IMPACT OF HARVESTING, NOISE AND DIFFUSION ON THE DYNAMICS OF A FOOD CHAIN MODEL WITH RATIO-DEPENDENT FUNCTIONAL RESPONSE III

G. Basava Kumar
M.N. Srinivas*
Department of Mathematics
School of Advanced Sciences
VIT University
Vellore-632 01, Tamil Nadu
India
basavakumarg@gmail.com
mnsrinivaselr@gmail.com

Abstract. The current article is associated with impact of noise and harvesting of a three species eco system which consists of prey–predator–top predator with Holling classification III. The stability of the given system is checked at the interior steady state and bionomial steady states are also evaluated. Model formulation of the optimal harvesting policy is given and its solution is derived at interior steady state by using Pontryagins Maximum principle. We also examined the population intensities of variations at the positive steady state due to environmental attribute, we have also highlighted the diffusive steadiness of the structure along with some numerical simulations.
Keywords: bionomic harvesting,optimal harvesting,prey,predator, stability, stochasticity white noise, diffusion.

1. Introduction

The prey-predator structure is a very affective model which has received extensive attention [1-5]. But, all these works not completely supported the effect of harvesting of species along with functional response. However in the real world almost all species have the age structure of prey, predator and top predator. Liu and Chen [6] surveyed the progression stage structured population dynamics. There are several debates on ratio dependent predation have drawn the attention of technologist is on Holling classification. Since this classification is necessary and essential for prey-predator interaction. It is also important for the dynamics of food web models, Bio pest controlling [7]. The choosing of the form of functional response gives a very interesting dynamics and surprising effects on statistical prediction.

Freedmen [8] discussed about the food chain model consisting of these species, food chain systems are very complex and interesting and are dependent in the en-

*. Corresponding author

vironment. Insignificant modifications in the characteristics of such type of food web system give the drastic consequences. Many food chain models have been thoroughly explored, many other authors discovered and examining models have three or four traffic levels [8]-[9] and they presented that the food chain models are rich in dynamics. In many field situation like tea plant-pest-top predator food chain have becomes extremely important and predator may determine the fitness of the plant and destroying pest[10]. In view of social necessities the utilization of organic assets and harvesting populace are usually practiced in yield life controlling, fishery and forestry etc. There is a wide range of interest use of bionomic modelling to gain insight in the technological utilization of well known fields like forestry, fisheries [11-13].In [13] Xiao et.al, presented the following model

$$x'(t) = rx\left(1 - \frac{x}{k}\right) - \frac{cxy}{my + x}, \quad y'(t) = y\left(-D - \frac{fx}{my + x}\right).$$

Our article is associated with the prey which is of profitable important. The predator and top predator are constantly participated in harvesting with an agency. This process do not disturb the prey populace directly. This article mainly concentrates on analysis of the changing aspects of tri-tropic food web constitutes of victim, hunter and chief hunter, and which is having ratio dependent three type functional response, along with harvesting efforts. Jostet al., [14] discussed and studied the underlying forces of a two species model with Holling classification. The main aim of the current article is to study the dynamics properties of the model system defined above, the effects of noise and diffusion on the above said structure are also analysed. Inspired by [15] and [16] we consider the tri tropic prey-predator-top predator system incorporating harvesting.

The organization of this article consists of several subdivisions. Subdivision 2 defines the elementary scientific model. Section 3 consists of boundedness of the structure and investigation of fixed points, whereas steadiness is analysed in section 4. Section 5 includes bionomic equilibrium analysis. Section 6 includes optimal harvesting approach. The consequences of noise of the given food web model termed as stochastic model is described in section 7. We also have given the spatiotemporal analysis in section 8. In section 9, numerical simulations are given. Finally concluding remarks are in section 10.

2. Mathematical model

We consider a computational mathematical model with harvesting, includes 3 species say a prey N_1, a predator N_2 and a top predator N_3. In this construction, we assume that top predator preys on predator only and predator, preys on prey species only. Also we assume that the nutrient recycling is not considered. In view of both biological and methodical point of view, this supposition is a remarkable and a real-world one. It is also assumed that a ratio dependent type-III Holling classification for both $N_1 \leftrightarrow N_2$ and $N_2 \leftrightarrow N_3$ interfaces. From the above technological discussion the framed system is as follows

$$N_1'(T) = r_1 N_1 \left(1 - \frac{N_1}{k_1}\right) - \frac{s_1 N_1^2 N_2}{(b_1 N_2^2 + N_1^2)c_1} - q_1 E_1 N_1, \tag{2.1}$$

$$N_2'(T) = \frac{s_1 N_1^2 N_2}{(b_1 N_2^2 + N_1^2)} - m_1 N_2 - \frac{s_2 N_2^2 N_3}{(b_2 N_3^2 + N_2^2)c_2} - q_2 E_2 N_2, \tag{2.2}$$

$$N_3'(T) = \frac{s_2 N_2^2 N_3}{(b_2 N_3^2 + N_2^2)} - m_2 N_3 - q_3 E_3 N_3, \tag{2.3}$$

where

$$N_1(0) > 0, N_2(0) > 0, N_3(0) > 0.$$

Here $N_i, i = 1,2,3$ represents bio mass densities of prey, predator and super predator species, $q_i, i = 1,2,3$ represents catchability coefficients of species respectively. $E_i, i = 1,2,3$ represents efforts of applied to harvest to prey, predator and super predators respectively, r_1 represents intrinsic growth rate of prey species, $s_i, i = 1,2$ represents maximal predator growth of predator and top predator, $m_i, i = 1,2$ represents natural death rate of predator and top predator, $b_i, i = 1,2$ represents half saturation constants of predator and top predator, $c_i, i = 1,2$ represents yield constants of prey and predator. The term $(N_1^2 N_2)/(b_1 N_2^2 + N_1^2)$represents ratio dependent type III functional response for prey and predator and $(N_2^2 N_3)/(b_1 N_3^2 + N_2^2)$ represents ratio dependent type III functional response for predator and super predator. Here, throughout our analysis we are assuming that $r_1 - q_1 E_1 > 0$, $p_2 - q_4 - q_3 E_3 > 0$, $p_1 - q_5 - q_2 E_2 > 0$. We make an observable supposition that all the attributes are positive. Since the densities of the population cannot be negative, the state space of system (2.1)-(2.3) is given by R_+^3 . To make a methodical analysis easy, we reduce (2.1)-(2.3) into a non-dimensionalized one by using

$$t = r_1 T, n_1 = \frac{N_1}{k_1}, n_2 = \frac{N_2 \sqrt{b_1}}{k_1}, n_3 = \frac{N_3 \sqrt{b_1 b_2}}{k_1}.$$

Then the system (2.1)-(2.3) takes the form

$$n_1'(t) = n_1(1 - n_1) - \frac{l_1 n_1^2 n_2}{n_1^2 + n_2^2} - \frac{q_1 E_1 n_1}{r_1}, \tag{2.4}$$

$$n_2'(t) = \frac{p_1 n_1^2 n_2}{n_1^2 + n_2^2} - q_5 n_2 - \frac{l_2 n_2^2 n_3}{n_3^2 + n_2^2} - \frac{q_2 E_2 n_2}{r_1}, \tag{2.5}$$

$$n_3'(t) = \frac{p_2 n_2^2 n_3}{n_3^2 + n_2^2} - q_4 n_3 - \frac{q_3 E_3 n_3}{r_1}, \tag{2.6}$$

where

$$n_1(0) > 0, n_2(0) > 0, n_3(0) > 0,$$

$$l_1 = \frac{s_1}{c_1 r_1 \sqrt{b_1}}, l_2 = \frac{s_2}{c_2 r_1 \sqrt{b_2}}, p_1 = \frac{s_1}{r_1}, q_5 = \frac{m_1}{r_1}, q_4 = \frac{m_2}{r_1}, p_2 = \frac{s_2}{r_1}.$$

3. Analysis of steady states

3.1 Steady states

It is obvious that the interior equilibrium point $E^*(n_1^*, n_2^*, n_3^*)$ of system (2.1)-(2.3) exist in the interior point of the first octant if there is a positive solution of the algebraic equations

$$\begin{aligned} n_1(1-n_1) - \frac{l_1 n_1^2 n_2}{n_1^2+n_2^2} - \frac{q_1 E_1 n_1}{r_1} &= 0, \\ \frac{p_1 n_1^2 n_2}{n_1^2+n_2^2} - q_5 n_2 - \frac{l_2 n_2^2 n_3}{n_3^2+n_2^2} - \frac{q_2 E_2 n_2}{r_1} &= 0, \\ \frac{p_2 n_2^2 n_3}{n_3^2+n_2^2} - q_4 n_3 - \frac{q_3 E_3 n_3}{r_1} &= 0. \end{aligned}$$

Then, by solving these equations, we get,

$$\begin{aligned} n_1^* &= \left(\frac{p_1 p_2 r_1^2 (r_1 - q_1 E_1) - l_1 r_1 \sqrt{L(p_1 p_2 r_1^2 - L)}}{p_1 p_2 r_1^3} \right), \\ n_2^* &= \left(\sqrt{\frac{p_1 p_2 r_1^2 - L}{L}} \right) n_1^*, \\ n_3^* &= \left(\sqrt{\frac{(p_2+q_4) r_1 - q_3 E_3}{q_4 r_1 + q_3 E_3}} \right) n_2^*, \end{aligned}$$

where

$$L = p_2 q_2 - l_2 r_1 \sqrt{(p_2 - q_2 - q_3 E_3)(q_4 + q_3 E_3)} - p_2 q_2 E_2.$$

For positiveness of these values, we must have,

$$p_1 p_2 r_1^2 > L, (p_2 + q_4) r_1 > q_3 E_3.$$

4. Stability analysis

4.1 Local stability analysis

Now to investigate the confined steadiness of inner steady state $E^*(n_1^*, n_2^*, n_3^*)$. Next we have to construct a matrix $M(E^*)$

$$M(E^*) = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}.$$

Where

$$b_{11} = \left(1 - 2n_1 - \frac{2 l_1 n_2^{*^3} n_1}{\left(n_1^{*^2} + n_2^{*^2}\right)^2} - \frac{q_1 E_1}{r_1} \right), b_{12} = \left(- \frac{2 l_1 n_1^{*^4}}{\left(n_1^{*^2} + n_2^{*^2}\right)^2} \right),$$

$$b_{13}=0, b_{21}=\left(\frac{2p_1{n_2^*}^3 n_1^*}{({n_1^*}^2+{n_2^*}^2)^2}\right),$$
$$b_{22}=\left(\frac{2p_1{n_1^*}^3}{({n_1^*}^2+{n_2^*}^2)^2}\right)-q_5-l_2\left(\frac{2{n_3^*}^3 n_2^*}{({n_2^*}^2+{n_3^*}^2)^2}\right)-\frac{q_2E_2}{r_1},$$
$$b_{23}=-\left(\frac{2{n_2^*}^4 l_2}{({n_2^*}^2+{n_3^*}^2)^2}\right), b_{31}=0, b_{32}=\left(\frac{2p_2n_2n_3^3}{({n_3^*}^2+{n_2^*}^2)^2}\right),$$
$$b_{33}=\left(\frac{p_2{n_2^*}^2({n_2^*}^2-{n_3^*}^2)}{({n_3^*}^2+{n_2^*}^2)^2}\right)-q_4-\frac{q_3E_3}{r_1}.$$

The analogous specific equation is

$$\lambda^3+A_1\lambda^2+A_2\lambda+A_3=0. \tag{4.1.1}$$

Where

$$A_1=-(b_{11}+b_{22}+b_{33})$$

$$A_1=-\left(\left(1-2n_1-\frac{2l_1{n_2^*}^3n_1}{\left({n_1^*}^2+{n_2^*}^2\right)^2}-\frac{q_1E_1}{r_1}\right)+\left(\begin{array}{c}\left(\frac{2p_1{n_1^*}^3}{({n_1^*}^2+{n_2^*}^2)^2}\right)\\ -q_5-\left(\frac{2l_2{n_3^*}^3n_2^*}{({n_2^*}^2+{n_3^*}^2)^2}\right)-\frac{q_2E_2}{r_1}\end{array}\right)+\left(\frac{p_2{n_2^*}^2({n_2^*}^2-{n_3^*}^2)}{({n_3^*}^2+{n_2^*}^2)^2}\right)-q_4-\frac{q_3E_3}{r_1}\right)$$

$$A_2=(b_{11}b_{22}+b_{11}b_{33}+b_{22}b_{33}-b_{23}b_{32}-b_{21}b_{12})$$

$$\begin{aligned}A_2=&\left(1-2n_1-\frac{2l_1{n_2^*}^3n_1}{\left({n_1^*}^2+{n_2^*}^2\right)^2}-\frac{q_1E_1}{r_1}\right)\cdot\\
&\cdot\left(\begin{array}{c}\left(\frac{2p_1{n_1^*}^3}{({n_1^*}^2+{n_2^*}^2)^2}\right)-q_5-l_2\left(\frac{2{n_3^*}^3n_2^*}{({n_2^*}^2+{n_3^*}^2)^2}\right)\\ -\frac{q_2E_2}{r_1}+\left(\frac{p_2{n_2^*}^2({n_2^*}^2-{n_3^*}^2)}{({n_3^*}^2+{n_2^*}^2)^2}\right)-q_4-\frac{q_3E_3}{r_1}\end{array}\right)\\
&+\left(\left(\frac{2p_1{n_1^*}^3}{({n_1^*}^2+{n_2^*}^2)^2}\right)-q_5-l_2\left(\frac{2{n_3^*}^3n_2^*}{({n_2^*}^2+{n_3^*}^2)^2}\right)-\frac{q_2E_2}{r_1}\right)\cdot\\
&\cdot\left(\left(\frac{p_2{n_2^*}^2({n_2^*}^2-{n_3^*}^2)}{({n_3^*}^2+{n_2^*}^2)^2}\right)-q_4-\frac{q_3E_3}{r_1}\right)\\
&-\left(-\left(\frac{2{n_2^*}^4l_2}{({n_2^*}^2+{n_3^*}^2)^2}\right)\left(\frac{2p_2n_2n_3^3}{({n_3^*}^2+{n_2^*}^2)^2}\right)+\left(-\frac{2l_1{n_1^*}^4}{\left({n_1^*}^2+{n_2^*}^2\right)^2}\right)\left(\frac{2p_1{n_2^*}^3n_1^*}{({n_1^*}^2+{n_2^*}^2)^2}\right)\right)\end{aligned}$$

$$A_3 = [b_{11}b_{23}b_{33} - b_{11}b_{23}b_{32} - b_{12}b_{21}b_{33}]$$

$$A_3 = \left(1 - 2n_1 - \frac{2l_1 n_2^{*^3} n_1}{\left(n_1^{*^2} + n_2^{*^2}\right)^2} - \frac{q_1 E_1}{r_1}\right) \cdot$$
$$\cdot \left(\left(\frac{2p_1 n_1^{*^3}}{(n_1^{*^2} + n_2^{*^2})^2}\right) - q_5 - l_2 \left(\frac{2n_3^{*^3} n_2^*}{(n_2^{*^2} + n_3^{*^2})^2}\right) - \frac{q_2 E_2}{r_1}\right) \cdot$$
$$\cdot \left(\frac{p_2 n_2^{*^2}(n_2^{*^2} - n_3^{*^2})}{(n_3^{*^2} + n_2^{*^2})^2}\right) - q_4 - \frac{q_3 E_3}{r_1}$$
$$- \left(\left(1 - 2n_1 - \frac{2l_1 n_2^{*^3} n_1}{\left(n_1^{*^2} + n_2^{*^2}\right)^2} - \frac{q_1 E_1}{r_1}\right)\left(-\frac{2n_2^{*^4} l_2}{(n_2^{*^2} + n_3^{*^2})^2}\right)\left(\frac{2p_2 n_2 n_3^3}{(n_3^{*^2} + n_2^{*^2})^2}\right)\right)$$
$$- \left(\left(-\frac{2l_1 n_1^{*^4}}{\left(n_1^{*^2} + n_2^{*^2}\right)^2}\right)\left(\frac{2p_1 n_2^{*^3} n_1^*}{(n_1^{*^2} + n_2^{*^2})^2}\right)\left(\left(\frac{p_2 n_2^{*^2}(n_2^{*^2} - n_3^{*^2})}{(n_3^{*^2} + n_2^{*^2})^2}\right) - q_4 - \frac{q_3 E_3}{r_1}\right)\right)$$

clearly, the proposed system is locally asymptotically stable by Routh-Hurwitz criteria.

4.2 Global steadiness

Theorem. The positive steady state (n_1^*, n_2^*, n_3^*) of the above proposed system is globally asymptotically stable if

$$V(t) = \left(n_1 - n_1^* - n_3^* \ln\left(\frac{n_1}{n_1^*}\right)\right) + l_1\left(n_2 - n_2^* - n_2^* \ln\left(\frac{n_2}{n_2^*}\right)\right)$$
$$+ l_2\left(n_3 - n_3^* - n_3^* \ln\left(\frac{n_3}{n_3^*}\right)\right),$$

$l_1 > 0$, $l_2 > 0$ and $P < \min(Q, R)$, where

$$P = \alpha\left(l_1 c_2 m_3 n_2^* + l_2 p_2 m_4 n_3^*\right) + \beta\left(c_1 m_1 n_1^* + l_1 p_1 m_2 n_2^*\right),$$
$$Q = 2l_1\left(\beta p_1 m_2 n_1^* + \alpha c_2 m_3 n_3^*\right),$$
$$R = 2\alpha l_2 p_2 m_4 y^* + \beta\left(2\alpha + c_1 m_1 y^*\right).$$

Proof. To check the global steadiness of inner equilibrium point, construct a Lyapunov function $V(t)$ as

$$V(t) = \left(\frac{n_1 - n_1^*}{n_1}\right) n_1(t) + l_1\left(\frac{n_2 - n_2^*}{n_2}\right) n_2(t) + l_2\left(\frac{n_3 - n_3^*}{n_3}\right) n_3(t)$$
$$V'(t) = (n_1 - n_2^*)\left[1 - n_1 - \frac{c_1 n_1 n_2}{n_1^2 + n_2^2} - \frac{q_1 E_1}{r}\right]$$
$$+ l_1 (n_2 - n_2^*)\left[\frac{p_1 n_1^2}{n_1^2 + n_2^2} - q_5 - \frac{c_2 n_2 n_3}{n_2^2 + n_3^2} - \frac{q_2 E_2}{r}\right]$$

$$+ l_2\left(n_3 - n_3^*\right)\left[\frac{p_2 n_2^2}{n_2^2 + n_3^2} - q_4 - \frac{q_3 E_3}{r}\right].$$

At the equilibrium point (n_1^*, n_2^*, n_3^*), we have

$$\begin{aligned} V'(t) &= \left(n_1 - n_2^*\right)\left[1 - n_1 - \frac{c_1 n_1 n_2}{n_1^2 + n_2^2} - 1 + n_1^* + \frac{c_1 n_1^* n_2^*}{n_1^{*2} + n_2^{*2}}\right] \\ &+ l_1\left(n_2 - n_2^*\right)\left[\frac{p_1 n_1^2}{n_1^2 + n_2^2} - \frac{c_2 n_2 n_3}{n_2^2 + n_3^2} - \frac{p_1 n_1^{*2}}{n_1^{*2} + n_2^{*2}} + \frac{c_2 n_2^* n_3^*}{n_2^{*2} + n_3^{*2}}\right] \\ &+ l_2\left(n_3 - n_3^*\right)\left[\frac{p_2 n_2^2}{n_2^2 + n_3^2} - \frac{p_2 n_2^{*2}}{n_2^{*2} + n_3^{*2}}\right]. \end{aligned}$$

Therefore, we have

$$\begin{aligned} V'(t) =& -\left(n_1 - n_2^*\right)^2 - \frac{c_1 n_1^*}{\alpha}\left(n_2 n_2^* - n_1 n_1^*\right)\left(n_1 - n_1^*\right)\left(n_2 - n_2^*\right) \\ & - \frac{c_1 n_2^*}{\alpha}\left(n_2 n_2^* - n_1 n_1^*\right)\left(n_1 - n_1^*\right)^2 \\ & + \frac{l_1 p_1 n_2^*}{\alpha}\left(n_1 n_2^* + n_1^* n_2\right)\left(n_1 - n_2^*\right)\left(n_2 - n_2^*\right) - \frac{l_1 p_1 n_1^*}{\alpha}\left(n_1 n_2^* + n_1^* n_2\right)\left(n_2 - n_2^*\right)^2 \\ & + \frac{l_1 c_2 n_2^*}{\beta}\left(n_3 n_3^* - n_2 n_2^*\right)\left(n_2 - n_2^*\right)\left(n_3 - n_3^*\right) - \frac{l_1 c_2 n_3^*}{\beta}\left(n_3 n_3^* - n_2 n_2^*\right)\left(n_2 - n_2^*\right)^2 \\ & + \frac{l_2 p_2 n_3^*}{\beta}\left(n_2 n_3^* + n_2^* n_3\right)\left(n_2 - n_2^*\right)\left(n_3 - n_3^*\right) \\ & - \frac{l_2 p_2 n_2^*}{\beta}\left(n_2 n_3^* + n_2^* n_3\right)\left(n_3 - n_3^*\right)^2, \end{aligned}$$

where

$$\begin{aligned} \alpha &= \left(n_1^2 + n_2^2\right)\left(n_1^{*2} + n_2^{*2}\right), \beta = \left(n_2^2 + n_3^2\right)\left(n_2^{*2} + n_3^{*2}\right), \\ V'(t) &\le \left(n_1 - n_1^*\right)^2\left[-1 + \frac{c_1 m_1 n_1^*}{2\alpha} - \frac{c_1 m_1 n_2^*}{\alpha} + \frac{l_1 p_1 m_2 n_2^*}{2\alpha}\right] \\ &+ \left(n_2 - n_2^*\right)^2\left[\frac{c_1 m_1 n_1^*}{2\alpha} + \frac{l_1 p_1 m_2 n_2^*}{2\alpha} - \frac{l_1 p_1 m_2 n_1^*}{\alpha} + \frac{l_1 c_2 m_3 n_2^*}{2\beta}\right. \\ &\left. - \frac{l_1 c_2 m_3 n_3^*}{\beta} + \frac{l_2 p_2 m_4 n_3^*}{2\beta}\right] \\ &+ \left(n_3 - n_3^*\right)^2\left[\frac{l_1 c_2 m_3 n_1^*}{2\beta} + \frac{l_2 p_2 m_4 n_3^*}{2\beta} - \frac{l_2 p_2 m_4 n_2^*}{\beta}\right], \end{aligned}$$

where $m_1 = n_2 n_2^* - n_1 n_1^* > 0, m_2 = n_1 n_2^* + n_1^* y, m_3 = n_3 n_3^* - n_2 n_2^* > 0, m_4 = n_2 n_3^* + n_2^* n_3$. The derivative of Lyapunov function becomes negative, if

$$\alpha(l_1 c_2 m_3 n_1^* + l_2 p_2 m_4 n_2^*) + \beta(c_1 m_1 n_1^* + l_1 p_1 m_2 n_2^*) < 2\alpha l_2 p_2 m_4 n_2^* + 2\beta(\alpha + c_1 m_1 n_2^*).$$

Hence proved.

5. Bionomic steadiness

The aim of this section is to capture the changing aspects of several live tools and data using inexpensive models. Let c_3 be the harvesting cost per unit effort for prey species. Let c_4 be the harvesting cost per unit effort for predator species. Let c_5 be the harvesting cost per unit effort for super species. Let p_3 be the price for unit biomass for prey species. Let p_4 be the price for unit biomass for predator species. Let p_5 be the price for unit biomass for super predator species. Thus net revenue is $NR = NR_1 + NR_2 + NR_3$. Where $NR_1 = (p_3q_1n_1 - c_3)E_1$. $NR_2 = (p_4q_2n_2 - c_4)E_2$, $NR_3 = (p_5q_3n_3 - c_5)E_3$. The Bionomic steady state is $((n_1)_\infty, (n_2)_\infty, (n_3)_\infty, (E_1)_\infty, (E_2)_\infty, (E_3)_\infty)$ and

$$n_1(1-n_1) - \frac{l_1n_1^2n_2}{n_1^2+n_2^2} - \frac{q_1E_1n_1}{r_1} = 0, \tag{5.1}$$

$$\frac{p_1n_1^2n_2}{n_1^2+n_2^2} - q_5n_2 - \frac{l_2n_2^2n_3}{n_3^2+n_2^2} - \frac{q_2E_2n_2}{r_1} = 0, \tag{5.2}$$

$$\frac{p_2n_2^2n_3}{n_3^2+n_2^2} - q_4n_3 - \frac{q_3E_3n_3}{r_1} = 0, \tag{5.3}$$

$$\begin{aligned} NR = NR_1 + NR_2 + NR_3 &= (p_3q_1n_1 - c_3)E_1 + (p_4q_2n_2 - c_4)E_2 \\ &+ (p_5q_3n_3 - c_5)E_3 = 0. \end{aligned} \tag{5.4}$$

To obtain the bionomic steady state , it is needed to verify the cases

Case (i). If $c_3 > p_3q_1n_1$,$c_4 > p_4q_2n_2$,$c_5 > p_5q_3n_3$, which is not nearer to our aim.

Case (ii). if $c_3 < p_3q_1n_1$,$c_4 < p_4q_2n_2$, $c_5 < p_5q_3n_3$, which is very nearer to our aim and recommended for the above working model.

$$(n_1)_\infty = \left(\frac{c_3}{p_3q_1}\right), \quad (n_2)_\infty = \left(\frac{c_4}{p_4q_2}\right), \quad (n_3)_\infty = \left(\frac{c_5}{p_5q_3}\right). \tag{5.5}$$

From (5.1), (5.2) and (5.3) we get

$$(E_1)_\infty = \frac{r_1}{q_1}\left(\frac{p_3q_1 - c_3}{p_3q_3} - \frac{l_1c_3^2c_4p_3p_1p_4q_2}{c_3^2p_4^2q_2^2 + c_4^2p_3^2q_1^2}\right), \tag{5.6}$$

$$(E_2)_\infty = \frac{r_1}{q_2}\left(\frac{p_1c_3^2p_4^2q_2^2}{c_3^2p_4^2q_2^2 + c_4^2p_3^2q_1^2} - q_5 - l_2\left(\frac{c_4c_5p_4p_5q_2q_3}{c_4^2p_5^2q_3^2 + c_5^2p_4^2q_2^2}\right)\right), \tag{5.7}$$

$$(E_3)_\infty = \frac{r_1}{q_3}\left(\frac{p_2c_4p_4q_2p_5^2q_3^2}{c_4^2p_5^2q_3^2 + c_5^2p_4^2q_2^2} - q_4\right). \tag{5.8}$$

For,$(E_1)_\infty$, $(E_2)_\infty$,$(E_3)_\infty$ are to be positive if

$$A_P > B_p, C_p > D_p, E_P > F_p, \tag{5.9}$$

where

$$A_p = \frac{p_3 q_1 - c_3}{p_3 q_3}, B_p = \frac{l_1 c_3^2 c_4 p_3 p_1 p_4 q_2}{c_3^2 p_4^2 q_2^2 + c_4^2 p_3^2 q_1^2}, C_p = \frac{p_1 c_3^2 p_4^2 q_2^2}{c_3^2 p_4^2 q_2^2 + c_4^2 p_3^2 q_1^2} - q_5,$$
$$D_p = l_2 \left(\frac{c_4 c_5 p_4 p_5 q_2 q_3}{c_4^2 p_5^2 q_3^2 + c_5^2 p_4^2 q_2^2} \right), E_P = \frac{p_2 c_4 p_4 q_2 p_5^2 q_3^2}{c_4^2 p_5^2 q_3^2 + c_5^2 p_4^2 q_2^2}, F_p = q_4.$$

Thus, bionomic steady state

$$\left((n_1)_\infty, (n_2)_\infty, (n_3)_\infty, (E_1)_\infty, (E_2)_\infty, (E_3)_\infty \right)$$

occurs if (5.9) satisfied.

6. Optimal harvesting strategy

This section deals with feasible harvesting procedure of (2.4)-(2.6). we use the pontryagin's principle to attain an optimal path. For this we construct present value function

$$J = \int_0^\infty P(n_1, n_2, n_3, E_1, E_2, E_3, t) e^{-\delta t}. \tag{6.1}$$

Where

$$\begin{aligned} P(n_1, n_2, n_3, E_1, E_2, E_3, t) &= (p_1 q_1 n_1 E_1 - c_3 E_1) + (p_2 q_2 n_2 E_2 - c_4 E_2) \\ &\quad + (p_3 q_3 n_3 E_3 - c_5 E_3) \end{aligned} \tag{6.2}$$

And δ is the instantaneous annual rate of the discount. To maximize J with respect to (2.4)-(2.6), consider the Hamiltonian function

$$\begin{aligned} H &= e^{-\delta t}(p_1 q_1 n_1 - c_3) E_1 + e^{-\delta t}(p_2 q_2 n_2 - c_4) E_2 + e^{-\delta t}(p_3 q_3 n_3 - c_5) E_3 \\ &+ \lambda_1 \left[n_1(1 - n_1) - \frac{l_1 n_1^2 n_2}{(n_2^2 + n_1^2)} - \frac{q_1 E_1 n_1}{r_1} \right] \\ &+ \lambda_2 \left(\frac{p_1 n_1^2 n_2}{(n_2^2 + n_1^2)} - q_5 n_2 - \frac{l_2 n_2^2 n_3}{(n_3^2 + n_2^2)} - \frac{q_2 E_2 n_2}{r_1} \right) \\ &+ \lambda_3 \left(\frac{p_2 n_2^2 n_3}{(n_3^2 + n_2^2)} - q_4 n_3 - \frac{q_3 E_3 n_3}{r_1} \right), \end{aligned} \tag{6.3}$$

where λ_1,λ_2,λ_3 are adjoint variables and E_1,E_2,E_3 are the control variables satisfying the constraints $0 \leq E_1 \leq (E_1)_{\max}, 0 \leq E_2 \leq (E_2)_{\max}, 0 \leq E_3 \leq (E_3)_{\max}$, and $\psi_1(t) = e^{-\delta t}[p_1 q_1 n_1 - c_3] - \lambda_1 q_1 n_1, \psi_2(t) = e^{-\delta t}[p_2 q_2 n_2 - c_4] - \lambda_2 q_2 n_2, \psi_3(t) = e^{-\delta t}[p_3 q_3 n_3 - c_5] - \lambda_3 q_3 n_3$.

In order to find a feasible steady state $((n_1)_\delta, (n_2)_\delta, (n_3)_\delta, (E_1)_\delta, (E_2)_\delta, (E_3)_\delta)$ if H is linear in E_1 and E_2 ideal control can be extreme, thus we have $E_i = (E_i)_{\max}, \psi_i(t) > 0$, $E_i = 0, \psi_i(t) < 0$, for $i = 1, 2, 3$.

If $\psi_1(t) = \psi_2(t) = \psi_3(t) = 0$, then $\lambda_1 e^{\delta t} = p_1 - \frac{c_3}{q_1 n_1}, \lambda_2 e^{\delta t} = p_2 - \frac{c_4}{q_2 n_2}, \lambda_3 e^{\delta t} = p_3 - \frac{c_5}{q_3 n_3}$

$$\frac{\partial H}{\partial E_i} = 0, \tag{6.4}$$

where $i = 1, 2, 3$ With the help of ideal controls, and (6.4) are necessary conditions for the maximization H.By pontryagin's rule, the equations are

$$\begin{aligned} \frac{\partial \lambda_1}{\partial t} &= -\frac{\partial H}{\partial n_1} - \left[e^{-\delta t} p_1 q_1 + \lambda_1 \left[1 - 2n_1 - \frac{l_1 n_2 2n_1 n_2^2}{(n_1^2+n_2^2)^2} - \frac{q_1 E_1}{r_1} \right] \right. \\ &\left. + \frac{\lambda_2 p_1 n_2 2n_1 n_2^2}{(n_1^2+n_2^2)^2} \right], \end{aligned} \tag{6.5}$$

$$\begin{aligned} \frac{\partial \lambda_2}{\partial t} &= -\frac{\partial H}{\partial n_2} = -\left[e^{-\delta t} p_2 q_2 + \lambda_1 \left[\frac{l_1 n_1^2 (n_2^2 - n_1^2)}{(n_1^2+n_2^2)^2} \right] \right. \\ &+ \lambda_2 \left[p_1 n_1^2 \left[\frac{n_2^2 - n_1^2}{(n_1^2+n_2^2)^2} \right] - q_5 \right] \Bigg] \\ &+ 2\lambda_2 \left[\left[\frac{l_2 n_3^3 n_2}{(n_3^2+n_2^2)^2} \right] - \frac{q_2 E_2}{r_1} \right] + \lambda_3 \left[\frac{2p_2 n_3^3 n_2}{(n_3^2+n_2^2)^2} \right], \end{aligned} \tag{6.6}$$

$$\begin{aligned} \frac{\partial \lambda_3}{\partial t} &= -\frac{\partial H}{\partial n_3} = -\left[e^{-\delta t} p_3 q_3 - \lambda_2 \left[\frac{l_2 n_2^2 (n_3^2 - n_2^2)}{(n_3^2+n_2^2)^2} \right] \right. \\ &\left. + \lambda_3 \left[\frac{p_2 n_2^2 (n_3^2 - n_3^2)}{(n_3^2+n_2^2)^2} - q_4 - \frac{q_3 E_3}{r_1} \right] \right]. \end{aligned} \tag{6.7}$$

At $E^*(n_1^*, n_2^*, n_3^*)$ and from (6.4),(6.5),(6.6) and (6.7)we have

$$\frac{d\lambda_1}{dt} + \lambda_1 M_1 = -e^{-\delta t} M_2, \frac{d\lambda_2}{dt} + \lambda_2 (M_3) = -e^{-\delta t} M_4, \frac{d\lambda_3}{dt} + \lambda_3 M_5 = -e^{-\delta t} M_6.$$

And solutions are

$$\lambda_1 = \frac{M_2}{(-M_1+\delta)} e^{-\delta t}, \lambda_2 = \frac{M_4}{(\delta - M_3)} e^{-\delta t}, \lambda_3 = \frac{M_6}{(\delta - M_5)} e^{-\delta t}, \tag{6.8}$$

where

$$\begin{aligned} M_1 &= \left[1 - 2n_1 - l_1 n_2 \frac{2n_1 n_2^2}{(n_1^2+n_2^2)^2} - \frac{q_1 E_1}{r_1} \right], \\ M_2 &= \left[p_1 q_1 + (p_2 - \frac{c_4}{q_2 n_2}) \left[p_1 n_2 \left[\frac{2n_1 n_2^2}{(n_1^2+n_2^2)^2} \right] \right] \right], \end{aligned}$$

$$M_3 = p_1 n_1^2 \left[\frac{n_2^2 - n_1^2}{(n_1^2 + n_2^2)^2}\right] - q_5 + 2\left[\frac{l_2 n_3^3 n_2}{(n_3^2 + n_2^2)^2}\right] - \frac{q_2 E_2}{r_1},$$
$$M_4 = -p_2 q_2 - (p_1 - \frac{c_3}{q_2 n_2})\left[\frac{l_1 n_1^2 (n_2^2 - n_1^2)}{(n_1^2 + n_2^2)^2}\right] + (p_3 - \frac{c_5}{q_3 n_3})\left[\frac{2p_2 n_3^3 n_2}{(n_3^2 + n_2^2)^2}\right],$$
$$M_5 = \frac{p_2 n_2^2 (n_3^2 - n_3^2)}{(n_3^2 + n_2^2)^2} - q_4 - \frac{q_3 E_3}{r_1},$$
$$M_6 = (p_3 q_3 - (p_2 - \frac{c_4}{q_2 n_2})\left[\frac{l_2 n_2^2 (n_3^2 - n_2^2)}{(n_3^2 + n_2^2)^2}\right].$$

It is obviously that $\lambda_1(t), \lambda_2(t), \lambda_3(t)$ are bounded as $t \to \infty$. From (6.4) & (6.8), we obtain a singular paths

$$p_1 - \frac{c_3}{q_1 N_1^*} = \frac{M_2}{\delta - M_1}, p_2 - \frac{c_4}{q_2 N_2^*} = \frac{M_4}{\delta - M_3}, p_3 - \frac{c_5}{q_3 N_3^*} = \frac{M_6}{\delta - M_5}.$$

These singular paths can be written as

$$F_1(N_1^*) = (p_1 - \frac{c_3}{q_1 N_1^*}) - \frac{M_2}{\delta - M_1},$$

$$F_2(N_2^*) = (p_2 - \frac{c_4}{q_2 N_2^*}) - \frac{M_4}{\delta - M_3}, F_3(N_3^*) = (p_3 - \frac{c_5}{q_3 N_3^*}) - \frac{M_5}{\delta - M_5}.$$

There exists a unique roots $n_i^* = (n_i)_\delta$,$i = 1, 2, 3$ of $F_i(n_i^*) = 0$in the interval $0 < (n_i)_\infty < k_i$ if the inequalities $F_i(0) < 0, F_i(k_i) > 0, F_i^1(n_i^*) > 0$ for $n_i^* > 0$.Therefore, the feasible steady state population are is $n_i^* = (n_i)_\delta, i = 1, 2, 3$.

Then the feasible efforts are

$$E_1^* = (E_1)_\delta = \frac{r_1}{q_1}\left((1 - n_1) - \frac{l_1 n_1 n_2}{n_1^2 + n_2^2}\right),$$
$$E_2^* = (E_2)_\delta = \frac{r_1}{q_2}\left(\frac{p_1 n_1^2}{n_1^2 + n_2^2} - q_5 - \frac{l_2 n_3 n_2}{n_3^2 + n_2^2}\right),$$
$$E_3^* = (E_3)_\delta = \frac{r_1}{q_3}\left(\frac{p_2 n_2^2}{n_3^2 + n_2^2} - q_4\right).$$

For $(E_i)_\delta > 0, i = 1, 2, 3$, are to be affirmative, if

$$(1 - n_1) > \frac{l_1 n_1 n_2}{n_1^2 + n_2^2}, \frac{p_1 n_1^2}{n_1^2 + n_2^2} > q_5 + \frac{l_2 n_3 n_2}{n_3^2 + n_2^2}, \frac{p_2 n_2^2}{n_3^2 + n_2^2} > q_4$$

exists. Hence $(n_i)_\delta$, $(E_i)_\delta, i = 1, 2, 3$ are resolute, and from (6.3),(6.4) and (6.8), we establish that $\lambda_i(t)e^{\delta i}, (i = 1, 2, 3)$do not vary with time in feasible steady state. Therefore they keep on bounded as $t \to \infty$.

7. The Stochastic model

Now, this section is meant for the extension of the deterministic model (2.4)-(2.6), which is obtained by adding noisy term. There are several ways in which environmental noise may be incorporated in the model system (2.4)-(2.6). External noise may arise from random fluctuations of finite number of parameters around some known mean values of the populace densities around some fixed values. The populace intensities of oscillations are calculated near the inner steady states due to environmental attributes by applying the method of [17] and [18]. Since the aquatic ecosystem which always has unsystematic fluctuations of the environment, it is difficult to define the usual phenomenon as a deterministic ideal. The stochastic investigation benefits us to get an extra intuition about the continuous changing aspects of any ecological unit. The deterministic model (2.4)-(2.6) with the effect of random noise of the environmental results in a stochastic system ((7.1)-(7.3)) given in the following discussion.

$$n_1'(t) = n_1(1-n_1) - \frac{l_1 n_1^2 n_2}{n_1^2+n_2^2} - \frac{q_1 E_1 n_1}{r_1} + \eta_1 \varphi_1(t), \tag{7.1}$$

$$n_2'(t) = \frac{p_1 n_1^2 n_2}{n_1^2+n_2^2} - q_5 n_2 - \frac{l_2 n_2^2 n_3}{n_1^2+n_2^2} - \frac{q_2 E_2}{r_1} n_2 + \eta_2 \varphi_2(t), \tag{7.2}$$

$$n_3'(t) = \frac{p_2 n_1^2 n_3}{n_3^2+n_2^2} - q_4 n_3 - \frac{q_3 E_3}{r_1} n_3 + \eta_3 \varphi_3(t), \tag{7.3}$$

where η_1, η_2, η_3 are the real constants and $\varphi_1(t) = [\varphi_1(t), \varphi_2(t), \varphi_3(t)]$ is a three dimensional Gaussian white noise process satisfying $E(\varphi_1(t)) = 0, i = 1,2,3$, $E[\varphi_i(t)\varphi_j(t)] = \delta_{ij}\delta(t-t^1)$, $i = j = 1,2,3$ where δ_{ij} is the Kronecker delta function, δ is the Dirac –delta function. In this analysis, we focus on the dynamics of the model (7.1)-(7.3) and we compute the population variances around E^*.

Let

$$n_1(t) = u_1(t) + S^*, n_2(t) = u_2(t) + P^*, n_3(t) = u_3(t) + T^*,$$

then

$$\frac{dn_1}{dt} = \frac{du_1}{dt}, \frac{dn_2}{dt} = \frac{du_2}{dt}, \frac{dn_3}{dt} = \frac{du_3}{dt}.$$

Utilizing the above and taking only the linear part, we have

$$\frac{du_1}{dt} = -u_1 s^* - u_2 (S^*)^2 + \eta_1 \varphi_1(t),$$

$$\frac{du_2}{dt} = u_2 (P^*)^2 + \eta_2 \varphi_2(t), \frac{du_3}{dt} = \eta_3 \varphi_3(t).$$

Now applying Fourier transform, we get

$$i\omega \tilde{u}_1(\omega) = -S^* \tilde{u}_1(\omega) - (S^*)^2 \tilde{u}_2(\omega) + \eta_1 \tilde{\varphi}_1(t),$$

$$i\omega \tilde{u}_2(\omega) = -(P^*)^2 \tilde{u}_2(\omega) + \eta_2 \tilde{\varphi}_2(t), i\omega \tilde{u}_3(\omega) = \eta_3 \tilde{\varphi}_3(t).$$

The matrix form can be written as

$$\widetilde{\varphi}(\omega) = A(\omega)\widetilde{u}(\omega) \tag{7.4}$$

where

$$A(\omega) = \begin{bmatrix} (i\omega + S^*) & (S^*)^2 & 0 \\ 0 & i\omega + (P^*)^2 & 0 \\ 0 & 0 & i\omega \end{bmatrix}, \widetilde{u}(\omega) = \begin{bmatrix} \widetilde{u_1}(\omega) \\ \widetilde{u_2}(\omega) \\ \widetilde{u_3}(\omega) \end{bmatrix}, \widetilde{\varphi}(\omega) = \begin{bmatrix} \eta_1\widetilde{\varphi_1}(\omega) \\ \eta_2\widetilde{\varphi_2}(\omega) \\ \eta_3\widetilde{\varphi_3}(\omega) \end{bmatrix}$$

(7.4) can also be expressed as $\widetilde{u}(\omega) = [A(\omega)]^{-1}\widetilde{\varphi}(\omega) = B(\omega)\widetilde{\varphi}(\omega)$, and $B(\omega) = [A(\omega)]^{-1}$. The solution of (7.4) is

$$\widetilde{u}(\omega) = \sum_{j=1}^{3} B_{ij}(\omega)\eta_j\widetilde{\varphi_j}(\omega), i = 1, 2, 3.$$

The strengths of oscillations of u_i, i= 1,2,3 are

$$\sigma_{u_i}^2 = \frac{1}{2\pi}\sum_{j=1}^{3}\int_{-\infty}^{\infty} \eta_j \left|B_{ij}(\omega)\right|^2 d\omega, i = 1, 2, 3,$$

where

$$B_{ij} = \frac{G_{ij}(\omega)}{\det A\,(\omega)}, i = 1, 2, 3.$$

Using (7.4), The populace variances of (7.1)-(7.3) are as follows,

$$\sigma_{u_i}^2 = \frac{\eta_1}{2\pi}\int_{-\infty}^{\infty} \frac{|G_{ij}(\omega)|^2}{|\det A(\omega)|^2}\, d\omega + \frac{\eta_2}{2\pi}\int_{-\infty}^{\infty} \frac{|G_{ij}(\omega)|^2}{|\det A(\omega)|^2} d\omega + \frac{\eta_3}{2\pi}\int_{-\infty}^{\infty} \frac{|G_{ij}(\omega)|^2}{|\det A(\omega)|^2} d\omega,$$

where

$$G_{mn} = X_{mn} + iY_{mn}, m, n = 1, 2, 3, X_{11} = -\omega^2, Y_{11} = \omega P^*, X_{12} = 0, Y_{12} = \omega S^*$$

$$X_{13} = 0, Y_{13} = 0, X_{21} = 0, Y_{21} = 0, X_{22} = \omega^2, Y_{22} = \omega S^*, X_{23} = 0, Y_{23} = 0,$$

$$X_{31} = 0, Y_{31} = 0, X_{32} = 0, Y_{32} = 0, X_{33} = -\omega^2 + S^*P^{*^2}, Y_{33} = \omega(S^* + P^{*^2}).$$

Hence the population variances of prey, predator and top predator are as

$$\sigma_{u_1}^2 = \frac{\eta_1}{2\pi}\int_{-\infty}^{\infty} \frac{|G_{11}(\omega)|^2}{|\det A(\omega)|^2} d\omega + \frac{\eta_2}{2\pi}\int_{-\infty}^{\infty} \frac{|G_{12}(\omega)|^2}{|\det A(\omega)|^2} d\omega,$$

$$\sigma_{u_2}^2 = \frac{1}{2\pi}\left\{\eta_2\int_{-\infty}^{\infty} \frac{|G_{22}(\omega)|^2}{|\det A(\omega)|^2} d\omega\right\}, \sigma_{u_3}^2 = \frac{1}{2\pi}\left\{\eta_3\int_{-\infty}^{\infty} \frac{|G_{33}(\omega)|^2}{|\det A(\omega)|^2} d\omega\right\}.$$

Now, we come across, the following cases.

Case (i). If $\eta_1 = \eta_2 = 0$, then

$$\sigma_{u_1}^2 = 0, \sigma_{u_2}^2 = 0, \sigma_{u_3}^2 = \frac{\eta_3}{2\pi}\int_{-\infty}^{\infty} \frac{((-\omega^2 + S^* P^{*^2})^2 + (\omega(S^* + P^*)^2))}{R^2(\omega) + I^2(w)} d\omega$$

Case(ii). If $\eta_2 = \eta_3 = 0$, then

$$\sigma_{u_1}^2 = \frac{\eta_1}{2\pi}\int_{-\infty}^{\infty} \frac{(\omega^4 + \omega^2 P^{*^2})}{R^2(\omega) + I^2(w)} d\omega, \sigma_{u_2}^2 = 0, \sigma_{u_3}^2 = 0$$

Case (iii). If $\eta_1 = \eta_3 = 0$, then

$$\sigma_{u_1}^2 = \frac{\eta_1}{2\pi}\int_{-\infty}^{\infty} \frac{(\omega^2 S^{*^2})}{R^2(\omega) + I^2(w)} d\omega, \sigma_{u_2}^2 = \frac{\eta_2}{2\pi}\int_{-\infty}^{\infty} \frac{(\omega^4 + \omega^2 S^{*^2})}{R^2(\omega) + I^2(w)} d\omega, \sigma_{u_3}^2 = 0.$$

Further, for stochastic system, populace variances play an affective role in stability analysis of the system. In the presence of environmental variable, the parameters of the system oscillate in populace densities around the inner state. Hence we conclude that inclusion of stochastic perturbation creates a significant change in the entire changing aspects and in the proposed model system due to change of responsive parameters can create more effective large environmental oscillations.

8. Diffusion analysis

The current article deals with a class of extended tri trophic prey predator systems in environmental science, modelled by diffusion equations. Although the dispersal system is a relatively simple model for the raid of prey species by predators in a spatial domain, the solutions exhibit an extensive spectrum of ecologically pertinent behaviour. Spatiotemporal dynamics includes chaos, target patterns [19,20,21]. The study of such spatiotemporal dynamics is an intensive area of research and there are still many unanswered questions concerning these solution types [21,22,23].By constructing a structure consists of prey ,predator and top predator system with constant harvesting rates .The populace of the system are prey, predator and top predator. The populations are subject to dispersal .The spread of the population is observed by the pattern. These are two kinds of spread (i) The propagation of continuous travelling population fronts of high species density. (ii) The formation & movement of paths of high density separated by areas with density close to zero. The actual dynamics of the species spread is a result of the inter play between diffusion and deterministic factors. We shall study the effect of diffusion of ecological population on the model system. Let us consider the diffusive equation system as

$$n_1'(t) = n_1(1-n_1) - \frac{l_1 n_1^2 n_2}{n_1^2+n_2^2} - \frac{q_1 E_1 n_1}{r_1} + D_1 \frac{\partial^2 n_1}{\partial u^2}, \tag{8.1}$$

$$n_2'(t) = \frac{p_1 n_1^2 n_2}{n_1^2+n_2^2} - q_5 n_2 - \frac{l_2 n_2^2 n_3}{n_1^2+n_2^2} - \frac{q_2 E_2}{r_1} n_2 + D_2 \frac{\partial^2 n_2}{\partial u^2}, \tag{8.2}$$

$$n_3'(t) = \frac{p_2 n_1^2 n_3}{n_3^2+n_2^2} - q_4 n_3 - \frac{q_3 E_3}{r_1} n_3 + D_3 \frac{\partial^2 n_3}{\partial u^2}. \tag{8.3}$$

In this D_1, D_2,D_3 represents the constant diffusion coefficients of the prey, predator and super predator. The model system (8.1)-(8.3) are inhomogeneous as the reaction diffusion system. For such introduction of the diffusion term of the populations, it has become a spatiotemporal dynamical system. We consider the following conditions of the population $n_1(u,t)$, $n_2(u,t)$ and $n_3(u,t)$ in $0 \le u \le L, L > 0$ as follows

$$\frac{\partial n_1(0,t)}{\partial t} = \frac{\partial n_1(L,t)}{\partial t} = \frac{\partial n_2(0,t)}{\partial t} = 0, \frac{\partial n_2(L,t)}{\partial t} = \frac{\partial n_3(0,t)}{\partial t} = \frac{\partial n_3(L,t)}{\partial t} = 0.$$

The zero isoclines of model equations (8.1)-(8.3) also give the steady state which are same as we have obtained for homogeneous system. Now we linearize the system (8.1)-(8.3) by putting $n_1 = n_1^* + N_{11}$,$n_2 = n_2^* + N_{22}$,$n_3 = n_3^* + N_{33}$, in view of inner steady state and we get

$$N_{11}'(t) = -l_1 r_1 N_1 N_2 n_1^* + D_1 \frac{\partial^2 n_1}{\partial u^2}, \tag{8.4}$$

$$N_{22}'(t) = D_2 \frac{\partial^2 n_2}{\partial u^2}, \tag{8.5}$$

$$N_3'(t) = D_3 \frac{\partial^2 n_3}{\partial u^2}. \tag{8.6}$$

The solution of (8.4)-(8.6) can be expressed as

$$\begin{aligned} &n_1(u,t) = \alpha_1 e^{\lambda t} e^{iku}, \\ &n_2(u,t) = \alpha_2 e^{\lambda t} e^{iku}, n_3(u,t) = \alpha_3 e^{\lambda t} e^{iku}. \end{aligned}$$

Then the model becomes

$$N_{11}'(t) = -l_1 r_1 N_1 N_2 n_1^* + D_1(-k^2 N_{11}), \tag{8.7}$$

$$N_{22}'(t) = D_2(-k^2 N_{22}), \tag{8.8}$$

$$N_{33}'(t) = D_3(-k^2 N_{33}). \tag{8.9}$$

The characteristic equation of (8.7)-(8.9) is

$$\lambda^3 + A\lambda^2 + B\lambda + C = 0 \tag{8.10}$$

where $A = k^2(D_1+D_2+D_3)+l_1r_1n_1^*N_{22}$, $C = l_1r_1n_1^*N_{22}D_1D_3k^4+D_1D_2D_3k^6$, $B = l_1r_1N_{22}n_1^*k^2(D_3+D_2)+k^4(D_1D_2+D_2D_3+D_1D_3)$.

By applying Routh-Hurwitz criterion, to satisfy and make it possible if and only if $A > 0$, $C > 0$, $D = C - AB < 0$ (which is definitely possible).

Theorem. The system in the absence of spatiotemporal attributes at the inner steady state (n_1^*, n_2^*, n_3^*)attains steadiness, then the corresponding uniform steady state of the model (8.1)-(8.3) in the presence of spatiotemporal attributes also attains steadiness.

Proof. Consider a function $V_1(t)$ as $V_1(t) = \int_0^R V(n_1,n_2,n_3)du$

$$V(n_1,n_2,n_3) = \left[(n_1-n_1^*) - n_1^*\ln(\frac{n_1}{n_1^*})\right] + l_1\left[(n_2-n_2^*) - n_2^*\ln(\frac{n_2}{n_2^*})\right] + l_2\left[(n_3-n_3^*) - n_3^*\ln(\frac{n_3}{n_3^*})\right],$$

$$V_1^1(t) = \int_0^R \left(\frac{\partial v}{\partial n_1}.\frac{\partial n_1}{\partial t} + \frac{\partial v}{\partial n_2}.\frac{\partial n_2}{\partial t} + \frac{\partial v}{\partial n_3}.\frac{\partial n_3}{\partial t}\right) du = I_1 + I_2$$

where $I_1 = \int_0^R \frac{dv}{dt} du$ and $I_2 = \int_0^R \left(D_1\frac{\partial v}{\partial n_1}\frac{\partial^2 n_1}{\partial u^2} + D_2\frac{\partial v}{\partial n_2}\frac{\partial^2 n_1}{\partial u^2} + D_3\frac{\partial v}{\partial n_3}\frac{\partial^2 n_3}{\partial u^2}\right) du$

$$I_2 {=} -D_1\int_0^R \frac{\partial^2 v}{\partial n_1}\left(\frac{\partial n_1}{\partial u}\right)^2 du - D_2\int_0^R \frac{\partial^2 v}{\partial n_2}\left(\frac{\partial n_2}{\partial u}\right)^2 du - D_3\int_0^R \frac{\partial^2 v}{\partial n_3}\left(\frac{\partial n_3}{\partial u}\right)^2 du$$

$$I_2 {=} -D_1\int_0^R \frac{n_1^*}{n_1}\left(\frac{\partial n_1}{\partial u}\right)^2 du - D_2\int_0^R \frac{n_2^*}{n_2}\left(\frac{\partial n_2}{\partial u}\right)^2 du - D_3\int_0^R \frac{n_3^*}{n_3}\left(\frac{\partial n_3}{\partial u}\right)^2 du$$

It is observed that, if $I_1 < 0$ then $\frac{dV_1}{dt} < 0$. Hence the theorem holds.

9. Numerical simulations

In this section we have validated the analytical findings through numerical data using Technical tool MATLAB

Example 1. For the parameters $l_1 = 8$, $l_2 = 5.3$, $p_1 = 3.1$, $p_2 = 2$, $q_5 = 0.3$, $q_4 = 0.6$, $E_1 = 10$, $E_2 = 15$, $E_3 = 5$, $r_1 = 10$, $q_1 = 0.15$, $q_2 = 0.01$, $q_3 = 0.01$

Example 2. For $l_1 = 8$, $l_2 = 11$, $p_1 = 10$, $p_2 = 2$, $q_5 = 1.5$, $q_4 = 1.5$, $E_1 = 2$, $E_2 = 3$, $E_3 = 4$, $r_1 = 4$, $q_1 = 0.15$, $q_2 = 0.01$, $q_3 = 0.01$

Example 3. For $l_1 = 2$, $l_2 = 11$, $p_1 = 10$, $p_2 = 2$, $q_5 = 1.5$, $q_4 = 1.5$, $E_1 = 2$, $E_2 = 3$, $E_3 = 4$, $r1 = 4$, $q1 = 0.15$, $q2 = 0.01$, $q_3 = 0.01$

Example 4. For $l_1 = 2, l_2 = 5.3, p_1 = 3.1, p_2 = 2, q_5 = 0.3$,$q_4 = 0.6$,$E_1 = 10$,$E_2 = 15, E_3 = 5$, $r_1 = 10$,$q_1 = 0.15$,$q_2 = 0.01, q_3 = 0.01$,$D_1 = 20$, $D_2 = 30$, $D_3 = 40$

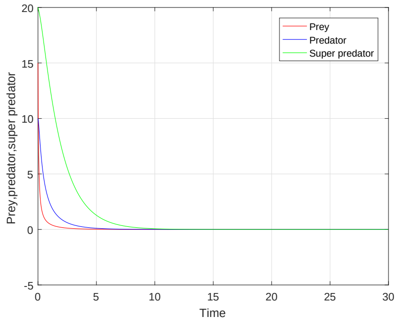

Figure 1.a

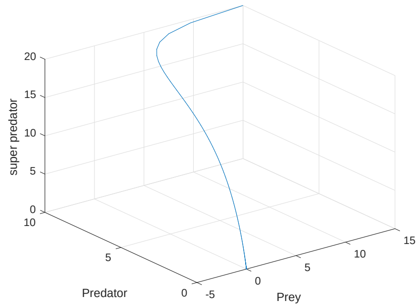

Figure 1.b

Figure 1: a:shows the differences in populace against time for the set of values of example 1.
Figure 1.b: shows graphically the differences in the population among N_1, N_2& N_3 predator values for the values of example 1

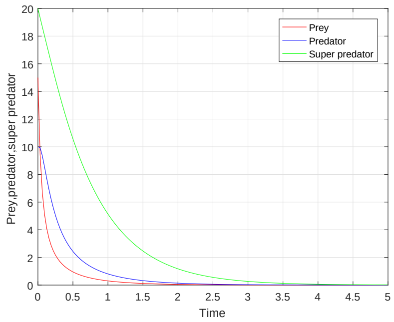

Figure 2.a

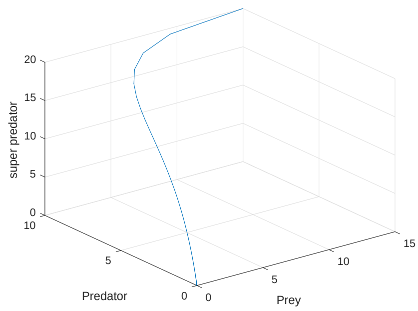

Figure 2.b

Figure 2: a:shows the differences in populace against time for the set of values of example 2.
Figure 2.b: shows graphically the differences in the population among N_1, N_2& N_3 predator values for the values of example 2

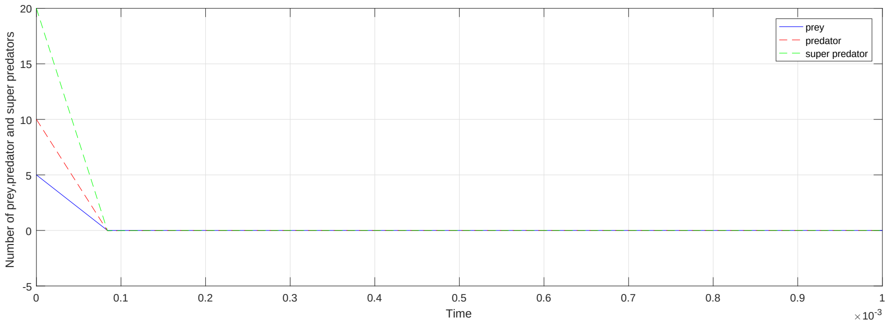

Figure 3

Figure 3: shows populace shades against time for the set of values of example 3.

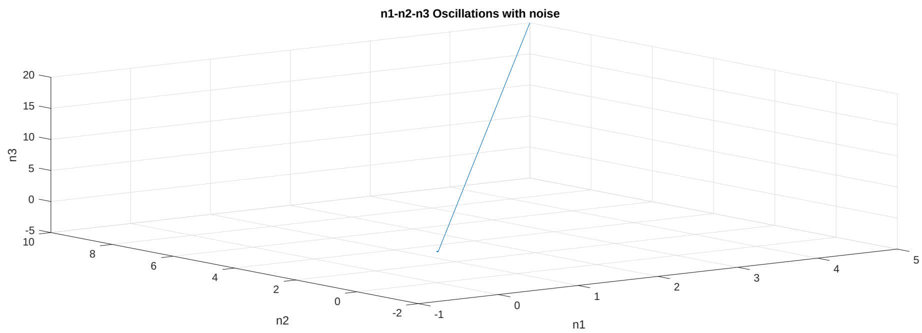

Figure 4

Figure 4: shows graphically how the populace shades differ among N_1, N_2, & N_3 for the set of values of example 3.

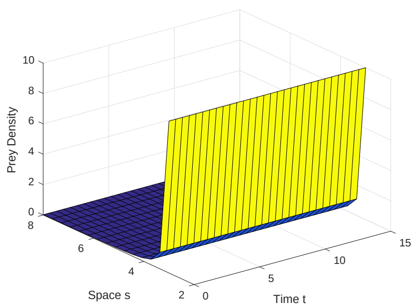

Figure 5.a

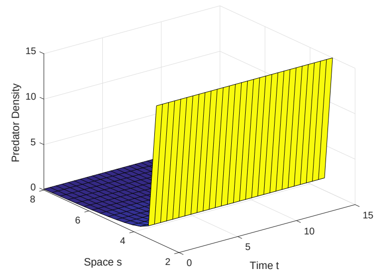

Figure 5.b

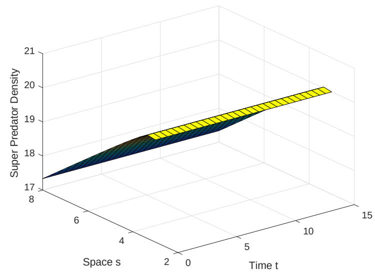

Figure 5.c

Figure 5: a , 5.b , 5.c represents the stable oscillations of $n_1(u,t)$, $n_2(u,t)$, and $n_3(u,t)$, respectively against time and space for the attributes of example 4.

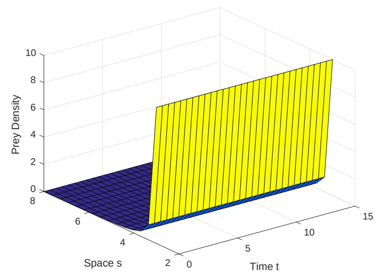

Figure 6.a

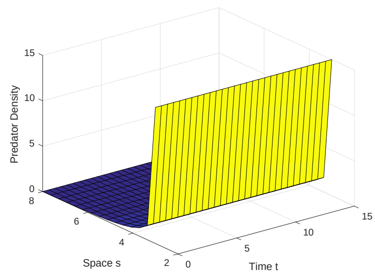

Figure 6.b

Figure 6: a , 6.b , 6.c represents the stable oscillations of $n_1(u,t)$, $n_2(u,t)$, and $n_3(u,t)$, respectively against time and space for the attributes of example 4.

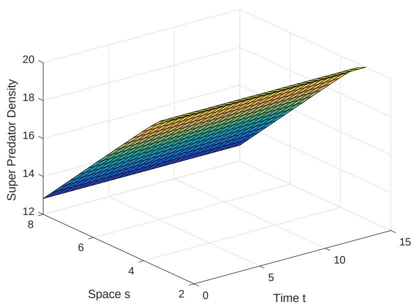

Figure 6.c

10. Concluding remarks

This article mainly concentrates and aims at the most interesting changing aspects of a prey, predator and top predator food chain model with functional response III. We obtain the possible equilibrium points and analysed. Bioeconomic and feasible harvesting strategies have been computed using maximum principle. It is shown that the dynamics of deterministic system in the figures (1.a), (1.b), (2.a) and(2.b). It is also studied about the stochasticity of the given system and observed the effect of environmental flux around the positive steady state in the presence of an environmental attribute. The population variances computed and stability concept is also analysed. Graphically Figures (3),(4) shows the impact of noise with suitable set of values. We also verified the steadiness of the spatiotemporal model of the system (8.1)-(8.3) graphically. Figures (5.a), (5.b), (5.c), (6.a), (6.b), (6.c) shows the spatiotemporal steadiness.

References

[1] I. Chen, *Mathematical ecology modeling and research methods Beijing*, Science Press, 1988.

[2] J.D. Murray, *Mathematical Biology,* Springer-Verlag, 1989.

[3] B.S. Goh, *Global stability in two species interaction*, J. Math. Biol., 3 (1976), 313.

[4] Y. Kuang, *Delay differential equations with applications in population dynamics*, New York Academic Press, 1993.

[5] Y. Takeuchi, *Global dynamics properties of Lokta-Volterra systems*, Singapore: World scientific, 1996.

[6] S. Liu, I. Chen, R. Agarwal, *Recent progress on stage-structured population dynamics*, Math Comput. Model, 2002 (36), 11-13, 1319-60.

[7] S.B. Hsu, T.W. Hwang and Y. Kuang, *A ratio dependent food chain model and its applications to biological control*, Mathematical Biosciences, 181 (2003), 55-83.

[8] H.I. Freedman and P. Waltman, *Mathematical analysis of some three-species food-chain models,* Math. Biosci., 33 (1977), 257-276.

[9] S.M. Moghadas and A.B. Gummel, *Dynamical and numerical analysis of a generalized food chain model*, Applied Math. and Comp., 142(I), 2003, 35-49.

[10] J.X. Loon, G. De Boer and M. Dicke, *Parasitoid-plant mutualism parasitoid attack of herbivore increases plant reproduction*, Entomol. Exp. Appl., 97 (2000), 219-227.

[11] F. Brauer, A.C. Soudack, *Stability regions in predator-prey systems with constant rate prey harvesting*, J. Math. Biol., 8 (1979), 319-337.

[12] C.W. Clark, *Mathematical Bio Economics,* The Optimal Management of Renewable Resources, seconded. Wiley, New York, 1990.

[13] D. Xiao, S. Ruan, *Bogadanov-Takens bifurcations in predator-prey systems with constant rate harvesting,* Fields Inst. Common., 21 (1999), 493-506.

[14] C. Jost and S.P. Ellner, *Testing for predator dependence in predator-prey dynamics:a non-parameteric approach*, Proc. R Soc. Land B., 267 (2000), 1611-1620.

[15] Manju Aagarwal and Vimesh Singh, *Rich dynamics of food chain model with ratio-dependent type-III function responses*, International Journal of Engineering, Science and Technology, 5 (2013), 106-123.

[16] Dongonei Xiao, Wenxia Li and Maoan Han, *Dynamics in a ratio-dependent predator-prey model with predator harvesting*, J.Math. Anal. Appl., 324 (2000), 14-29.

[17] R.M. Nisbet, W.S.C. Gurney, *Modelling fluctuating populations*, John wiley, Newyork, 1982.

[18] M. Carletti, *Numerical solution of stochastic differential problems in bio sciences*, J. Compute. Appl. Math., 185 (2006), 422-440.

[19] N. Kopell and L.N. Howard, *Plane wave solutions to reaction-diffusion equations,* Stud. Appl. Math., 42 (1973), 291-328.

[20] N. Kopell and L.N. Howard, *Target patterns and spiral solutions to reaction-diffusion equations with more than one space dimension*, Adv. Appl. Math., 2 (1981), 417-449.

[21] A.B. Medvinsky, S.V. Petrovskii, I.A. Tikhonovi, H. Malchow, and B.-L Li, *Spatio temporal complexity of plankton and fish dynamics*, SIAM Rev., 44 (2002), 311-370.

[22] J.D. Murray, *Mathematical Biology*, Biomathematics Texts, vol. 19, Springer, Berlin, 1993.

[23] J.A. Sherratt, B.T. Eagan, and M.A. Lewis, *Oscillations and chaos behind predator-prey invasion: Mathematical artefact or ecological reality?*, Phil, Trans. R. Soc. Lond. B 352, 1997, 21-38.

Accepted: 1.03.2017