and y, respectively, such that $U \oplus V \subseteq U$,

(ii) *topological effect algebra* if the operation $\oplus$ and $'$ are continuous.

Example 3.2. (i) Let $E = [0, 1]$ be effect algebra as Example 2.1(i) with the interval topology $\mathcal{T}$ of R. Then E is a topological effect algebra. For this, let $W = (a \oplus b - \varepsilon, a \oplus b + \varepsilon)$ be an open neighborhood of $a \oplus b$. It is clearly

$U = (a - \varepsilon/2, a + \varepsilon/2)$ and $V = (b - \varepsilon/2, b + \varepsilon/2)$ are two open neighborhoods of x and y, respectively. Since $U \oplus V = (a \oplus b - \varepsilon, a \oplus b + \varepsilon) \subseteq W$, hence $\oplus$ is continuous. Now we prove that $\prime$ is continuous, too. Let $W = (a' - \varepsilon, a' + \varepsilon)$ be an neighborhood of a' and $\delta \leq \varepsilon$. Since $(a - \delta, a + \delta)$ is a neighborhood of a and

$$\begin{aligned} \prime(a - \delta, a + \delta) &= ((a + \delta)', (a - \delta)') = (1 - (a + \delta), 1 - (a - \delta)) \\ &= ((1 - a) - \delta, (1 - a) + \delta) \\ &= (a' - \delta, a' + \delta) \\ &\subseteq W. \end{aligned}$$

Hence E is a topological effect algebra.

(ii) Consider effect algebra C_4 as Example 2.1(ii). Then we have $C_4 = \{0, 1/4, 2/4, 3/4, 1\}$. It is clearly $\mathcal{T} = \{\phi, \{0\}, \{0, 1/4\}, C_4\}$ is a topology on C_4. Now $(C_4, \mathcal{T})$ is a paratopological effect algebra. For this, it is enough to show that the mapping $\oplus$ is continuous. Let W be an open neighborhood of $a \oplus b$. If $a \oplus b \geq 1/2$, then C_4 is the unique open neighborhood of $a \oplus b$. Hence we can consider C_4 as the open neighborhood of a and b such that $C_4 \oplus C_4 \subseteq C_4$. If $a = b = 0$, then $a \oplus b = 0$, and so $W = \{0, 1/4\}$ or $\{0\}$ or C_4. Consider $U = V = \{0\}$. Thus, $U \oplus V = \{0\} \subseteq W$. If $a = 0$ and $b = 1/4$, then $a \oplus b = 1/4$ and so $W = \{0, 1/4\}$ or C_4, which are the neighborhoods of $1/4$. Now, if $U = \{0\}$ and $V = \{0, 1/4\}$, then $U \oplus V = \{0, 1/4\} \subseteq W$. Therefore, the mapping $\oplus$ is continuous, and so C_4 is a paratopological effect algebra. But C_4 is not a topological effect algebra, because $(3/4)' \in \{0, 1/4\}$, C_4 is the only open neighborhood of $3/4$ and $C_4' = C_4 \nsubseteq W = \{0, 1/4\}$. Therefore, the mapping $\prime$ is not continuous.

Theorem 3.3. *Let $\mathcal{T}$ be a topology on E. Then $(E, \mathcal{T})$ is a topological effect algebra if and only if the function $F : E \times E \to E$ defined by $F(a, b) = a \oplus b'$ is continuous.*

Proof. Suppose $(E, \mathcal{T})$ is a topological effect algebra. Since the mapping $i(a) = a$ and $\prime(a) = a'$ on E are continuous, then the mapping $f(a, b) = (i(a), \prime(b)) = (a, b')$ is continuous on $E \times E$, too. Hence $F = \oplus \circ f$ is continuous. Conversely, let F be continuous. Then for any $b \in E$, $K = F(0, b)$ is continuous. We define $h : E \to E \times E$ by $h(b) = (0, b)$. Clearly, h is continuous. Hence, $\prime = K \circ h$ is continuous. Now we show that $\oplus$ is continuous. Let $f : E \times E \to E \times E$ be defined by $f(a, b) = (a, b')$. Since the identity map and $\prime$ are continuous, f is continuous, too. Now, since $\oplus = F \circ f$, so $(F \circ f)(a, b) = F(a, b') = a \oplus (b')' = a \oplus b$. Hence $\oplus$ is continuous. $\square$

Notation. For $a \in E$, we define the maps $T_a, L_a, R_a : E \to E$ as follows:

$$T_a(x) = a \oplus x \ , \ L_a(x) = a \ominus x \ , \ R_a(x) = x \ominus a.$$

Proposition 3.4. Let $(E, \mathcal{T})$ be a paratopological effect algebra. Then $(E, \mathcal{T})$ is a topological effect algebra if and only if the mapping $\prime$ is an open map.

Proof. ($\Leftarrow$) Let $(E,\mathcal{T})$ be a paratopological effect algebra and mapping $'$ be an open map in $(E,\mathcal{T})$. We show that for any $a \in E$, $'$ is continuous, for any $a \in U \in \mathcal{T}$, there exists $V \in \mathcal{T}$ such that $a' \in V$ and $V' \subseteq U$. Put $V = U'$, so $a' \in V$. Since $'$ is an open map, hence $V \in \mathcal{T}$. On the other hand by Proposition 2.3(i), we have $V' = (U')' = U \subseteq U$.

($\Rightarrow$) The proof is clear. □

Theorem 3.5. *Let $(E,\mathcal{T})$ be a topological effect algebra. Then the operation $\ominus$ is continuous.*

Proof. Let $(E,\mathcal{T})$ be a topological effect algebra. Then the mapping $f(a,b) = (b,a')$ on $E \times E$ is continuous. On the other hand, since the maps $\oplus$ and $'$ are continuous, thus $' \circ \oplus \circ f$ is continuous. Now by Proposition 2.3(viii), we have;

$$(' \circ \oplus \circ f)(a,b) = '(\oplus(f(a,b))) = '(\oplus(b,a')) = '(b \oplus a') = (b \oplus a')' = a \ominus b.$$

Hence $\ominus$ is continuous. □

Theorem 3.6. *Let $\mathcal{T}$ be a topology on E and $\ominus$ and $'$ are continuous on it. Then $(E,\mathcal{T})$ is a topological effect algebra.*

Proof. Let $f : E \times E \to E \times E$ is defined by $f(a,b) = (a',b)$, for any $a,b \in E$. Since $'$, $\ominus$ and f are continuous on E, thus $' \circ \ominus \circ f$ is continuous, too. Hence

$$(' \circ \ominus \circ f)(a,b) = '(\ominus(a',b)) = '(a' \ominus b) = (a' \ominus b)' = b \oplus (a')' = b \oplus a.$$

and so $\oplus$ is continuous on E. □

Lemma 3.7. *Let $(E,\mathcal{T})$ be a topological effect algebra. Then:*

(i) for any $a,b \in E$, there is a continuous map f on E such that $f(a) = b$,

(ii) the mapping $'(a) = a'$ on E is a homeomorphism,

(iii) for any $a \in E$ if L_a or R_a be an open map, then T_a is an open map, too.

Proof. (i) Let $a,b \in E$. As E is a topological effect algebra, then T_b and T_a are continuous. Then $f = T_b \circ T_a$ is continuous, too. Now by Proposition 2.3(xi, iv),

$$f(a) = T_b \circ L_a(a) = T_b(a \ominus a) = T_b(0) = 0 \oplus b = b.$$

(ii) By Proposition 2.3(i), $(x')' = x$, for each $x \in E$. Then $'$ is an invertible map and $(')^{-1} = '$. Since $'$ is continuous, thus $'$ is a homeomorphism.

(iii) Let $a \in E$ and R_a be an open map. Suppose that U is an open set of $\mathcal{T}$. By (ii), U' is an open set, too. Since R_a is an open map, then $R_a(U') = U' \ominus a$ is open. Again by (ii), $(U' \ominus a)'$ is open. On the other hand by Proposition 2.3(ix), $(U' \ominus a)' = a \oplus (U')' = a \oplus U$. Hence $a \oplus U$ or $T_a(U)$ is open and this means that T_a is an open map. The proof of the other case is similar. □

Proposition 3.8. Let $\mathcal{N}$ be a fundamental system of open neighborhoods of 0 in a topological effect algebra $(E, \mathcal{T})$. If $1 \in \cap\mathcal{N}$, then there exists a fundamental system $\mathcal{N}_0$ of open neighborhoods of 0 such that $U = U'$, for each $U \in \mathcal{N}_0$.

Proof. Let $\mathcal{N}_0 = \{V \cap V' : V \in \mathcal{N}\}$. Since $1 \in V$, we get $0 \in V'$ and so $0 \in V \cap V'$. By Lemma 3.7(ii), the map $'$ is a homeomorphism, so for each $V \in \mathcal{N}$, $V \cap V'$ is an open neighborhood of 0. Let O be an open neighborhood of 0. Because $0 \oplus 0 = 0 \in O$, there is $W \in \mathcal{N}$ such that $W = 0 \oplus W \subseteq O$. On the other hand, since $0' = 1 \in W$, there is $V \in \mathcal{N}$ such that $V' \subseteq W$. Hence $V \cap V' \subseteq V' \subseteq W \subseteq O$. This shows that $\mathcal{N}_0$ is a fundamental system of open neighborhoods of 0. Clearly, for each $V \in \mathcal{N}$,

$$(V \cap V')' = V' \cap (V')' = V' \cap V.$$

Hence for any $U \in \mathcal{N}_0$, we have $U = U'$. □

Definition 3.9. Let $\mathcal{N}$ be a family of subsets in E. We call $\mathcal{N}$ is a *system of 0* if $0 \in \cap\mathcal{N}$ and

(i) for every $x \in U \in \mathcal{N}$, there is $V \in \mathcal{N}$ such that $x \oplus V \subseteq U$,
(ii) for every $U \in \mathcal{N}$, there is $V \in \mathcal{N}$ such that $V \oplus V \subseteq U$,
(iii) for every $U, V \in \mathcal{N}$, there is $W \in \mathcal{N}$ such that $W \subseteq U \cap V$.

Proposition 3.10. Let $\mathcal{N}$ be a fundamental system of open neighborhoods of 0 in a paratopological effect algebra $(E, \mathcal{T})$. Then $\mathcal{N}$ is a system of 0 in E.

Proof. Clearly, $0 \in \cap\mathcal{N}$. Let $a \in U \in \mathcal{N}$. Because $\oplus$ is continuous and $a \oplus 0 = a \in U$, then there exists an open neighborhood W of 0 such that $a \oplus W \subseteq U$. Since $\mathcal{N}$ is a fundamental system of open neighborhoods of 0, there is $V \in \mathcal{N}$ such that $0 \in V \subseteq W$. Hence, $a \oplus V \subseteq a \oplus W \subseteq U$. Now, let $U \in \mathcal{N}$. Because $0 \oplus 0 = 0 \in U$ and the mapping $\oplus$ is continuous, there are open neighborhoods W_0 and W_1 of 0 such that $W_0 \oplus W_1 \subseteq U$. We consider $W = W_0 \cap W_1$. As $\mathcal{N}$ is a fundamental system of open neighborhoods of 0, there is $V \in \mathcal{N}$ such that $0 \in V \subseteq W$. Hence $V \oplus V \subseteq W_0 \oplus W_1 \subseteq U$. Finally, let $U, V \in \mathcal{N}$. Since $\mathcal{N}$ is a fundamental system of neighborhoods of 0, hence it is closed under finite intersection, that is $W \subseteq U \cap V$. Therefore, $\mathcal{N}$ is a system of 0 in E. □

Theorem 3.11. *Let $\mathcal{N}$ be a system of 0 in effect algebra E. Then there exists a topology $\mathcal{T}$ on E such that $(E, \mathcal{T})$ is a paratopological effect algebra.*

Proof. Let $\mathcal{T} = \{W \subseteq E : \forall x \in W, \exists U \in \mathcal{N}\ s.t.\ x \oplus U \subseteq W\}$. First we show that, $x \oplus U \in \mathcal{T}$, for each $x \in E$ and $U \in \mathcal{N}$. Let $y \in x \oplus U$. Then $y = x \oplus a$, for some $a \in U$. Since $\mathcal{N}$ is a system, there is $V \in \mathcal{N}$ such that $a \oplus V \subseteq U$. Hence $y \oplus V = x \oplus a \oplus V \subseteq x \oplus U$ which implies that $x \oplus U \in \mathcal{T}$. Let $\{W_i : i \in I\}$ be a subfamily of $\mathcal{T}$. We show that $\cup W_i \in \mathcal{T}$. Let $x \in \cup W_i$. Then there exists $i \in I$ such that $x \in W_i$. Hence, there exists $U \in \mathcal{N}$ such that $x \oplus U \subseteq W_i$. Thus $x \oplus U \subseteq \cup W_i$.

Now let $W_1, W_2 \in \mathcal{T}$ and $W = W_1 \cap W_2$. We prove that $W \in \mathcal{T}$. For $x \in W$, there exist $U_1, U_2 \in \mathcal{N}$ such that $x \oplus U_1 \subseteq W_1$ and $x \oplus U_2 \subseteq W_2$. Since $\mathcal{N}$ is a system of 0, there is $U \in \mathcal{N}$ such that $U \subseteq U_1 \cap U_2$. Because

$$x \oplus U \subseteq x \oplus (U_1 \cap U_2) \subseteq (x \oplus U_1) \cap (x \oplus U_2) \subseteq W_1 \cap W_2 = W,$$

then $\mathcal{T}$ is a topology on E. Now, it is enough to prove that $\oplus$ is continuous. Let $z = x \oplus y$ and W be an open neighborhood of z. Then there is $U \in \mathcal{N}$ such that $z \oplus U \subseteq W$. By definition, there exists $V \in \mathcal{N}$ such that $V \oplus V \subseteq U$. On the other hands $x \oplus V$ and $y \oplus V$ are two open neighborhoods of x and y, respectively, such that

$$(x \oplus V) \oplus (y \oplus V) = x \oplus y \oplus V \oplus V \subseteq x \oplus y \oplus U = z \oplus U \subseteq W.$$

Therefore, $(E, \mathcal{T})$ is a paratopological effect algebra. $\square$

Example 3.12. Let $E = [0, 1]$ be effect algebra as Example 2.1(i) and $\mathcal{N} = \{[0, 1/n) : n \in N\}$. It is clearly that $\mathcal{N}$ is a system of 0 in E. By Theorem 3.11, $\mathcal{T} = \{W \subseteq E : \forall x \in W, \exists U \in \mathcal{N} \text{ s.t. } x \oplus U \subseteq W\}$ is a topology on E. We show that the operation $\oplus$ is continuous. Let $x \oplus y \in W \in \mathcal{T}$. Then there exists $n \in N$ such that $x \oplus y \oplus [0, 1/n) \subseteq W$. Consider $U = [x, x \oplus 1/k)$ and $V = [y, y \oplus 1/m)$ such that $k, m > 2$ and $1/k + 1/m \leq 1/n$. Thus, U and V are two open neighborhoods of x and y,, respectively such that $U \oplus V \subseteq W$. Therefore $(E, \mathcal{T})$ is a paratopological effect algebra.

4. Uniformity on effect algebras

In this section we study the concept of uniformity on effect algebras and define a topology on effect algebras by Riesz ideals.

Notations. Let X be a nonempty set and U, V are nonempty subsets of $X \times X$. Then we define:

$U \circ V = \{(a, b) : (c, b) \in U, (a, c) \in V, \text{ for some } c \in X\}$,

$U^{-1} = \{(a, b) : (b, a) \in U\}$,

$\triangle = \{(a, a) : a \in X\}$.

Definition 4.1 ([10])**.** By a *uniformity* on X we shall mean a nonempty collection $\mathcal{U}$ of subset of $X \times X$ which satisfies the following conditions:

(U_1) $\triangle \subseteq U$ for any $U \in \mathcal{U}$,

(U_2) if $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{U}$,

(U_3) if $U \in \mathcal{U}$, then there exists $V \in \mathcal{U}$ such that $V \circ V \subseteq U$,

(U_4) if $U, V \in \mathcal{U}$, then $U \cap V \in \mathcal{U}$,

(U_5) if $U \in \mathcal{U}$, and $U \subseteq V \subseteq X \times X$, then $V \in \mathcal{U}$.

If $\mathcal{U}$ is a uniformity on X, then the pair $(X, \mathcal{U})$ is called a *uniform structure* or *uniform space*.

Notations. Let Λ be a family of Riesz ideals in E, such that it is closed under intersection. Then for any $I \in \Lambda$, we define

$$U_I = \{(a,b) \in E \times E : a \backsim_I b\} \quad , \quad \mathcal{U}_\Lambda = \{U \subseteq E \times E : \exists I \in \Lambda \ s.t.\ U_I \subseteq U\}$$

Theorem 4.2. $(E, \mathcal{U}_\Lambda)$ *is a uniform space.*

Proof. (U_1): Since I is a Riesz ideal of E, $a \backsim_I a$, for any $a \in E$. Hence $\Delta \subseteq U_I \subseteq U$, for all $U \in \mathcal{U}_\Lambda$.

(U_2) : For any $U \in \mathcal{U}_\Lambda$, we have

$$\begin{aligned}(a,b) \in U^{-1} \quad &\Leftrightarrow \quad (b,a) \in U \Leftrightarrow (b,a) \in U_I \subseteq U, \exists I \in \Lambda \Leftrightarrow b \backsim_I a, \exists I \in \Lambda \\ &\Leftrightarrow \quad a \backsim_I b, \exists I \in \Lambda \Leftrightarrow (a,b) \in U_I, \exists I \in \Lambda \\ &\Leftrightarrow \quad (a,b) \in U\end{aligned}$$

(U_3) : For any $U \in \mathcal{U}_\Lambda$, the transitivity of $\sim_I$ implies that $U \circ U \subseteq U$.

(U_4) : For any $U, V \in \mathcal{U}_\Lambda$, there exist $I, J \in \Lambda$ such that $U_I \subseteq U$ and $U_J \subseteq V$. We claim that $U_I \cap U_J = U_{I \cap J}$. Let $(a,b) \in U_I \cap U_J$. Then $a \sim_I b$ and $a \sim_J b$. Hence, for any $i \in I$ and $a, b \in E, i \leq a \oplus b$ implies that there exist $a_1, b_1 \in I$ such that $a_1 \leq a$ and $b_1 \leq b$ with $i \leq a_1 \oplus b_1$. Furthermore, for any $j \in J$ and $a, b \in E, j \leq a \oplus b$ implies that there are $a_2, b_2 \in J$ such that $a_2 \leq a$ and $b_2 \leq b$ with $j \leq a_2 \oplus b_2$. If $K = I \cap J$, then $a \sim_K b$ and this means that $(a,b) \in U_{I \cap J}$. Conversely, let $(a,b) \in U_{I \cap J}$. Then $a \sim_{I \cap J} b$. Hence, for any $k \in I \cap J$ and $a, b \in E, k \leq a \oplus b$ implies that there are $a_1, b_1 \in I \cap J$ such that $a_1 \leq a$ and $b_1 \leq b$ with $k \leq a_1 \oplus b_1$. Since $k \in I$ we have $a \sim_I b$, which means that $(a,b) \in U_I$. Now, since $k \in J$, similarly $(a,b) \in U_J$ and so $(a,b) \in U_I \cap U_J$. Hence $U_I \cap U_J = U_{I \cap J}$. Since $I, J \in \Lambda$, then $I \cap J \in \Lambda$ and $U_I \cap U_J = U_{I \cap J} \subseteq U \cap V$. Therefore $U \cap V \in U_\Lambda$.

(U_5) : Let $U \in \mathcal{U}_\Lambda$ and $U \subseteq V \subseteq E \times E$. Then there exists $U_I \subseteq U \subseteq V$, which means that $V \in \mathcal{U}_\Lambda$. □

Theorem 4.3. *Let for any $a \in E$, $U[a] = \{b \in E : (a,b) \in U\}$. Then*

$$\mathcal{T}_\Lambda = \{G \subseteq E : \forall a \in G, \exists U \in \mathcal{U}_\Lambda \ s.t.\ U[a] \subseteq G\}$$

is a topology on E.

Proof. Clearly, $\phi, E \in \mathcal{T}_\Lambda$. Let Σ be a non-empty directed set, $\{G_\alpha\}_{\alpha \in \Sigma}$ be a subfamily of $\mathcal{T}_\Lambda$, and $a \in \bigcup_{\alpha \in \Sigma} G_\alpha$. Then, there exists $\alpha_0 \in \Sigma$ such that $a \in G_{\alpha_0}$. Since $G_{\alpha_0} \in \mathcal{T}_\Lambda$, there exist $U \in \mathcal{U}_\Lambda$ and $a \in U[a] \subseteq G_{\alpha_0} \subseteq \bigcup_{\alpha \in \Sigma} G_\alpha$. Hence $\bigcup_{\alpha \in \Sigma} G_\alpha \in \mathcal{T}_\Lambda$. Let $G_1, G_2 \in \mathcal{T}_\Lambda$, and $a \in G_1 \cap G_2$. Then there are $U_1, U_2 \in \mathcal{U}_\Lambda$ such that $a \in U_1[a]$ and $a \in U_2[a]$. By (U_4), $U_1 \cap U_2 \in \mathcal{U}_\Lambda$ and $a \in (U_1 \cap U_2)[a] \subseteq G_1 \cap G_2$. Therefore $G_1 \cap G_2 \in \mathcal{T}_\Lambda$. Hence $\mathcal{T}$ is a topology on E. □

Lemma 4.4. *Let $a, b \in E$ and $I \in \Lambda$. Then $U_I[a \oplus b] = U_I[a] \oplus U_I[b]$.*

Proof. Let $t \in U_I[a \oplus b]$. Then, $(t, a \oplus b) \in U_I$ and $t \sim_I a \oplus b$. Since I is a Riesz ideal of E, there are $t_1, t_2 \in E$ such that $t = t_1 \oplus t_2$, $t_1 \oplus t_2 \sim_I a \oplus b$ and $t_1 \sim_I a, t_2 \sim_I b$. Thus $t_1 \in U_I[a], t_2 \in U_I[b]$ and $t_1 \oplus t_2 \in U_I[a] \oplus U_I[b]$. Hence $t \in U_I[a] \oplus U_I[b]$ and so $U_I[a \oplus b] \subseteq U_I[a] \oplus U_I[b]$. By the similar way, we have $U_I[a] \oplus U_I[b] \subseteq U_I[a \oplus b]$. Therefore $U_I[a \oplus b] = U_I[a] \oplus U_I[b]$. □

Theorem 4.5. $(E, \mathcal{T}_\Lambda)$ *is a topological effect algebra.*

Proof. We prove that operations $\oplus$ and $\prime$ are continuous on E. Let G be an open set of E and $a \oplus b \in G$. Then there exists $U \in \mathcal{U}_\Lambda$ such that $U[a \oplus b] \subseteq G$, and there exists Riesz ideal I of E such that $U_I[a \oplus b] \subseteq U[a \oplus b]$. On the other hand, By Lemma 4.4, we have $U_I[a \oplus b] = U_I[a] \oplus U_I[b]$. Thus $\oplus(U_I[a], U_I[b]) = U_I[a] \oplus U_I[b] = U_I[a \oplus b] \subseteq U[a \oplus b] \subseteq G$ and this means that $\oplus$ is continuous. Now we show the mapping $\prime : E \to E$ by $\prime(a) = a'$ is continuous. Let G be an open set of E and a' in E. Then there exists $U \in \mathcal{U}_\Lambda$ such that $U[a'] \subseteq G$, and there is a Riesz ideal I of E such that $U_I[a'] \subseteq U[a'] \subseteq G$. Since $I = I'$, $\prime(U_I[a]) = U_{I'}[a'] = U_I[a'] \subseteq U[a'] \subseteq G$. Hence $\prime$ is continuous. □

Proposition 4.6. For any $i \in \Lambda$ and $x \in E$, $U_I[x]$ is a clopen subset of E.

Proof. Since the pair $(E, \mathcal{U}_\Lambda)$ is a uniform space, $U_I[x]$ is an open subset of E. We prove $U_I[x]$ is closed in E. Let $y \in (U_I[x])^c$. Hence $y \nsim_I x$ and so $U_I[x] \cap U_I[y] = \phi$. This means that $U_I[y] \subseteq (U_I[x])^c$, and so $U_I[x]$ is closed. Therefore, $U_I[x]$ is a clopen subset of E. □

Recall that a uniform space $(X, \mathcal{U})$ is called totally bounded if for each $U \in \mathcal{U}$, there exist $x_1, x_2, ..., x_n \in X$ such that $X = \bigcup_{i=1}^n U[x_i]$,

Theorem 4.7. *The following statements are equivalent:*
(i) *topological space* $(E, \mathcal{T}_\Lambda)$ *is compact.*
(ii) *uniformity space* $(E, \mathcal{U}_\Lambda)$ *is totally bounded.*

Proof. $(i) \Rightarrow (ii)$ This is clear by [10].

$(ii) \Rightarrow (i)$ Let $\{U_\alpha\}_{\alpha \in \Sigma}$ be a cover open sets of E and $J = \bigcap_{I \in \Lambda} I$. Since $(E, \mathcal{U}_\Lambda)$ is totally bounded and Λ is closed under intersection, we have $J \in \Lambda$ and there exists $x_1, x_2, ..., x_n \in E$ such that $E = \bigcup_{i=1}^n U_J[x_i]$. Now for any $1 \leq i \leq n$, there exists $\alpha_i \in \Sigma$ such that $x_i \in U_{\alpha_i}$. Hence $U_J[x_i] \subseteq U_{\alpha_i}$ and so

$$E = \bigcup_{i=1}^{n} U_J[x_i] \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}.$$

Therefore, E is compact. □

Corollary 4.8. *In uniform space* $(E, \mathcal{U}_\Lambda)$, *E is compact if and only if E is totally bounded.*

5. Completion of an effect algebra

The study of completion in uniformity spaces are very important. Hence in this section we are going to get some conditions that completion for uniformity space be an effect algebra.

Note. In this section, we suppose Λ is closed under finite intersection.

Definition 5.1 ([9])**.** Let $(X,\mathcal{U})$ be a uniformity space. Then uniformity space $(X^*,\mathcal{U}^*)$ is called a *completion of* $(X,\mathcal{U})$ if:
(i) $(X^*,\mathcal{U}^*)$ is complete.
(ii) for any uniformity space $(Y',\mathcal{U}')$ and map $i:(X,\mathcal{U})\to(Y,\mathcal{U}')$ there exist an extension $i^*:(X^*,\mathcal{U}^*)\to(Y,\mathcal{U}')$ such that: $i^*|_X=i$.

Recall that, for a nonempty set $\sum$, we say that $(\sum,\leq)$ is a *directed set*, if the binery relation $\leq$ on $\sum$ has the following properties: (i): for every $\alpha\in\Sigma$, $\alpha\leq\alpha$, (ii): if $\alpha\leq\beta$ and $\beta\leq\gamma$, then $\alpha\leq\gamma$, (iii): for any $\alpha,\beta\in\Sigma$ there exists a $\gamma\in\Sigma$ such that $\alpha\leq\gamma$ and $\beta\leq\gamma$. A *net* in a topological space X is a function from a non-empty directed set Σ to the space X. Nets will be denoted by $\{x_\alpha\}_{\alpha\in\Sigma}$, where x_α is the point of X assigned to the element $\alpha\in\Sigma$. The net of $\{x_\alpha\}_{\alpha\in\Sigma}$ is called a *Cauchy net* if and only if for each $U\in\mathcal{U}$, there exists $\alpha_0\in\Sigma$ such that for all $\alpha,\beta\geq\alpha_0$, then $(x_\alpha,x_\beta)\in U$. (See [9])

Lemma 5.2. *Let Σ be a nonempty directed set and $\widetilde{E}=\{\{x_\alpha\}_{\alpha\in\Sigma}:\{x_\alpha\}_{\alpha\in\Sigma}$ is a Cauchy net in $E\}$. Suppose that the binary relation $\sim$ on $\widetilde{E}$ is defined by $\{x_\alpha\}_{\alpha\in\Sigma}\sim\{y_\beta\}_{\beta\in\Sigma}$ if and only if for all $U\in\mathcal{U}_\Lambda$, there exist $\alpha_0,\beta_0\in\Sigma$ such that for all $\alpha\geq\alpha_0$ and $\beta\geq\beta_0$, $(x_\alpha,y_\beta)\in U$. Then $\sim$ is an equivalence relation on $\widetilde{E}$.*

Proof. Since $(E,\mathcal{U}_\Lambda)$ is a uniform space, for any $U\in\mathcal{U}_\Lambda$ and $\alpha\in\Sigma$, $\Delta=(x_\alpha,x_\alpha)\in U$. Hence, $\{x_\alpha\}_{\alpha\in\Sigma}\sim\{x_\alpha\}_{\alpha\in\Sigma}$. Let $\{x_\alpha\}_{\alpha\in\Sigma}\sim\{y_\beta\}_{\beta\in\Sigma}$. Then for any $U\in\mathcal{U}_\Lambda$, there are $\alpha_0,\beta_0\in\Sigma$ such that for all $\alpha\geq\alpha_0$ and $\beta\geq\beta_0$, $(x_\alpha,y_\beta)\in U$. Since $U\in\mathcal{U}_\Lambda$, we get $U^{-1}\in\mathcal{U}_\Lambda$ and so $(y_\beta,x_\alpha)\in U^{-1}$. Thus $\{y_\beta\}_{\beta\in\Sigma}\sim\{x_\alpha\}_{\alpha\in\Sigma}$.

Now, let $\{x_\alpha\}_{\alpha\in\Sigma}\sim\{y_\beta\}_{\beta\in\Sigma}$ and $\{y_\beta\}_{\beta\in\Sigma}\sim\{z_\gamma\}_{\gamma\in\Sigma}$. Hence for all $U\in\mathcal{U}_\Lambda$, there exist $\alpha_0,\beta_0\in\Sigma$ such that for all $\alpha\geq\alpha_0$ and $\beta\geq\beta_0$, $(x_\alpha,y_\beta)\in U$ and there exist $\beta_1,\gamma_0\in\Sigma$ such that for all $\beta\geq\beta_1$ and $\gamma\geq\gamma_0$, $(y_\beta,z_\gamma)\in U$. By Definition 4.1(U_3), for all $\alpha\geq\alpha_0$ and $\beta\geq\beta_0,\beta_1$ and $\gamma\geq\gamma_0$, $(x_\alpha,z_\gamma)\in U$. Thus $\{x_\alpha\}_{\alpha\in\Sigma}\sim\{z_\gamma\}_{\gamma\in\Sigma}$. Therefore $\sim$ is an equivalence relation. □

Notations. Let $E^*=\widetilde{E}/\sim$, for any $U\in\mathcal{U}_\Lambda$,

$$U^*=\{([\{x_\alpha\}],[\{y_\beta\}]):\exists\alpha_0,\beta_0\in\Sigma;\forall\alpha\geq\alpha_0,\beta\geq\beta_0,(x_\alpha,y_\beta)\in U\}$$

and $\mathcal{U}^*_\Lambda=\{U^*:\ U\in\mathcal{U}_\Lambda\}$.

Theorem 5.3. *$(E^*,\mathcal{U}^*_\Lambda)$ is a uniform space.*

Proof. (U_1) : Let $[\{x_\alpha\}_{\alpha\in\Sigma}] \in E^*$. Since $\{x_\alpha\}_{\alpha\in\Sigma} \sim \{x_\alpha\}_{\alpha\in\Sigma}$ for any $U^* \in \mathcal{U}^*_\Lambda$, there exists $\alpha \geq \alpha_0$ such that $(x_\alpha, x_\alpha) \in U$ and so $\Delta \in \mathcal{U}^*_\Lambda$.

(U_2): Let $U^* \in \mathcal{U}^*_\Lambda$. Then there exist $\{x_\alpha\}_{\alpha\in\Sigma}, \{y_\beta\}_{\beta\in\Sigma} \in \widetilde{E}$ such that there are $\alpha \geq \alpha_0$ and $\beta \geq \beta_0$ which $(x_\alpha, y_\beta) \in U$. Since $(E,\mathcal{U}_\Lambda)$ is a uniform space, hence $U = U^{-1}$ and so $(x_\alpha, y_\beta) \in U^{-1}$. Then $(x_\alpha, y_\beta) \in (U^{-1})^* = (U^*)^{-1}$. Therefore, $(U^*)^{-1} \in \mathcal{U}^*_\Lambda$

(U_3): For any $U^* \in \mathcal{U}^*_\Lambda$, the transitivity of $\sim$ implies that $U^* \circ U^* \subseteq U^*$.

(U_4): Let $U^*, V^* \in \mathcal{U}^*_\Lambda$. Then there exist $\{x_\alpha\}_{\alpha\in\Sigma}, \{y_\beta\}_{\beta\in\Sigma} \in \widetilde{E}$ such that there are $\alpha_0, \beta_0 \in \sum$ which for all $\alpha \geq \alpha_0$ and $\beta \geq \beta_0$, $(x_\alpha, y_\beta) \in U$. Similarly there are $\alpha_1, \beta_1 \in \sum$ such that for all $\alpha \geq \alpha_1$ and $\beta \geq \beta_1$, $(x_\alpha, y_\beta) \in V$. Thus for all $\alpha \geq \alpha_0, \alpha_1$ and $\beta \geq \beta_0, \beta_1$, we get $(x_\alpha, y_\beta) \in U \cap V$. Hence $U^* \cap V^* \in \mathcal{U}^*_\Lambda$.

(U_5) : Let $U^* \in \mathcal{U}^*_\Lambda$ and $U^* \subseteq V^* \subseteq E^* \times E^*$. Since $(x_\alpha, y_\beta) \in U$ and $U \subseteq V$, $(x_\alpha, y_\beta) \in V$. Hence $V^* \in \mathcal{U}^*_\Lambda$. □

Lemma 5.4. *For any $I \in \Lambda$, $U^*_I([\{x_\alpha \oplus y_\beta\}]) = U^*_I([\{x_\beta\}]) \oplus U^*_I([\{y_\beta\}])$.*

Proof. Let $I \in \Lambda$. Then $\{z_\alpha \oplus t_\beta\} \in U^*_I([\{x_\alpha \oplus y_\beta\}]) \Leftrightarrow z_\alpha \oplus t_\beta \sim_I x_\alpha \oplus y_\beta \Leftrightarrow z_\alpha \sim_I x_\alpha, t_\beta \sim_I y_\beta \Leftrightarrow [\{z_\alpha\}] \in U^*_I([\{x_\alpha\}])$ and $[\{t_\beta\}] \in U^*_I([\{y_\beta\}]) \Leftrightarrow \{z_\alpha \oplus t_\beta\} \in U^*_I([\{x_\alpha\}]) \oplus U^*_I([\{y_\beta\}])$ □

Notation. We denote:

$$\mathcal{T}_{\mathcal{U}^*_\Lambda} = \{G \subseteq E^* : \forall [\{x_\alpha\}] \in G, \exists U^* \in \mathcal{U}^*_\Lambda \ \ s.t. \ \ U^*([\{x_\alpha\}]) \subseteq G\}.$$

Theorem 5.5. *$(E^*, \mathcal{T}_{\mathcal{U}^*_\Lambda})$ is a topological effect algebra.*

Proof. Let $\oplus : E^* \times E^* \to E^*$ has been defined by $([\{x_\alpha\}], [\{y_\beta\}]) \mapsto [\{x_\alpha \oplus y_\beta\}]$. We show that the operation $\oplus$ is continuous. Let G be an open set of E^* such that $[\{x_\alpha \oplus y_\beta\}] \in G$. Then there is $U^* \in \mathcal{U}^*_\Lambda$ such that $U^*([\{x_\alpha \oplus y_\beta\}]) \subseteq G$. Hence there exists a Riesz ideal $I \in \Lambda$ such that $U^*_I([\{x_\alpha \oplus y_\beta\}]) \subseteq U^*([\{x_\alpha \oplus y_\beta\}]) \subseteq G$. By Lemma 5.4, we have $U^*_I([\{x_\alpha\}]) \oplus U^*_I([\{y_\beta\}]) \subseteq G$, where $U^*_I([\{x_\alpha\}])$ and $U^*_I([\{y_\beta\}])$ are two open neighbourhoods of $[\{x_\alpha\}]$ and $[\{y_\beta\}]$, respectively, and this means that the operation $\oplus$ is continuous. Now we prove the mapping $\prime : E^* \to E^*$ which is defined by $[\{x_\alpha\}] \mapsto [\{x'_\alpha\}]$ is continuous. Let G be an open set of E^* and $[\{x'_\alpha\}] \in G$. Hence there is $U^* \in \mathcal{U}^*_\Lambda$ such that $U^*([\{x'_\alpha\}]) \subseteq G$. Then, there exists $I \in \Lambda$ such that $U^*_I([\{x'_\alpha\}) \subseteq U^*([\{x'_\alpha\}]) \subseteq G$. On the other hand, $\prime(U^*_I([\{x_\alpha\}])) = U^*_I([\{x'_\alpha\}])$ and $U^*([\{x_\alpha\}])$ is an open neighborhood of $[\{x_\alpha\}]$. So $\prime$ is continuous. Thus $(E^*, \mathcal{T}_{\mathcal{U}^*_\Lambda})$ is a topological effect algebra. □

Theorem 5.6. *$(E^*, \mathcal{U}^*_\Lambda)$ is a completion of $(E, \mathcal{U}_\Lambda)$.*

Proof. Let $i : E \to E^*$ is defined by $x \mapsto [\{x_\alpha\}_{\alpha\in\Sigma}]$, where $x_\alpha = x$, for any $\alpha \in \Sigma$. Clearly if $i(x) = i(y)$ then $x = y$. Now we show that $i(E)$ is dense in E^*. Let $U^*([\{x_\alpha\}]) \in \mathcal{T}_{\mathcal{U}^*_\Lambda}$. Hence there is $\alpha_0 \in \Sigma$ such that for all $\alpha \geq \alpha_0$, $x_\alpha \in U[x]$ and so $x_\alpha \to x$. Hence $\{x_\alpha\}_{\alpha\in\Sigma} \sim \{t_\alpha\}_{\alpha\in\Sigma}$, where $t_\alpha = x$ for any

$\alpha \in \Sigma$ and this means $\{t_\alpha\} \in U^*([\{x_\alpha\}]) \cap i(E)$. Thus $U^*([\{x_\alpha\}]) \cap i(E) \neq \phi$. At present we prove that $(E^*, \mathcal{U}_\Lambda^*)$ is complete. Let $\{[(x_\alpha^\gamma)]\}_{\gamma \in \Sigma}$ be a Cauchy net in E^*. We show this net is convergent. As this net is Cauchy, so for every $U \in \mathcal{U}_\Lambda$ there is $\alpha_0 \in \Sigma$ such that for all $\alpha \geq \alpha_0, x_\alpha^\gamma \in U$. Now we define the net of $\{z_t\}_{t \in \Sigma}$ with $z_t = x_{\alpha_0}^\gamma$. We show that the net $\{[(x_\alpha^\gamma)_{\alpha \in \Sigma}]\}_{\gamma \in \Sigma}$ is convergent to $[\{z_t\}_{t \in \Sigma}]$. For this purpose, let U^* be a neighborhood of $[\{z_t\}_{t \in \Sigma}]$. Hence for all $\alpha \geq \alpha_0, [\{x_\alpha^\gamma\}] \in U^*([\{z_t\}])$ and the proof is completed. □

6. Conclusion

In this paper we have studied topological structuers on effect algebras and we introduced the notions of topological and paratopological effect algebras. Next researches can study separation axioms on topological effect algebras, quotient effect algebras and many of the other concepts of topology.

References

[1] R.A. Borzooei, N. Kohestani, G.R. Rezaei, *Metrizability on (Semi)Topological BL-algebras*, Soft Computing, 16(10), (2012), 1681-1690.

[2] R.A. Borzooei, G.R. Rezaei, N. Kouhestani, *On (semi)topological BL-algebra*, Iranian Journal Mathematical Sciences and Informatics, 6(1), (2001), 59-77.

[3] N. Bourbaki, *Elements of Mathematics General topology*, Addison-Wesley Publishing Company, 1966.

[4] S. Gudder, *Examples, Problems, and Results in Effect Algebras*, 35(11), (1996), 2365-2376.

[5] S. Gudder, S. Pulmannova, *Quotients of partial abelian monoids*, Algebra Universalis, 38, (1997), 395-421.

[6] D.J. Foulis, M.K. Bennett, *Effect algebras and unsharp quantumlogics*, Foundations of Physics, 24(10), (1994), 1331-1352.

[7] G. Jenca, *Notes on R_1-Ideals in Generalized Effect Algebras*, Algebra Universalis, 43, (2000), 302-319.

[8] G. Jenca, S. Pulmannova, *Ideals and quotients in lattice ordered effect algebras*, Soft Computing, 5(5), (2001), 376-380.

[9] K.D. Joshi, *Introduction to General Topology*, New Age International Publisher, (1997).

[10] J.L. Kelly, *General Topology*, Springer-Verlag, New York, (1955).

[11] M. Haveshki, E. Eslami, A. Borumand Saeid, *A topology induced by uniformity on BL-algebras,* Math. Log. Quart, 53(2), (2007), 162-169.

[12] T. Hosain, *Introduction to topological groups*, W.B.Sunders Company, 1966.

[13] Z. Ma, *Note on ideals of effect algebras*, Information Sciences, 179(5), (2009), 505-507.

[14] O. Zahiri, R.A. Borzooei, *Topology on BL-algebras*, Fuzzy Sets and Systems, 289, (2016), 137-150.

[15] O. Zahiri, R.A. Borzooei, *Semitopological BL-algebras and MV-algebras*, Demonstratio Mathematica, Vol. XLVII, No 3 (2014), 522-538.

Accepted: 16.03.2017