CHARACTERIZATIONS OF ORDERED SEMIHYPERGROUPS BASED ON ORDERED FUZZY POINTS

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Abstract. In this paper, we introduce the concepts of quasi-prime and quasi-semiprime fuzzy left hyperideals of ordered semihypergroups, and investigate their related properties. Furthermore, we give some characterizations of strongly semisimple ordered semihypergroups in terms of ordered fuzzy points and fuzzy left hyperideals. Especially, we prove that an ordered semihypergroup S is strongly semisimple if and only if every fuzzy left hyperideal of S can be expressed as the intersection of all quasi-prime fuzzy left hyperideals of S containing it.

Keywords: ordered semihypergroup, ordered fuzzy point, quasi-prime fuzzy left hyperideal, quasi-semiprime fuzzy left hyperideal, strongly semisimple ordered semihypergroup.

1. Introduction

The important concept of a fuzzy set put forth by L.A. Zadeh in 1965 [33] has opened up keen insights and applications in a wide range of scientific fields. Since its inception, the theory of fuzzy sets has developed in many directions and found applications in a wide variety of fields. The study of fuzzy sets and its application to various mathematical contexts has given rise to what is now

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commonly called “fuzzy mathematics”. Fuzzy algebra is an important branch of fuzzy mathematics. The study of fuzzy algebraic structures was started with the introduction of the concept of fuzzy subgroups of a group in the pioneering paper of A. Rosenfeld [24]. The fuzzy algebraic structures play an important role in Mathematics with wide applications in computer sciences, coding theory, theoretical physics, information sciences and topological spaces [11, 21]. Since then, fuzzy sets have been applied to diverse branches of algebra. In [16], N. Kehayopulu and M. Tsingelis applied the concept of fuzzy sets to the theory of ordered semigroups. Then they defined “fuzzy” analogue of several notations, which appeared to be useful in the theory of ordered semigroups. The theory of fuzzy sets on ordered semigroups has been recently developed. For more details, the reader is referred to [17, 18, 25, 28, 31].

In 1934, F. Marty introduced the theory of hyperstructures [20]. He analyzed different properties of hypergroups and applied them to the theory of groups. Thus one can say that hypergroups are suitable generalization of classical groups. Later on, many researchers have worked on algebraic hyperstructures and generalized various classical algebraic structures, for example [9, 15]. One of the main reason which attracts researches towards hyperstructures is its unique property that in hyperstructures composition of two elements is a set, while in classical algebraic structures the composition of two elements is an element. Thus hyperstructures are natural extension of classical algebraic structures. After the pioneering work of F. Marty, algebraic hyperstructures have been intensively studied, both from the theoretical point of view and especially for their applications in other fields such as Euclidean and non-Euclidean geometries, graphs and hypergraphs, fuzzy sets, automata, cryptography, artificial intelligence, codes, probabilities, lattices and so on (see [3]). Several papers and books have been written on algebraic hyperstructures theory, for example, see [5, 8, 9, 14, 29].

We noticed that the relationships between the fuzzy sets and algebraic hyperstructures have been already considered by P. Corsini, B. Davvaz, V. Leoreanu, W.A. Dudek, J. Zhan, K. Hila and others, for instance, the reader can refer to [1, 4, 6, 10, 13, 19, 32, 34, 35]. Recently, D. Heidari and B. Davvaz [12] applied the theory of hyperstructures to ordered semigroups and introduced the concept of ordered semihypergroups, which is a generalization of the concept of ordered semigroups. Also see [7, 22, 23, 26]. It is now natural to investigate similar type of the existing fuzzy subsystems of ordered semihypergroups. As a further study of ordered semihypergroups theory, we attempt in the present paper to study fuzzy left hyperideals of ordered semihypergroups in detail.

The rest of this paper is organized as follows. In Section 2, we recall some basic definitions and results of ordered semihypergroups which will be used throughout this paper. In Section 3, we introduce the concepts of quasi-prime and quasi-semiprime fuzzy left hyperideals in ordered semihypergroups, and give some characterizations of them. We also introduce the concept of fuzzy m-systems of an ordered semihypergroup S, and prove that a fuzzy left hyperideal f of S is quasi-prime if and only if $1 - f$ is a fuzzy m-system of S. In Section 4, some characterizations of strongly semisimple ordered semihypergroups based on ordered fuzzy points and fuzzy left hyperideals are given. In particular, it is proven that an ordered semihypergroup S is strongly semisimple if and only if every fuzzy left hyperideal of S can be expressed as the intersection of all quasi-prime fuzzy hyperideals of S containing it.

2. Preliminaries and some notations

In this section, we present some definitions and results which will be used throughout this paper.

Recall that a *hypergroupoid* $(S, \circ)$ is a nonempty set S together with a hyperoperation, that is a map $\circ : S \times S \to P^*(S)$, where $P^*(S)$ denotes the set of all nonempty subsets of S (see [2]). The image of the pair (x, y) is denoted by $x \circ y$. If $x \in S$ and A, B are nonempty subsets of S, then $A \circ B$ is defined by $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$. The notations $A \circ x$ and $x \circ A$ are used for $A \circ \{x\}$ and $\{x\} \circ A$, respectively.

We say that a hypergroupoid $(S, \circ)$ is a *semihypergroup* if the hyperoperation "$\circ$" is associative, that is, $(x \circ y) \circ z = x \circ (y \circ z)$ for all $x, y, z \in S$ (see [3]).

As we know, an ordered semigroup $(S, \cdot, \leq)$ is a semigroup $(S, \cdot)$ with an order relation "$\leq$" such that $a \leq b$ implies $xa \leq xb$ and $ax \leq bx$ for any $x \in S$. In the following, we shall extend the concept of ordered semigroups to the hyper version, and introduce the concept of ordered semihypergroups from [12].

Definition 2.1. An algebraic hyperstructure $(S, \circ, \leq)$ is called an *ordered semihypergroup* (also called *po-semihypergroup* in [12]) if $(S, \circ)$ is a semihypergroup and $(S, \leq)$ is a partially ordered set such that: for any $x, y, a \in S$, $x \leq y$ implies $a \circ x \preceq a \circ y$ and $x \circ a \preceq y \circ a$. Here, if $A, B \in P^*(S)$, then we say that $A \preceq B$ if for every $a \in A$ there exists $b \in B$ such that $a \leq b$. In particular, if $A = \{a\}$, then we write $a \preceq B$ instead of $\{a\} \preceq B$.

Clearly, every ordered semigroup can be regarded as an ordered semihypergroup (see [26]).

Throughout this paper, unless otherwise mentioned, S will denote an ordered semihypergroup.

Let S be an ordered semihypergroup. For $\emptyset \neq H \subseteq S$, we define

$$(H] := \{t \in S \mid t \leq h \text{ for some } h \in H\}.$$

For $H = \{a\}$, we write $(a]$ instead of $(\{a\}]$.

By a *subsemihypergroup* of an ordered semihypergroup S we mean a nonempty subset A of S such that $A \circ A \subseteq A$. A nonempty subset A of an ordered semihypergroup S is called a *left* (resp. *right*) *hyperideal* of S if (1) $S \circ A \subseteq A$ (resp. $A \circ S \subseteq A$) and (2) If $a \in A$ and $S \ni b \leq a$, then $b \in A$. If A is both a left and a right hyperideal of S, then it is called a (*two-sided*) *hyperideal* of S (see [12]). We denote by $L(a)$ the left hyperideal of S generated by a $(a \in S)$. One can easily prove that $L(a) = (a \cup S \circ a]$. Let L be a left hyperideal of an ordered semihypergroup S. L is called *quasi-prime* if for any two left hyperideals L_1, L_2 of S such that $L_1 \circ L_2 \subseteq L$, we have $L_1 \subseteq L$ or $L_2 \subseteq L$.

Lemma 2.2 ([27]). *Let S be an ordered semihypergroup. Then the following statements hold*:

(1) $A \subseteq (A], \quad \forall A \subseteq S$.

(2) *If $A \subseteq B \subseteq S$, then $(A] \subseteq (B]$.*

(3) $(A] \circ (B] \subseteq (A \circ B]$ *and* $((A] \circ (B]] = (A \circ B], \quad \forall A, B \subseteq S$.

(4) $((A]] = (A], \quad \forall A \subseteq S$.

(5) *For every left hyperideal T of S, we have $(T] = T$.*

(6) *If A, B are left hyperideals of S, then $(A \circ B]$ and $A \cap B$ are left hyperideals of S.*

(7) *For every $a \in S$, $(S \circ a]$ is a left hyperideal of S.*

(8) *For any two nonempty subsets A, B of S such that $A \preceq B$, we have $C \circ A \preceq C \circ B$ and $A \circ C \preceq B \circ C$ for any nonempty subset C of S.*

Definition 2.3. Let M be a nonempty subset of an ordered semihypergroup S. M is called a *m-system* if for any $a, b \in M$, there exists $x \in S$ such that $(a \circ x \circ b] \cap M \neq \emptyset$.

We next state some fuzzy logic concepts.

Let S be an ordered semihypergroup. By a *fuzzy subset* of S, we mean a function from S into the real closed interval [0,1], that is, $f : S \to [0, 1]$. For an ordered semihypergroup S, the fuzzy subset 1 of S is defined as follows:

$$1: \ S \to [0, 1], \ x \mapsto 1(x) := 1, \ \forall x \in S.$$

Let f and g be two fuzzy subsets of S. Then the inclusion relation $f \subseteq g$ is defined by $f(x) \leq g(x)$ for all $x \in S$, and $1 - f, f \cap g, f \cup g$ are defined by

$$\begin{aligned}
&(1 - f)(x) = 1 - f(x),\\
&(f \cap g)(x) = f(x) \wedge g(x),\\
&(f \cup g)(x) = f(x) \vee g(x),
\end{aligned}$$

for all $x \in S$, respectively. We denote by $F(S)$ the set of all fuzzy subsets of S. One can easily show that $(F(S), \subseteq, \cap, \cup)$ forms a complete lattice with the maximum element 1 and the minimum element 0, which is a mapping from S into $[0, 1]$ defined by

$$0: \ S \to [0, 1], \ x \mapsto 0(x) := 0, \ \forall x \in S.$$

Let $(S, \circ, \leq)$ be an ordered semihypergroup. For $x \in S$, we define $H_x := \{(y, z) \in S \times S | \ x \preceq y \circ z\}$. For any $f, g \in F(S)$, the product $f * g$ of f and g is defined by

$$(\forall x \in S) \ (f * g)(x) = \begin{cases} \bigvee_{(y,z) \in H_x} [f(y) \wedge g(z)], & \text{if } H_x \neq \emptyset, \\ 0, & \text{if } H_x = \emptyset. \end{cases}$$

As we know, the multiplication "$*$" on $F(S)$ is associative and $(F(S), *, \subseteq)$ forms an ordered semigroup (see [27]).

Let S be an ordered semihypergroup. A fuzzy subset f of S is called a *fuzzy left* (resp. *right*) *hyperideal* of S if

(1) $x \leq y$ implies $f(x) \geq f(y)$, for all $x, y \in S$, and

(2) $\bigwedge_{z \in x \circ y} f(z) \geq f(y)$ (resp. $\bigwedge_{z \in x \circ y} f(z) \geq f(x)$) for all $x, y \in S$. Equivalently, $1 * f \subseteq f$ (resp. $f * 1 \subseteq f$).

A *fuzzy hyperideal* of S is a fuzzy subset of S which is both a fuzzy left and a fuzzy right hyperideal of S (see [23, 27]).

Lemma 2.4. *Let $\{f_i \mid i \in I\}$ be a family of fuzzy left hyperideals of an ordered semihypergroup S. Then $f := \bigcup_{i \in I} f_i$ is a fuzzy left hyperideal of S, where $(\bigcup_{i \in I} f_i)(x) = \bigvee_{i \in I}(f_i(x))$.*

Proof. The proof is straightforward verification, and hence we omit the details.

Definition 2.5. Let S be an ordered semihypergroup and $f \in F(S)$. The set

$$f_t := \{x \in S \mid f(x) \geq t\}, \text{ where } t \in (0, 1]$$

is called a *level subset* of f.

Lemma 2.6 ([27]). *Let S be an ordered semihypergroup and $f \in F(S)$. Then f is a fuzzy left hyperideal of S if and only if the level subset f_t $(t \in (0, 1])$ of f is a left hyperideal of S for $f_t \neq \emptyset$.*

Let A be a nonempty subset of an ordered semihypergroup S. We define a fuzzy subset λf_A $(\lambda \in (0, 1])$ of S as follows:

$$(\forall x \in S)\ \lambda f_A(x) = \begin{cases} \lambda, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

Clearly, λf_A is a generalization of the characteristic mapping f_A of A.

Lemma 2.7 ([27]). *Let A, B be any nonempty subsets of an ordered semihypergroup S. Then the following statements are true:*

(1) *$A \subseteq B$ if and only if $\lambda f_A \subseteq \lambda f_B$.*

(2) *$\lambda f_A * \lambda f_B = \lambda f_{(A \circ B]}$. In particular, $f_A * f_B = f_{(A \circ B]}$.*

(3) *A is a left hyperideal of S if and only if λf_A is a fuzzy left hyperideal of S.*

Let S be an ordered semihypergroup, $a \in S$ and $\lambda \in [0, 1]$. An *ordered fuzzy point* a_λ of S is defined by the rule that

$$(\forall x \in S)\ a_\lambda(x) = \begin{cases} \lambda, & \text{if } x \in (a], \\ 0, & \text{if } x \notin (a]. \end{cases}$$

It is evident that every ordered fuzzy point of S is a fuzzy subset of S. For any fuzzy subset f of S, we also denote $a_\lambda \subseteq f$ by $a_\lambda \in f$ in the sequel (see [27]).

Definition 2.8 ([27]). Let f be a fuzzy subset of an ordered semihypergroup S. We define $(f]$ by the rule that

$$(f](x) = \bigvee_{y \geq x} f(y),$$

for all $x \in S$. A fuzzy subset f of S is called *strongly convex* if $f = (f]$.

Lemma 2.9 ([27]). *If f is a strongly convex fuzzy subset of an ordered semihypergroup S, then $f = \bigcup_{a_\lambda \in f} a_\lambda$.*

Lemma 2.10 ([27]). *Let a_λ, b_μ $(\lambda > 0, \mu > 0)$ be ordered fuzzy points of an ordered semigroup S, and $f, g, h \in F(S)$. Then the following statements are true:*

(1) $(\forall x \in S)\ (1 * a_\lambda)(x) = \begin{cases} \lambda, & \text{if } x \in (S \circ a], \\ 0, & \text{if } x \notin (S \circ a], \end{cases}$ *and $1 * a_\lambda$ is a fuzzy left hyperideal of S.*

(2) $(a_\lambda * b_\mu) * c_\delta = a_\lambda * (b_\mu * c_\delta) = \bigcup_{d \in (a \circ b \circ c]} d_{\lambda \wedge \mu \wedge \delta}$ *for any ordered fuzzy point a_λ, b_μ and c_δ of S.*

(3) $L(a_\lambda) = a_\lambda \cup 1 * a_\lambda$.

(4) $(L(a_\lambda))^2 \subseteq 1 * a_\lambda$.

(5) *If S is commutative, then $f * 1 = 1 * f$.*

(6) $(g \cup h) * f = (g * f) \cup (h * f)$.

The reader is referred to [3, 30] for notation and terminology not defined in this paper.

3. Quasi-prime and quasi-semiprime fuzzy left hyperideals of ordered semihypergroups

In what follows, we denote by Z^+ the set of positive integers. In the current section we define and study the quasi-prime and quasi-semiprime fuzzy left hyperideals of ordered semihypergroups, and give some characterizations of them.

Definition 3.1. Let S be an ordered semihypergroup. A fuzzy left hyperideal f of S is called *quasi-prime* if for any two fuzzy left hyperideals g and h of S, $g * h \subseteq f$ implies $g \subseteq f$ or $h \subseteq f$.

Theorem 3.2. *Let L be a nonempty subset of an ordered semihypergroup S. Then a left hyperideal L is quasi-prime if and only if the characteristic function f_L of L is a quasi-prime fuzzy left hyperideal of S.*

Proof. Let L be a quasi-prime left hyperideal of S. Then, by Lemma 2.7(3), f_L is a fuzzy left hyperideal of S. For any two fuzzy left hyperideals g and h of S, if $g * h \subseteq f_L$, then $g \subseteq f_L$ or $h \subseteq f_L$. In fact, if $g \not\subseteq f_L$ and $h \not\subseteq f_L$, then there exist $x, y \in S$ such that $g(x) > f_L(x), h(y) > f_L(y)$. Thus we have

$$g(x) > 0, h(y) > 0, f_L(x) = f_L(y) = 0.$$

It implies that $x, y \notin L$. We now show that there exists $s \in S$ such that $(x \circ s \circ y] \not\subseteq L$. Indeed, if $(x \circ S \circ y] \subseteq L$, then $(S \circ x] \circ (S \circ y] \subseteq L$. By Lemma 2.2(7), $(S \circ x]$ and $(S \circ y]$ are left hyperideals of S. Since L is a quasi-prime left hyperideal of S, it can be obtained that $(S \circ x] \subseteq L$ or $(S \circ y] \subseteq L$. Let $(S \circ x] \subseteq L$. Then

$$(L(x))^2 = (x \cup S \circ x] \circ (x \cup S \circ x] \subseteq (S \circ x] \subseteq L.$$

Thus $x \in L(x) \subseteq L$, which is impossible. From $(S \circ y] \subseteq L$, similarly, we get a contradiction. Now if $a \in (x \circ s \circ y]$ such that $a \notin L$, then $f_L(a) = 0$, and there exists $z \in s \circ y$ such that $a \preceq x \circ z$. Thus

$$\begin{aligned}(g * h)(a) &= \bigvee_{(p,q) \in H_a} [g(p) \wedge h(q)] \geq g(x) \wedge h(z) \\ &\geq g(x) \wedge (\bigwedge_{z \in s \circ y} h(z)) \geq g(x) \wedge h(y) > 0,\end{aligned}$$

which contradicts the fact that $g * h \subseteq f_L$. Therefore, f_L is a quasi-prime fuzzy left hyperideal of S.

Conversely, suppose that f_L is a quasi-prime fuzzy left hyperideal of S. Let L_1, L_2 are left hyperideals of S such that $L_1 \circ L_2 \subseteq L$. Then, by Lemma 2.2, $(L_1 \circ L_2] \subseteq (L] = L$. Thus, by Lemma 2.7, we have

$$f_{L_1} * f_{L_2} = f_{(L_1 \circ L_2]} \subseteq f_L.$$

By hypothesis and Lemma 2.7(3), since f_L is quasi-prime, it can be shown that $f_{L_1} \subseteq f_L$ or $f_{L_2} \subseteq f_L$, which implies that $L_1 \subseteq L$ or $L_2 \subseteq L$. This completes the proof.

Lemma 3.3. *Let S be an ordered semihypergroup. If f is a nonconstant quasi-prime fuzzy left hyperideal of S, then $|Im(f)| = 2$.*

Proof. Since f is a nonconstant quasi-prime fuzzy left hyperideal of S, we have $|Im(f)| \geq 2$. Suppose $|Im(f)| \geq 3$. Then there exist $x, y, z \in S$ such that $f(x), f(y)$ and $f(z)$ are different from each other. Without loss of generality , it can be assumed that

$$f(x) < f(y) < f(z).$$

Thus there exist $r, t \in (0, 1)$ such that

$$f(x) < r < f(y) < t < f(z).$$

Then, for any $u \in S$, we have

$$(L(x_r) * L(y_t))(u) = \begin{cases} r \wedge t = r, & u \in (L(x) \circ L(y)], \\ 0, & \text{otherwise.} \end{cases}$$

If $u \in (L(x) \circ L(y)]$, then there exist $a \in L(x), b \in L(y)$ such that $u \in (a \circ b]$, and there exists $c \in a \circ b$ such that $u \leq c$. Since f is a fuzzy left hyperideal of S, we have

$$f(u) \geq \bigwedge_{\substack{c \in a \circ b \\ u \leq c}} f(c) \geq \bigwedge_{c \in a \circ b} f(c) \geq f(b).$$

Since $b \in L(y) = (y \cup S \circ y] = (y] \cup (S \circ y]$, we have $b \in (y]$ or $b \in (S \circ y]$. Similar to the previous proof, it can be obtained that $f(b) \geq f(y)$. Hence $f(u) \geq f(y) > r$. It follows that $L(x_r) * L(y_t) \subseteq f$. Thus $L(x_r) \subseteq f$ or $L(y_t) \subseteq f$ because f is a quasi-prime fuzzy left hyperideal of S. Let $L(x_r) \subseteq f$. Then we have $f(x) \geq L(x_r)(x) = r$, which is impossible. From $L(y_t) \subseteq f$, similarly, we get a contradiction. This completes the proof.

Theorem 3.4. *Let S be an ordered semihypergroup. If f is a nonconstant quasi-prime fuzzy left hyperideal of S, then there exists $x_0 \in S$ such that $f(x_0) = 1$.*

Proof. By Lemma 3.3, $|Im(f)| = 2$. If $f(x) \neq 1$ for all $x \in S$, then $Im(f) = \{s, t\},\ s < t < 1$. Hence there exist $x, y \in S$ and $m \in (0, 1]$ such that

$$f(x) = s < t = f(y) < m \leq 1.$$

Let $t_1, t_2 \in (0, 1)$ such that $s < t_1 < t < t_2 < m$. Then by the similar way of the proof of Lemma 3.3, we have $L(x_{t_1}) * L(y_{t_2}) \subseteq f$. Since f is a quasi-prime fuzzy left hyperideal of S, we have $L(x_{t_1}) \subseteq f$ or $L(y_{t_2}) \subseteq f$. It implies that $f(x) \geq t_1$ or $f(y) \geq t_2$, which is impossible. Thus there exists $x_0 \in S$ such that $f(x_0) = 1$.

Theorem 3.5. *Let S be an ordered semihypergroup. If f is a quasi-prime fuzzy left hyperideal of S, then the level subset f_t $(t \in (0, 1])$ of f is a quasi-prime left hyperideal of S for $f_t \neq \emptyset$.*

Proof. Suppose that f is a quasi-prime fuzzy left hyperideal of S. By Lemma 2.6, for any $t \in (0, 1]$, f_t is a left hyperideal of S for $f_t \neq \emptyset$. To prove that f_t is quasi-prime, let L_1 and L_2 be left hyperideals of S such that $L_1 \circ L_2 \subseteq f_t$, and let $g = tf_{L_1}$ and $h = tf_{L_2}$. Then, by Lemma 2.7(3), g and h are fuzzy left hyperideals of S. Furthermore, we have $g * h \subseteq f$, that is, $(g * h)(x) \leq f(x)$ for all $x \in S$. Indeed, if $(g * h)(x) = 0$, then it is obvious. If $(g * h)(x) \neq 0$, then $H_x \neq \emptyset$, and there exist $y, z \in S$ such that $x \preceq y \circ z$, $0 < g(y) \wedge h(z) \leq t$. Thus $y \in L_1$ and $z \in L_2$, and so $x \in (L_1 \circ L_2] \subseteq (f_t] = f_t$, and $f(x) \geq t$. Consequently,

$$(g * h)(x) = \bigvee_{(y,z) \in H_x} [g(y) \wedge h(z)\}] \leq t \leq f(x).$$

Hence it can be obtained that $g * h \subseteq f$. Since f is a quasi-prime fuzzy left hyperideal of S, it can be followed that $g \subseteq f$ or $h \subseteq f$. Say $g \subseteq f$, then for any $x \in L_1, g(x) = t \leq f(x)$, and $x \in f_t$. Thus $L_1 \subseteq f_t$. Similarly, say $h \subseteq f$, we have $L_2 \subseteq f_t$. Therefore, f_t is a quasi-prime left hyperideal of S for $f_t \neq \emptyset$.

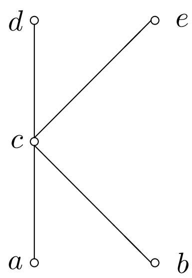
Example 3.6. We consider a set $S := \{a, b, c, d, e\}$ with the following hyperoperation " $\circ$ " and the order " $\leq$ ":

$\circ$	a	b	c	d	e
a	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$
b	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$
c	$\{a, b\}$	$\{a, b\}$	$\{c\}$	$\{c\}$	$\{e\}$
d	$\{a, b\}$	$\{a, b\}$	$\{c\}$	$\{d\}$	$\{e\}$
e	$\{a, b\}$	$\{a, b\}$	$\{c\}$	$\{c\}$	$\{e\}$

$\leq := \{(a, a), (a, c), (a, d), (a, e), (b, b), (b, c), (b, d), (b, e), (c, c), (c, d), (c, e), (d, d), (e, e)\}$.

We give the covering relation "$\prec$" and the figure of S as follows:

$$\prec = \{(a, c), (b, c), (c, d), (c, e)\}.$$



Then $(S, \circ, \leq)$ is an ordered semihypergroup. With a small amount of effort one can verify that the sets $\{a, b\}, \{a, b, c, d\}, \{a, b, c, e\}$ and S are all quasi-prime left hyperideals of S. Now let f be a fuzzy subset of S such that $f(a) = f(b) = 0.8, f(c) = f(d) = 0.7, f(e) = 0.6$. Then

$$f_t = \begin{cases} S, & \text{if } t \in (0, 0.6], \\ \{a, b, c, d\}, & \text{if } t \in (0.6, 0.7], \\ \{a, b\}, & \text{if } t \in (0.7, 0.8], \\ \emptyset, & \text{if } t \in (0.8, 1]. \end{cases}$$

Thus all nonempty level subsets f_t $(t \in (0, 1])$ of f are quasi-prime left hyperideals of S and by Theorem 3.5, f is a quasi-prime fuzzy left hyperideal of S.

By Theorems 3.4 and 3.5, we immediately obtain the following corollary:

Corollary 3.7. *Let S be an ordered semihypergroup. If f is a nonconstant quasi-prime fuzzy left hyperideal of S, then f_1 is a quasi-prime left hyperideal of S.*

Remark 3.8. The inverse of Theorem 3.5 is not true. For example, let L be a quasi-prime left hyperideal of S, $L \neq S$, and

$$f(x) = \begin{cases} \lambda, & \text{if } x \in L, \\ 0, & \text{if } x \notin L, \end{cases}$$

for any $x \in S$, where $0 < \lambda < 1$. Then f is a fuzzy left hyperideal of S. For any $t \in (0, 1]$, if $f_t \neq \emptyset$, then $f_t = L$, which is a quasi-prime left hyperideal of S. But f is not quasi-prime since $f_1 = \emptyset$.

Now, quasi-prime fuzzy left hyperideals of ordered semihypergroups can be characterized.

Theorem 3.9. *Let f be a nonconstant fuzzy subset of an ordered semihypergroup S. Then f is a quasi-prime fuzzy left hyperideal of S if and only if f satisfies the following conditions:*

(1) $|Im(f)| = 2$.

(2) $f_1 \neq \emptyset$, *and* f_1 *is a quasi-prime left hyperideal of* S.

Proof. Suppose that f is a nonconstant quasi-prime fuzzy left hyperideal of S. Then, by Lemma 3.3, Theorem 3.4 and Corollary 3.7, the conditions (1) and (2) hold.

Conversely, assume that the conditions (1) and (2) hold. Since $|Im(f)| = 2$, by hypothesis we have $Im(f) = \{t, 1\}$ $(t < 1)$. Thus

(A) f is a fuzzy left hyperideal of S. To prove this assertion, let $x, y \in S$. We consider the following two cases:

Case 1. If $y \in f_1$, then $f(y) = 1$, and by (2), we have $x \circ y \subseteq S \circ f_1 \subseteq f_1$, which implies that $f(z) = 1$ for any $z \in x \circ y$. Hence $\bigwedge_{z \in x \circ y} f(z) = 1 = f(y)$.

Case 2. If $y \notin f_1$, then $f(y) = t$. Consequently, by hypothesis, $\bigwedge_{z \in x \circ y} f(z) \geq t = f(y)$.

Thus, in both cases, $\bigwedge_{z \in x \circ y} f(z) \geq f(y)$ for all $x, y \in S$. Furthermore, let $x, y \in S$ such that $x \leq y$. Then $f(x) \geq f(y)$. In fact, if $y \notin f_1$, then $f(y) = t \leq f(x)$. If $y \in f_1$, then, since f_1 is a left hyperideal of S, we have $x \in f_1$. Thus $f(x) = 1 = f(y)$.

(B) f is quasi-prime. In fact, let g and h be fuzzy left hyperideals of S such that $g * h \subseteq f$. We claim that $g \subseteq f$ or $h \subseteq f$. If $g \not\subseteq f$ and $h \not\subseteq f$, then there exist $x, y \in S$ such that $g(x) > f(x)$ and $h(y) > f(y)$. Hence $x, y \notin f_1$, which implies $(x \circ S \circ y] \not\subseteq f_1$. Otherwise, by Lemma 2.2, we have

$$(S \circ x] \circ (S \circ y] \subseteq (S \circ (x \circ S \circ y]] \subseteq (S \circ f_1] \subseteq (f_1] \subseteq f_1.$$

Since f_1 is a quasi-prime left hyperideal of S, by Lemma 2.2(7) we have $(S \circ x] \subseteq f_1$ or $(S \circ y] \subseteq f_1$. Say $(S \circ x] \subseteq f_1$, we can deduce that $(L(x))^2 \subseteq (S \circ x] \subseteq f_1$. It follows that $x \in L(x) \subseteq f_1$ because f_1 is quasi-prime. Impossible. Say $(S \circ y] \subseteq f_1$, similarly, we get a contradiction. Thus $(x \circ S \circ y] \not\subseteq f_1$, and there exists $a \in (x \circ S \circ y]$ such that $a \notin f_1$. Then $f(a) = t$ and there exists $s \in S$ such that $a \preceq x \circ s \circ y$. Thus there exists $b \in s \circ y$ such that $a \preceq x \circ b$. Since $x, y \notin f_1$, by hypothesis we have $f(x) = f(y) = t$. Consequently,

$$\begin{aligned}(g * h)(a) &= \bigvee_{(u,v)\in H_a} [g(u) \wedge h(v)] \geq g(x) \wedge h(b) \\ &\geq g(x) \wedge (\bigwedge_{b \in s\circ y} h(b)) \geq g(x) \wedge h(y) \\ &\quad \text{(Since } h \text{ is a fuzzy left hyperideal of } S\text{)} \\ &> f(x) \wedge f(y) = t = f(a),\end{aligned}$$

which contradicts the fact that $g * h \subseteq f$. Therefore, f is a quasi-prime fuzzy left hyperideal of S.

Definition 3.10. Let S be an ordered semihypergroup. A fuzzy left hyperideal f of S is called *proper* if $f \neq 1$.

Theorem 3.11. *Let S be an ordered semihypergroup. If f is a nonconstant quasi-prime fuzzy left hyperideal of S, then there exists a proper quasi-prime fuzzy left hyperideal g of S such that $f \subset g$.*

Proof. Let f be a nonconstant quasi-prime fuzzy left hyperideal of S. By Theorem 3.9, there exists $x_0 \in S$ such that $f(x_0) = 1$, and $Im(f) = \{t, 1\}$, where $t < 1$. Let g be a fuzzy subset of S defined by

$$(\forall x \in S) \;\; g(x) = \frac{1}{2} f(x) + \frac{1}{2}.$$

Then, it is easy to show that g is a fuzzy left hyperideal of S, and $|Im(g)| = 2$.

On the other hand, since $g_1 = f_1$, by Theorem 3.9, g_1 is a quasi-prime left hyperideal of S and g is a quasi-prime fuzzy left hyperideal of S. Let $y \in S$ such that $f(y) = t$. Then

$$f(y) < \frac{1}{2}(f(y) + 1) = g(y) < 1,$$

which implies that g is a proper quasi-prime fuzzy left hyperideal of S and $f \subset g$. The proof is completed.

We now characterize the quasi-prime fuzzy left hyperideals by ordered fuzzy points.

Theorem 3.12. *Let S be an ordered semihypergroup. Then a fuzzy left hyperideal f of S is quasi-prime if and only if for any two ordered fuzzy points x_r, y_t of S $(r > 0, t > 0)$, $x_r * 1 * y_t \subseteq f$ implies that $x_r \in f$ or $y_t \in f$.*

Proof. Let x_r and y_t are ordered fuzzy points of S such that $x_r * 1 * y_t \subseteq f$. Then

$$(1 * x_r) * (1 * y_t) = 1 * (x_r * 1 * y_t) \subseteq 1 * f \subseteq f.$$

By Theorem 2.10(1), $1 * x_r$ and $1 * y_t$ are fuzzy left hyperideals of S. Since f is quasi-prime, we have $1 * x_r \subseteq f$ or $1 * y_t \subseteq f$. Say $1 * x_r \subseteq f$, then, by Theorem 2.10(4), $(L(x_r))^2 \subseteq 1 * x_r \subseteq f$. Thus $x_r \in L(x_r) \subseteq f$. Similarly, say $1 * y_t \subseteq f$, we have $y_t \in L(y_t) \subseteq f$.

Conversely, let g, h be fuzzy left hyperideals of S such that $g * h \subseteq f$. If $g \not\subseteq f, h \not\subseteq f$, then there exist $x, y \in S$ such that $g(x) > f(x), h(y) > f(y)$. Let $r = g(x), t = h(y)$. Then $r > 0, t > 0, x_r \in g, y_t \in h$, since h is a fuzzy left hyperideal of S, we have

$$x_r * 1 * y_t \subseteq g * 1 * h \subseteq g * h \subseteq f.$$

By hypothesis, $x_r \in f$ or $y_t \in f$. If $x_r \in f$, then $f(x) \geq r = g(x)$, which is impossible. Similarly, if $y_t \in f$, then we get a contradiction. Therefore, f is a quasi-prime fuzzy left hyperideal of S.

In order to characterize the quasi-prime fuzzy left hyperideals of ordered semihypergroups, we need the following concept.

Definition 3.13. Let S be an ordered semihypergroup. A fuzzy subset f of S is called *fuzzy m-system* if for any $s, t \in [0, 1)$ and $a, b \in S$, $f(a) > s, f(b) > t$ imply that there exists $x \in S$ such that $f(y) > s \vee t$ for some $y \in (a \circ x \circ b]$.

Theorem 3.14. *Let M be a nonempty subset of an ordered semihypergroup S. Then M is a m-system of S if and only if the characteristic function f_M of M is a fuzzy m-system of S.*

Proof. For any $s, t \in [0, 1)$ and $a, b \in S$, if $f_M(a) > s,\ f_M(b) > t$, then $a, b \in M$. Since M is a m-system of S, there exists $x \in S$ such that $(a \circ x \circ b] \cap M \neq \emptyset$. Let $y \in (a \circ x \circ b] \cap M$. Then $f_M(y) = 1$. Hence $f_M(y) > s \vee t$ for some $y \in (a \circ x \circ b]$. It thus follows that f_M is a fuzzy m-system of S.

Conversely, suppose that f_M is a fuzzy m-system of S. Let $a, b \in M$. Then $f_M(a) = f_M(b) = 1$. Thus for any $s, t \in [0, 1)$, we have

$$f_M(a) > s,\ f_M(b) > t,$$

which imply that there exists an element $x \in S$ such that $f_M(y) > s \vee t$ for some $y \in (a \circ x \circ b]$ and that $f_M(y) = 1$, that is, $y \in M$. It can be followed that $(a \circ x \circ b] \cap M \neq \emptyset$. Hence M is a m-system of S.

Theorem 3.15. *Let f be a proper fuzzy left hyperideal of an ordered semihypergroup S. Then f is quasi-prime if and only if $1 - f$ is a fuzzy m-system of S.*

Proof. Suppose that f is a quasi-prime fuzzy left hyperideal of S. For any $s, t \in [0, 1)$, $a, b \in S$, if $(1-f)(a) > s, (1-f)(b) > t$, then $f(a) < 1-s,\ f(b) < 1-t$. It implies that $a_{1-s} \notin f$ and $b_{1-t} \notin f$. Since f is a quasi-prime fuzzy left hyperideal of S, by Theorem 3.12, Lemmas 2.9 and 2.10(2), there exists an ordered fuzzy point x_r of S such that

$$a_{1-s} * x_r * b_{1-t} = \bigcup_{y \in (a \circ x \circ b]} y_{(1-s)\wedge(1-t)\wedge r} \notin f.$$

Thus, there exists $y \in (a \circ x \circ b]$ such that

$$f(y) < (1-s) \wedge (1-t) \wedge r \leq (1-s) \wedge (1-t) = 1 - (s \vee t).$$

which implies that $(1-f)(y) > s \vee t$. We have thus shown that $1-f$ is a fuzzy m-system of S.

Conversely, assume that $1-f$ is a fuzzy m-system of S. Let a_s, b_t of S $(t > 0, s > 0)$ such that $a_s * 1 * b_t \subseteq f$. If $a_s \notin f$ and $b_t \notin f$, then there exist $a_1 \in (a], b_1 \in (b]$ such that $f(a_1) < s, f(b_1) < t$. Thus we have

$$(1-f)(a_1) > 1-s, (1-f)(b_1) > 1-t.$$

By hypothesis, there exists an element $x \in S$ such that

$$(1-f)(y) > (1-s) \vee (1-t) = 1 - s \wedge t$$

for some $y \in (a_1 \circ x \circ b_1]$, that is, $f(y) < s \wedge t$. Since S be an ordered semihypergroup, it can be obtained that $y \in (a \circ x \circ b]$. It thus follows, by Lemma 2.10(2), that $a_s * x_{s\wedge t} * b_t = \bigcup_{y \in (a \circ x \circ b]} y_{s \wedge t} \notin f$, which is a contradiction. Consequently, f is a quasi-prime fuzzy left hyperideal of S.

In the following we shall define and study the quasi-semiprime fuzzy left hyperideals of ordered semihypergroups.

Definition 3.16. Let S be an ordered semihypergroup. A fuzzy left hyperideal f of S is called *quasi-semiprime* if for any fuzzy left hyperideal g of S, $g * g \subseteq f$ implies $g \subseteq f$.

Lemma 3.17. *If f and g are fuzzy left hyperideals of an ordered semihypergroup S, then $f * g$ is also a fuzzy left hyperideal of S.*

Proof. Let f, g be two fuzzy left hyperideals of S. Then we have

$$1 * (f * g) = (1 * f) * g \subseteq f * g.$$

Furthermore, if $x \leq y$, then $(f * g)(x) \geq (f * g)(y)$. Indeed, if $H_y = \emptyset$, then $(f*g)(y) = 0$. Since $f*g$ is a fuzzy subset of S, we have $(f*g)(x) \geq 0 = (f*g)(y)$. If $H_y \neq \emptyset$, then, since $x \leq y$, we have $H_y \subseteq H_x$. Thus we have

$$(f * g)(y) = \bigvee_{(u,v) \in H_y} [f(u) \wedge g(v)] \leq \bigvee_{(u,v) \in H_x} [f(u) \wedge g(v)] = (f * g)(x).$$

Therefore, $f * g$ is a fuzzy left hyperideal of S.

Theorem 3.18. *Let S be an ordered semihypergroup and f a fuzzy left hyperideal of S. Then f is quasi-semiprime if and only if for any fuzzy left hyperideal g of S, $g^n \subseteq f,\ n \in Z^+$ implies that $g \subseteq f$.*

Proof. $\Longleftarrow$. This is obvious.

$\Longrightarrow$. Let f be a quasi-semiprime fuzzy left hyperideal of S. Here we prove the result by induction. Clearly the result holds for $n = 2$. Let $k \geq 2$ be any positive integer and let the result holds for every positive integer $n,\ 1 \leq n \leq k$. We claim that $g^{k+1} \subseteq f$ implies $g \subseteq f$. We consider the following two cases:

Case 1. If k is odd, let $k = 2m + 1$. Then $g^{k+1} = g^{2(m+1)} = (g^{m+1})^2$.

Case 2. If k is even, let $k = 2m$. Then, by Lemma 3.17, we have

$$g^{k+1} = g^{2m+1} \supseteq 1 * g^{2m+1} \supseteq g * g^{2m+1} = g^{2m+2} = (g^{m+1})^2.$$

Thus, in both cases, if $g^{k+1} \subseteq f$, then $g^{m+1} \subseteq f$. Since $m+1 \leq k$, the induction hypothesis insures that $g \subseteq f$. The proof is completed.

Remark 3.19. By Theorem 3.18, we have characterized quasi-semiprime fuzzy left hyperideals of an ordered semihypergroup S. The characterization, however, make no reference to the grade of membership of an element of S. The purpose of following theorem is to characterize quasi-semiprime fuzzy left hyperideal in terms of its effect on the elements of S. We shall see that the following theorem is simpler to use.

Theorem 3.20. *Let S be an ordered semihypergroup and f a fuzzy left hyperideal of S. Then f is quasi-semiprime if and only if $f(a) = \bigwedge_{b \in (a \circ S \circ a]} f(b)$ for all $a \in S$.*

Proof. Assume that $f(a) = \bigwedge_{b \in (a \circ S \circ a]} f(b)$ for any $a \in S$. Let g be any fuzzy left hyperideal of S such that $g * g \subseteq f$. If $g \not\subseteq f$, then there exists $a \in S$ such that $g(a) > f(a)$. Since $f(a) = \bigwedge_{b \in (a \circ S \circ a]} f(b)$, there exists $t \in S$ such that $b \preceq a \circ t \circ a$ and $f(a) = f(b)$. Then there exists $c \in a \circ t \circ a$ such that $b \leq c$, and there exists $x \in t \circ a$ such that $c \in a \circ x$. Since f is a fuzzy left hyperideal of S, we have

$$f(c) \leq f(b) = f(a) < g(a).$$

Furthermore, according to $g * g \subseteq f$, we have

$$\begin{aligned} g(a) &> f(c) \geq (g * g)(c) = \bigvee_{(u,v) \in H_c} [g(u) \wedge g(v)] \\ &\geq g(a) \wedge g(x) \geq g(a) \wedge (\bigwedge_{x \in t \circ a} g(x)) \\ &\geq g(a) \wedge g(a) = g(a), \end{aligned}$$

which is a contradiction.

Conversely, suppose that f is a quasi-semiprime fuzzy left hyperideal of S. If $f(a) \neq \bigwedge_{b\in(a\circ S\circ a]} f(b)$ for some $a \in S$, then $f(a) < \bigwedge_{b\in(a\circ S\circ a]} f(b)$. In fact, for any $b \in (a \circ S \circ a]$, there exists $t \in S$ such that $b \preceq a \circ t \circ a$. Then there exists $c \in a \circ t \circ a$ such that $b \leq c$, and there exists $x \in a \circ t$ such that $c \in x \circ a$. By the fact that f is a fuzzy left hyperideal of S, we have

$$f(b) \geq f(c) \geq \bigwedge_{c\in x\circ a} f(c) \geq f(a).$$

Let $\bigwedge_{b\in(a\circ S\circ a]} f(b) = m$. Define a fuzzy subset g of S as follows:

$$(\forall x \in S)\ g(x) = \begin{cases} m, & \text{if } x \in (S \circ a], \\ 0, & \text{if } x \notin (S \circ a]. \end{cases}$$

Then, by Lemma 2.10(1), g is a fuzzy left hyperideal of S. Furthermore, we can show that $g * g \subseteq f$. It is enough to prove that $(g * g)(x) \leq f(x)$ for all $x \in S$. Indeed, if $(g * g)(x) = 0$, then it is obvious that $(g * g)(x) \leq f(x)$. Let $(g * g)(x) = m$. Then we have

$$\bigvee_{(y,z)\in H_x} [g(y) \wedge g(z)] = m,$$

which means there exist $u, v \in (S \circ a]$ such that $x \preceq u \circ v$. Put $u \preceq s \circ a,\ v \preceq t \circ a$ for some $s, t \in S$. Then, by Lemma 2.2(8), we have

$$x \preceq u \circ v \preceq (s \circ a) \circ (t \circ a) = s \circ (a \circ t \circ a),$$

and there exists $y \in a \circ t \circ a$ such that $x \preceq s \circ y$. Then there exists $z \in s \circ y$ such that $x \leq z$. Since f is a fuzzy left hyperideal of S, we have

$$\begin{aligned} f(x) &\geq \bigwedge_{\substack{x\leq z\\ z\in s\circ y}} f(z) \geq \bigwedge_{z\in s\circ y} f(z) \geq f(y) \geq \bigwedge_{y\in a\circ t\circ a} f(y) \\ &\geq \bigwedge_{y\in(a\circ t\circ a]} f(y) \geq \bigwedge_{y\in(a\circ S\circ a]} f(y) = m = (g * g)(x). \end{aligned}$$

It implies that $g * g \subseteq f$. By hypothesis, $g \subseteq f$. Again define a fuzzy subset h of S as follows:

$$(\forall x \in S)\ h(x) = \begin{cases} m, & \text{if } x \in L(a), \\ 0, & \text{if } x \notin L(a). \end{cases}$$

Clearly, $h = mf_{L(a)}$. Then, by Lemma 2.7(3), h is a fuzzy left hyperideal of S. Moreover, $h * h \subseteq f$. Indeed, since $(h * h)(x) = \bigvee_{x\preceq x_1\circ x_2}[h(x_1) \wedge h(x_2)] = m$ only if there exist $u, v \in L(a)$ such that $x \preceq u \circ v$. We can easily verify that

$$x \preceq u \circ v \subseteq S \circ a,$$

which implies that $x \in (S \circ a]$. Thus $(h * h)(x) = m$ implies $g(x) = m$. Consequently, $h * h \subseteq g \subseteq f$. Since f is quasi-semiprime, by Proposition 3.7 we have $h \subseteq f$. Thus $m = h(a) \leq f(a)$, which contradicts the fact that $f(a) < m$. This completes the proof.

Definition 3.21. Let S be an ordered semihypergroup. A fuzzy subset f of S is called *fuzzy n-system* if for any $s \in [0, 1)$ and $a \in S$, $f(a) > s$ implies that there exists $x \in S$ such that $f(y) > s$ for some $y \in (a \circ x \circ a]$.

We now give characterizations of quasi-semiprime fuzzy left hyperideals of an ordered semihypergroup.

Theorem 3.22. *Let S be an ordered semihypergroup and f a proper fuzzy left hyperideal of S. Then the following statements are equivalent:*

(1) *f is quasi-semiprime.*

(2) *For every ordered fuzzy point x_r of S $(r > 0)$, $x_r * 1 * x_r \subseteq f$ implies that $x_r \in f$.*

(3) *$1 - f$ is a fuzzy n-system of S.*

Proof. The proof is similar to that of Theorems 3.12 and 3.15 with a slight modification, we omit it.

4. Characterizations of strongly semisimple ordered semihypergroups

In this section, we investigate mainly the properties of strongly semisimple ordered semihypergroups. In particular, we discuss the characterizations of strongly semisimple ordered semihypergroups by fuzzy left hyperideals generated by ordered fuzzy points.

Definition 4.1. An ordered semihypergroup S is called *strongly semisimple* if $(L^2] = L$ holds for every left hyperideal L of S.

Lemma 4.2. *Let S be an ordered semihypergroup. Then the following statements are equivalent*:

(1) *S is strongly semisimple.*

(2) *$a \in (S \circ a \circ S \circ a]$ for all $a \in S$.*

Proof. (1) $\Rightarrow$ (2). Let $a \in S$. Then, by Lemma 2.2, we have

$$\begin{aligned} (L(a) \circ L(a)] &= ((a \cup S \circ a] \circ (a \cup S \circ a]] \\ &= ((a \cup S \circ a) \circ (a \cup S \circ a)] \\ &= (a \circ a \cup a \circ S \circ a \cup S \circ a \circ a \cup S \circ a \circ S \circ a] \\ &\subseteq (S \circ a]. \end{aligned}$$

Thus, by (1), we have

$$\begin{aligned} a &\in L(a) = (L(a) \circ L(a)] = (((L(a) \circ L(a)] \circ (L(a) \circ L(a)]] \\ &\subseteq ((S \circ a] \circ (S \circ a]] = ((S \circ a) \circ (S \circ a)] = (S \circ a \circ S \circ a]. \end{aligned}$$

(2) $\Rightarrow$ (1). Let L be a left hyperideal of S. Then $(L^2] = (L \circ L] \subseteq (S \circ L] \subseteq (L] = L$. On the other hand, let $x \in L$. Then, by (2) and Lemma 2.2, we have

$$\begin{aligned} x &\in (S \circ x \circ S \circ x] \subseteq (S \circ L \circ S \circ L] \\ &= ((S \circ L) \circ (S \circ L)] \subseteq (L \circ L] = (L^2], \end{aligned}$$

which means that $L \subseteq (L^2]$. Therefore, S is strongly semisimple.

Now, we give some characterizations of a strongly semisimple ordered semihypergroup by ordered fuzzy points and fuzzy left hyperideals.

Theorem 4.3. *Let S be an ordered semihypergroup. Then the following statements are equivalent:*

(1) *S is strongly semisimple.*

(2) *$f \cap g \subseteq f * g$ for all fuzzy left hyperideals f and g of S.*

(3) *$f * f = f$ for every fuzzy left hyperideal f of S.*

(4) *$(L(a_r))^2 = L(a_r)$ for every ordered fuzzy point a_r of S.*

(5) *$a_r \in S \circ a_r \circ S \circ a_r$ for every ordered fuzzy point a_r of S.*

(6) *Every fuzzy left hyperideal of S is quasi-semiprime.*

(7) *Every fuzzy left hyperideal of S is the intersection of all quasi-prime fuzzy left hyperideals of S containing it.*

Proof. (1) $\Rightarrow$ (2). Let S be a strongly semisimple ordered semihypergroup and $a \in S$. Then, by Lemma 5.2, we have $a \in (S \circ a \circ S \circ a]$, and there exist $x, y \in S$ such that $a \preceq x \circ a \circ y \circ a$. Then there exist $b \in x \circ a, c \in y \circ a$ such that $a \preceq b \circ c$. For any two fuzzy left hyperideals f and g of S, we have

$$\begin{aligned} (f * g)(a) &= \bigvee_{(u,v) \in H_a} [f(u) \wedge g(v)] \geq f(b) \wedge g(c) \\ &\geq (\bigwedge_{b \in x \circ a} f(b)) \wedge (\bigwedge_{c \in y \circ a} g(c)) \\ &\geq f(a) \wedge g(a) = (f \cap g)(a), \end{aligned}$$

which implies that $f \cap g \subseteq f * g$.

(2) $\Rightarrow$ (3). Let f be any fuzzy left hyperideal of S. Then, by (2), we have $f * f \supseteq f \cap f = f$. On the other hand, since f is a fuzzy left hyperideal of S, it can be obtained that $f * f \subseteq 1 * f \subseteq f$. Therefore, $f * f = f$.

(3) $\Rightarrow$ (1). Let L be any left hyperideal of S. Then, by Lemma 2.7(3), the characteristic function f_L of L is a fuzzy left hyperideal of S. Thus, by (2), we have $f_L * f_L = f_L$. By Lemma 2.7(2), we have $f_{(L^2]} = f_L$, and thus $(L^2] = L$. Hence S is a strongly semisimple ordered semihypergroup.

(3) $\Rightarrow$ (4). Clearly.

(4) $\Rightarrow$ (5). Let a_r be any ordered fuzzy point of S. By (4), $(L(a_r))^2 = L(a_r)$. Then we have

$$a_r \in L(a_r) = (L(a_r))^4 = (L(a_r))^3 * L(a_r).$$

By Lemma 2.10(4), $(L(a_r))^2 \subseteq 1 * a_r$. Then

$$(L(a_r))^3 = (L(a_r))^2 * L(a_r) \subseteq 1 * a_r * 1.$$

Thus we have

$$\begin{aligned}(L(a_r))^4 &= (L(a_r))^3 * L(a_r)\\ &\subseteq (1 * a_r * 1) * (a_r \cup 1 * a_r)\\ &= 1 * a_r * 1 * a_r \cup 1 * a_r * 1 * 1 * a_r \text{ (By Lemma 2.10(6))}\\ &\subseteq 1 * a_r * 1 * a_r.\end{aligned}$$

Therefore, $a_r \in 1 * a_r * 1 * a_r$.

(5) $\Rightarrow$ (6). Suppose that f is a fuzzy left hyperideal of S. Let g be a fuzzy left hyperideal of S such that $g * g \subseteq f$. Then, for any $a_r \in g$, by (5), we have

$$a_r \in 1 * a_r * 1 * a_r \subseteq 1 * g * 1 * g \subseteq g * g \subseteq f.$$

By Lemma 2.9, $g = \bigcup_{a_r \in g} a_r$, and thus $g \subseteq f$. Consequently, f is quasi-semiprime.

(6) $\Rightarrow$ (3). Let f be any fuzzy left hyperideal of S. Then, by Lemma 3.17, $f * f$ is also a fuzzy left hyperideal of S. Since $f * f \subseteq f * f$, by (6) we have $f \subseteq f * f$. Clearly, $f * f \subseteq f$. It thus follows that $f * f = f$.

(2) $\Rightarrow$ (7). Let f be a fuzzy left hyperideal of S, and let

$$\mathcal{N} = \{g_\alpha \mid g_\alpha \text{ is a quasi-prime fuzzy left hyperideal of } S \text{ such that } f \subseteq g_\alpha\}.$$

We claim that $f = \bigcap_{g_\alpha \in \mathcal{N}} g_\alpha$. Indeed, it is obvious that $f \subseteq \bigcap_{g_\alpha \in \mathcal{N}} g_\alpha$. Conversely, for any $a_r \in \bigcap_{g_\alpha \in \mathcal{N}} g_\alpha$, if $a_r \notin f$, then $r > 0, f(a) < r$. Let

$$\mathcal{B} = \{h_\beta \mid h_\beta \text{ is a fuzzy left hyperideal of } S \text{ such that } f \subseteq h_\beta, f(a) = h_\beta(a)\}.$$

Clearly, $\mathcal{B} \neq \emptyset$ because $f \in \mathcal{B}$. Thus $(\mathcal{B}, \subseteq)$ is an ordered set. Let $\mathcal{C}$ be a chain in $\mathcal{B}$. Then, by Lemma 2.4, the set $\bigcup_{h_\beta \in \mathcal{C}} h_\beta$ is a fuzzy left hyperideal of S and $f \subseteq \bigcup_{h_\beta \in \mathcal{C}} h_\beta$. Since for any $h_\beta \in \mathcal{C}$, $f(a) = h_\beta(a)$, we have

$$(\bigcup_{h_\beta \in \mathcal{C}} h_\beta)(a) = f(a).$$

Thus the fuzzy left hyperideal $\bigcup_{h_\beta \in \mathcal{C}} h_\beta$ is an upper bound of $\mathcal{C}$ in $\mathcal{B}$. By Zorn's Lemma, $\mathcal{B}$ has a maximal element. Denote it by $h_{\max}$. Then $a_r \notin h_{\max}$. We now show that $h_{\max}$ is a quasi-prime fuzzy left hyperideal of S. Let f_1 and f_2 be two fuzzy left hyperideals of S with $f_1 * f_2 \subseteq h_{\max}$. Then, by (2), we have

$$f_1 \cap f_2 \subseteq f_1 * f_2 \subseteq h_{\max}.$$

It thus follow that $h_{\max} = h_{\max} \cup (f_1 \cap f_2) = (h_{\max} \cup f_1) \cap (h_{\max} \cup f_2)$. We claim that $h_{\max} = h_{\max} \cup f_1$ or $h_{\max} = h_{\max} \cup f_2$, that is, $f_1 \subseteq h_{\max}$ or $f_2 \subseteq h_{\max}$. In fact, by $h_{\max} = (h_{\max} \cup f_1) \cap (h_{\max} \cup f_2)$, we have

$$f(a) = h_{\max}(a) = (h_{\max} \cup f_1)(a) \wedge (h_{\max} \cup f_2)(a).$$

This implies $(h_{\max} \cup f_1)(a) = f(a)$ or $(h_{\max} \cup f_2)(a) = f(a)$. Since $h_{\max}$ is maximal with respect to the property that $f \subseteq h_{\max}$ and $h_{\max}(a) = f(a)$, we have $h_{\max} = h_{\max} \cup f_1$ or $h_{\max} = h_{\max} \cup f_2$. Hence $h_{\max}$ is a quasi-prime fuzzy left hyperideal of S. Thus, by hypothesis, $a_r \in h_{\max}$. This is a contradiction. Therefore, $f = \bigcap_{g_\alpha \in \mathcal{N}} g_\alpha$.

(7) $\Rightarrow$ (3). Let f be any fuzzy left hyperideal of S. Then, by Lemma 3.17, $f * f$ is also a fuzzy left hyperideal of S. By (7), we have

$$f * f = \bigcap_{g \in \mathcal{M}} g,$$

where $\mathcal{M}$ is the set of all quasi-prime fuzzy left hyperideals of S containing $f * f$. Furthermore, we prove that $f * f = f$. In fact, for any $g \in \mathcal{M}$, clearly, $f * f \subseteq g$. Since g is quasi-prime, it can be obtained that $f \subseteq g$. Then we have $f \subseteq \bigcap_{g \in \mathcal{M}} g = f * f$. On the other hand, since f is a fuzzy left hyperideal of S, we have $f * f \subseteq 1 * f \subseteq f$. Thus $f * f = f$.

Theorem 4.4. *Let S be a commutative ordered semihypergroup. Then the fuzzy left hyperideals of S are quasi-prime if and only if they form a chain and S is strongly simisimple.*

Proof. Suppose that the fuzzy left hyperideals of S are quasi-prime. Let g and h be fuzzy left hyperideals of S. By Lemma 3.17, $g * h$ is a fuzzy left hyperideal of S. Then, by hypothesis, $g * h$ is quasi-prime. From $g * h \subseteq g * h$, by Lemma 2.10(5), we have $g \subseteq g * h \subseteq 1 * h \subseteq h$ or $h \subseteq g * h \subseteq g * 1 = 1 * g \subseteq g$. Thus the fuzzy left hyperideals of S form a chain. Moreover, for any fuzzy left hyperideal f of S, obviously, $f * f \subseteq f$. Since $f * f \subseteq f * f$, by hypothesis we have $f \subseteq f * f$. It thus follows that $f * f = f$. By Theorem 4.3, S is strongly simisimple.

Conversely, assume that f is a fuzzy left hyperideal of S. Let g, h be any fuzzy left hyperideals of S such that $g * h \subseteq f$. By hypothesis, we have $g \subseteq h$ or $h \subseteq g$. Say $g \subseteq h$, then, by Theorem 4.3, $g = g * g \subseteq g * h \subseteq f$. Similarly, say $h \subseteq g$, we have $h \subseteq f$. Therefore, f is quasi-prime.

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