GENERALIZATIONS OF PRIME TERNARY SUBSEMIMODULES OF TERNARY SEMIMODULES

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Abstract. Let R be a commutative semiring with $1 \neq 0$ and all semimodules are unital. Weakly prime ternary subsemimodules of ternary semimodules have been studied. In this paper we introduce the concept of almost prime ternary subsemimodule of a ternary semimodule over a ternary semiring as a new generalization of prime ternary subsemimodule. We will give some of its properties, characteristics and its relationship among other algebraic structures. Also we carry out this concept under multiplication ternary semimodules.
Keywords: almost prime submodule; subtractive ternary subsemimodule, partitioning ternary subsemimodule, weakly prime subsemimodule, quotient ternary semimodule.

1. Introduction

Through out this paper all semrings are commutative with $1 \neq 0$ and all semimodules are unitary. Weakly prime ideals have been introduced by [1], and almost prime ideals were introduced by [2] and studied by [3]. Later on, these concepts has been studied in modules and semirings by many authors [4, 5, 6, 7]. Further they are extended for semimodules by [8, 9]. The concept of weakly prime ternary subsemimodule of a ternary semimodule over a ternary semiring has introduced by [10]. In this paper we introduce The concept of weakly prime ternary subsemimodule of a ternary semimodule over a ternary semiring and obtain some properties and characteristics of almost prime ternary subsemimodules. For definitions of monoid and semiring see [11, 12] and for ternary semiring see [13, 14]. Throughout, all ternary semirings are commutative with

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$1 \neq 0$. Denote that $Z_0^+(N)$ to be the set of all positive integers where as $Z_0^-(Z^-)$ be the set of negative integers. An ideal I of a ternary semiring R is called a subtractive ideal (=k-ideal) if $a, a+b \in I$ and $b \in R$, then $b \in I$. A proper ideal P of a ternary semiring R is said to be prime if $abc \in P$, then either $a \in P$ or $b \in P$, or $c \in P$. A proper ideal P of a ternary semiring R is said to be weakly prime if $0 \neq abc \in P$, then either $a \in P$ or $b \in P$, or $c \in P$. A proper ideal P of a ternary semiring R is said to be almost prime if $abc \in P - P^2$, then either $a \in P$ or $b \in P$, or $c \in P$. Let R be a ternary semiring. A left ternary R-semimodule is a commutative monoid $(M, +)$ with additive identity 0_M where the function $R \times R \times M \to M$, defined by $(r_1, r_2, x) \mapsto r_1 r_2 x$ called ternary scalar multiplication, which satisfies the following conditions for all elements r_1, r_2, r_3 and r_4 of R and all elements x and y of M:

(1) $(r_1 r_2 r_3) r_4 x = r_1 (r_2 r_3 r_4) x = r_1 r_2 (r_3 r_4 x)$;

(2) $r_1 r_2 (x + y) = r_1 r_2 x + r_1 r_2 y$;

(3) $r_1 (r_2 + r_3) x = r_1 r_2 x + r_1 r_3 x$;

(4) $(r_1 + r_2) r_3 x = r_1 r_3 x + r_2 r_3 x$;

(5) $1_R 1_R x = x$;

(6) $r_1 r_2 0_M = 0_M = 0_R r_2 x = r_1 0_R x$;

Throughout, by a ternary R-semimodule we mean a left ternary semimodule over a ternary semiring R. By [15], every ternary semiring R is ternary $(Z_0^-, +, \cdot)$-semimodule. A ternary subsemimodule N of a ternary R-semimodule M is called subtractive ternary subsemimodule (=ternary k-subsemimodule) if $x, x+y \in N$, $y \in M$, then $y \in N$.

Since $0 = 0$ is a subtractive ternary subsemimodule of a ternary R-semimodule, then $(0 : m)$ and $(0 : M)$ are subtractive ideals of R where $m \in M$.

Following [16, Theorem 3.4], for any two subtractive ideals I and J of a ternary semiring R, their union is subtractive ideal of R if and only if their union equals one of the subtractive ideals. Following [10], A proper ternary semimodule N of a ternary R-semimodule M is said to be prime if $r_1 r_2 m \in N$, where, $r_1 r_2 \in R$, $m \in M$, then either $r_1 \in (N : M)$ or $r_2 \in (N : M)$, or $m \in N$. Also, a proper ternary subsemimodule N of a ternary R-semimodule M is said to be weakly prime if $0 \neq r_1 r_2 m \in N$, where $r_1, r_2 \in R$ and $m \in M$, then either $r_1 \in (N : M)$ or $r_2 \in (N : M)$, or $m \in N$.

2. Almost prime ternary subsemimodules

In this section we introduce the concept of almost prime ternary subsemimodule of a ternary semimodule over a ternary semiring and obtain some of its properties and characterizations.

Definition 2.1. *A proper ternary subsemimodule N of a ternary $R-$semimodule M is said to be almost prime ternary if, whenever $r_1, r_2 \in R$ and $m \in M$ such that $r_1 r_2 m \in N - (N : M)^2 N$, then either $r_1 \in (N : M)$ or $r_2 \in (N : M)$ or $m \in N$.*

Clearly, any weakly prime ternary subsemimodule of ternary $R-$semimodule is almost prime ternary subsemimodule. However, the convers is not necessarily ture, for example we consider $\mathbb{Z}_0^- -$semimodule $M = \mathbb{Z}_{-24}$ and the proper ternary cyclic subsemimodule N of M generated by $-\bar{8}$, clearly $(N : M)^2 N = N$ and so N is almost prime ternary subsemimodule.

In contrast $\bar{0} \neq (-4)(-4)(-\bar{4}) \in N$ but $-\bar{4} \notin N$ and $-4 \notin (N : M)$ which is not weakly.

Recall that, a ternary R-semimodule M is called a cancellation ternary R-semimodule if for all ideals I and J of R, $IRM = JRM$ implies that $I = J$. Following [17], let M be a ternary $R-$semimodule and N be a ternary subsemimodule of M, then N is called idempotent in M if $(N : M)^2 N = N$. Thus, any proper idempotent ternary subsemimodule of M is almost prime ternary subsemimodule of M. If M is a multiplication ternary $R-$semimodule and N_1,N_2,N_3 are ternary subsemimodules of M such that $N_1 = IRM$,$N_2 = JRM$,$N_3 = KRM$ for some ideals I,J,K of R , then the ternary multiplication of N_1,N_2 and N_3 is defined in [18], $N_1 N_2 N_3 = (IRM)(JRM)(KRM) = (IJK)RM$. In particular, we have

$$N^3 = NNN = [(N : M)RM]\,[(N : M)RM]\,[(N : M)RM] = (N : M)^3 RM.$$

So, a submodule N is idempotent in M if and only if $N = N^3$.

Recall that, a ternary subsemimodule N of a ternary R-semimodule is called a pure ternary subsemimodule if $I^2 N = N \cap IRM$ for any ideal of R. Following [19], we can prove that if N is a pure ternary subsemimodule in a multiplication R-semimodule M with pure ternary annihilator, then N is idempotent in M almost prime ternary.

Next theorem present the relationship between weakly prime ternary subsemimodule and almost prime ternary subsemimodule.

Theorem 2.2. *Let M be a ternary $R-$semimodule and N be a proper subsemimodule of M. Then N is almost prime in M if and only if $N/(N : M)^2 N$ is weakly prime in $M/(N : M)^2 N$.*

Proof. Suppose that N is almost prime ternary subsemimodule in M. Let r_1,$r_2 \in R$ and $m \in M$, such that $\bar{0} \neq r_1 r_2 (m + (N : M)^2 N) \in N/(N : M)^2 N$ in $M/(N : M)^2 N$. Then $r_1 r_2 m \in N - (N : M)^2 N$ and so either $r_1 \in (N : M)$ or $r_2 \in (N : M)$ or $m \in N$ because N is almost prime. Hence, either

$$r_1 \in (N : M) = (N/(N : M)^2 N : M/(N : M)^2 N)$$

or

$$r_2 \in (N : M) = (N/(N : M)^2 N : M/(N : M)^2 N)$$

or $m + (N : M)^2N \in N/(N : M)^2N$ and so $N/(N : M)^2N$ is weakly prime in $M/(N : M)^2N$. Conversely, assume that $N/(N : M)^2N$ is weakly prime ternary in $M/(N : M)^2N$ and let r_1,$r_2 \in R$ and $m \in M$ such that $r_1r_2m \in N - (N : M)^2N$. Then

$$\bar{0} \neq r_1r_2(m + (N : M)^2N) \in N/(N : M)^2N$$

and hence either

$$r_1 \in (N/(N : M)RN : M/(N : M)RN) = (N : M)$$

or

$$r_2 \in (N/(N : M)^2N : M/(N : M)^2N) = (N : M)$$

or $m + (N : M)^2N \in N/(N : M)^2N$ (and so $m \in N$) so N is almost prime ternary. □

Note that, we can generalize Definition 2.1 as follows.

Definition 2.3. *A proper ternary subsemimodule N of a ternary $R-$semimodule M is said to be n-almost prime ternary if, whenever $r_1, r_2 \in R$ and $m \in M$ such that $r_1r_2m \in N - (N : M)^{n-1}N$, then either $r_1 \in (N : M)$ or $r_2 \in (N : M)$ or $m \in N$ ($n \geq 3$).*

Theorem 2.4. *Let M be a ternary $R-$semimodule and N be a proper ternary subsemimodule of M. Then for $n \geq 3$ the following statements hold.*

(1) *For $r \in R - (N : M)$, $(N : (r)) = N \cup ((N : M)^{n-1}N : (r))$.*

(2) *For $r \in R-(N : M)$, $(N : (r)) = N$ or $(N : (r)) = ((N : M)^{n-1}N : (r))$.*

Proof. (1) Suppose that N is an almost prime ternary subsemimodule such that r,$r' \notin (N : M)$ for all $r' \in R$. Let $m \in (N : \langle r\rangle)$ so that $rr'm \in N$. If $rr'm \notin (N : M)^{n-1}N$, then N being almost prime implies that $m \in N$. Suppose that $rr'm \in (N : M)^{n-1}N$. Then $m \in ((N : M)^{n-1}N : (r))$ and so $(N : (r)) \subseteq N \cup ((N : M)^{n-1}N : (r))$. The other containment holds for any subsemimodule N.

(2) It is well known that if a ternary subsemimodule is the union of two ternary subsemimodule, then it is equal one of them. □

Theorem 2.5. *Let N be a proper ternary subsemimodule of a ternary R-semimodule M. Then for $n \geq 3$ the following are equivalent.*

1) *N is n-almost prime ternary subsemimodule of M.*

2) *For any ideal A and B of R and ternary subsemimodule K of M with $ABK \subseteq N - (N : M)^{n-1}N$, we have $A \subseteq (N : M)$ or $B \subseteq (N : M)$ or $K \subseteq N$.*

Proof. (1)⇒(2) Suppose that N is an n-almost prime ternary subsemimodule of M. Let $ABK \subseteq N - (N : M)^{n-1}N$, where A, B are ideals of R and K is a ternary subsemimodule of M, $B \nsubseteq (N : M)$ and $K \nsubseteq N$. Choose $r_2 \in B$ and $x \in K$ such that $r_2 \notin (N : M)$ and $x \notin N$. Let $r_1 \in A$. Then $r_1r_2x \in ABK \subseteq$

$N-(N:M)^{n-1}N$. Since N is n-almost prime ternary subsemimodule, then either $r_1 \in (N:M)$ or $r_2 \in (N:M)$ or $x \in K$, and so $r_1 \in (N:M)$. Hence $I \subseteq (N:M)$.

(2)⇒(1) Let r_1, $r_2 \in R$ and $m \in M$ such that $r_1r_2m \in N-(N:M)^{n-1}N$. Suppose that $A=(r_1)=RRr_1$, $B=(r_2)=RRr_2$ and $K=(m)=RRm$. Then $ABK \subseteq N-(N:M)^{n-1}N$. So either $A \subseteq (N:M)$ or $B \subseteq (N:M)$ or $K \subseteq N$ and hence $r_1 \in (N:M)$ or $r_2 \in (N:M)$ or $m \in N$. Thus N is an n-almost prime ternary subsemimodule of M. □

Definition 2.6. *Let M be a ternary R-semimodule, $S(M)$ be the set of all submodules of M, and $\phi: S(M) \to S(M) \cup \{\emptyset\}$ be a function. Then a proper subsemimodule N of M is called ϕ-prime subsemimodule if for $r_1, r_2 \in R$ and $m \in M$ with $r_1r_2m \in N-\phi(N)$, then either $r_1 \in (N:M)$ or $r_2 \in (N:M)$ or $m \in N$.*

For example, define $\phi_n : S(M) \to S(M) \cup \{\emptyset\}$ with $\phi_n(N)=(N:M)^{n-1}N$, for all $N \in S(M)$; ($n \geq 3$). Hence for $n \geq 3$, ϕ_n-prime submodule of M is an n-almost prime ternary subsemimodule and ϕ_2-almost prime ternary subsemimodule is an almost prime. Recall that, a ternary subsemimodule N of a ternary R-semimodule M is called subtractive ternary subsemimodule(=ternary k-subsemimodule) if $x, x+y \in N$ and $y \in M$, then $y \in N$.

Lemma 2.7 ([16]). *Let I and J be subtractive ideals of a ternary semiring R. Then $I \cup J$ is subtractive ideal of R if and only if $I \cup J = I$ or $I \cup J = J$.*

Theorem 2.8. *Let R be a ternary semiring and M be a ternary R-semimodule. Let N be a proper ternary subtractive subsemimodule of M. Then the following statements are equivalent:*

(1) *If $x \in M-N$, then $(N:x)=(N:M)\cup(\phi(N):x)$;*

(2) *If $x \in M-N$, then $(N:x)=(N:M)$ or $(N:x)=(\phi(N):x)$.*

Proof. (1) Let $x \in M-N$ and $r \in (N:x)$. Then $rr'x \in N$ for all $r' \in R$ and so $r1m \in N$. If $r1x \in \phi(N)$, then for all $a \in R$, we have $rax = arx = 1a(r1x) \in \phi(N)$ and so $r \in (\phi(N):x)$. Suppose that $r1x \notin \phi(N)$. Then either $r \in (N:M)$ or $1 \in (N:M)$ or $x \in N$, because N is ϕ-prime subtractive ternary subsemimodule. But $1 \in (N:M)$ is impossible and $x \notin N$. Thus $r \in (N:M)$. Hence, $(N:x)=(N:M)\cup(\phi(N):x)$.

(2) Let $(N:x)=(N:M)\cup(\phi(N):x)$ for $x \in M-N$. Then either $(N:x)=(N:M)$ or $(N:x)=(\phi(N):x)$ by Lemma 2.7. Therefore, the inclusion follows. □

Corollary 2.9. *Let R be a ternary semiring and M be a ternary R-semimodule. Let N be an almost prime subtractive ternary subsemimodule of M. Then the following statements hold.*

(1) *If $x \in M-N$, then $(N:x)=(N:M)\cup((N:M)^2N:x)$.*

(2) *If $x \in M-N$, then $(N:x)=(N:M)$ or $(N:x)=((N:M)^2N:x)$.*

Proof. The proof follows from previous Theorem by taking $\phi(N) = (N : M)^2N$. □

Recall that, a ternary R-semimodule M is called a multiplication ternary R-semimodule if for each ternary subsemimodule N of M, there exists an ideal I of R such that $N = IRM$ (or equivalently, $N = (N : M)RM$).

Also, if N is a ternary subsemimodule of an $R-$semimodule M, then the radical of N (denoted by $M-$rad N) is defined as the intersection of all prime ternary subsemimodules of M containing N. It is well known that if M is a multiplication ternary R-semimodule, then $M - radN = \sqrt{(N : M)}RM$, where $\sqrt{(N : M)}$ denotes the radical of ideal $(N : M)$ in R.

Theorem 2.10. *Let M be a ternary multiplication R-semimodule. If N is a ternary subsemimodule of M, then, $N \subseteq M$-rad$((N : M)^2N)$. Moreover, if N is a prime ternary subsemimodule of M, then, $N = M$-rad$((N : M)^2N)$*

Proof. As M is a ternary multiplication $R-$semimodule, then, $M - rad((N : M)^2N) = \sqrt{((N : M)^2N : M)}RM$.

As $(N : M)^3 \subseteq ((N : M)^2N : M)$, then $(N : M) \subseteq \sqrt{((N : M)^2N : M)}$ and so $N = (N : M)RM \subseteq \sqrt{((N : M)^2N : M)}RM = M - rad((N : M)^2N)$. Moreover, let N be a prime ternary in M. If $r \in \sqrt{((N : M)^2N : M)}$, then, $r^n \in ((N : M)^2N : M) \subseteq (N : M)$ for some integer n. As $(N : M)$ is prime in R, then $r \in (N : M)$ and so $\sqrt{((N : M)^2N : M)} \subseteq (N : M)$. Therefore, $\sqrt{((N : M)^2N : M)}RM \subseteq (N : M)RM = N$ and the required holds. □

Recall that, an R-semimodule M is called faithful if $Ann(M) = 0$ and is called a cancelation ternary R-semimodule if for all ideals I and J of R, $IRM = JRM$ implies that $I = J$.

Lemma 2.11. *let N be a ternary subsemimodule of finitely generated faithful multiplication (and so cancellation) $R-$semimodule M . Then, we have $(I^2N : M) = I^2(N : M)$ for every ideal I of R.*

Proof. As M is multiplication ternary $R-$semimodule,then, $I^2(N : M)RM = I^2N = (I^2N : M)RM$. The result follows because M is a cancellation semimodule. □

Theorem 2.12. *Let M be a finitely generated faithful multiplication $R-$semimodule and N be a proper subsemimodule of M. The following are equivalent*

(1) *N is almost prime ternary in M;*

(2) *$(N : M)$ is almost prime ternary ideal of R;*

(3) *$N = QRM$ for some almost prime ternary ideal Q of R.*

Proof. (1) ⇒ (2) Suppose that N is an almost prime ternary and let $a, b, c \in R$ such that $abc \in (N : M) - (N : M)^3$. Then $(abc)dM = a(bcd)M = ab(cdM) \subseteq N - (N : M)^2N$ for all $d \in R$. Indeed, if $(abc)dM \subseteq (N : M)^2N$ for all $d \in R$, then by Lemma 2.11, $abc \in ((N : M)^2N : M) = (N : M)^3$, a contradiction.

Now, N is almost prime ternary which implies that $a \in (N : M)$ or $b \in (N : M)$ or $cdM \subseteq N$ for all $d \in R$ (and so $c \in (N : M)$). Hence, $(N : M)$ is almost prime ternary in R.

(2) $\Rightarrow$ (1) In this direction, we need M to be just a multiplication ternary R-subsemimodule. Let $r_1, r_2 \in R$ and $m \in M$, such that $r_1r_2m \in N-(N : M)^2M$. Then $r_1r_2((m) : M) \subseteq ((r_1r_2m) : M)) \subseteq (N : M)$. Moreover, $r_1r_2((m) : M) \nsubseteq (N : M)^3$ because otherwise, if $r_1r_2((m) : M) \subseteq (N : M)^3 \subseteq ((N : M)^2 : M)$, then $r_1r_2(m) = r_1r_2((m) : M)RM \subseteq (N : M)^2N$, a contradiction. As $(N : M)$ is almost prime ternary in R, then, either $r_1 \in (N : M)$ or $r_2 \in (N : M)$ or $((m) : M) \subseteq (N : M)$. In third case, we obtain $(m) = ((m) : M)RM \subseteq (N : M)RM = N$ and so N is almost prime ternary in M.

(2) $\iff$ (3) We choose $Q = (N : M)$. □

Lemma 2.13. *Let R be a semiring, M a faithful multiplication ternary R-semimodule, N a proper ternary subsemimodule of M and I a finitely generated faithful multiplication ternary ideal of R. then the following statements are equivalent.*

(1) *N is weakly prime ternary subsemimodule;*

(2) *$(N : M)$ is a weakly ternary prime ideal of R;*

(3) *$N = QRM$ for some weakly prime ternary ideal Q of R.*

Proof. (1)$\Rightarrow$(2) Suppose that N is a weakly prime ternary subsemimodule of M. Let $a, b, c \in R$ such that $0 \neq abc \in (N : M)$. Then $(abc)RM = ab(cRM) \subseteq N$. Since M is faithful, then $abcRM \neq 0$ and since N is a weakly prime ternary, then $a \in (N : M)$ or $b \in (N : M)$ or $cRM \in$ (and hence $c \in (N : M)$).

(2)$\Rightarrow$(1) Let $(N : M)$ be a weakly prime ideal ternary of R. If $0 \neq r_1r_2m \in N$, where $r_1, r_2 \in R$, $m \in M$, then $r_1r_2(RRm : M) \subseteq (RRr_1r_2m : M) \subseteq (N : M)$. Since M is a multiplication, then $r_1r_2(RRm : M) \neq 0$. As $(N : M)$ is weakly prime ternary, then, either $r_1 \in (N : M)$ or $r_2 \in (N : M)$ or $(RRm : M) \subseteq (N : M)$. In third case, we obtain $RRm = (RRm : M)RM \subseteq (N : M)RM = N$ and so N is weakly prime ternary in M.

(1)$\Rightarrow$ (3). Take $Q = (N : M)$. □

In the following two theorems, we give a new characterization of weakly prime (resp.; almost prime) subsemimodules of finitely generated faithful multiplication ternary R-semimodules.

Theorem 2.14. *Let M be a finitely generated faithful multiplication ternary R-semimodule and P be a proper ternary subsemimodule of M. Then P is weakly prime ternary in M if and only if whenever N, K and L are ternary subsemimodules of M such that $0 \neq NKL \subseteq P$, then, $N \subseteq P$ or $K \subseteq P$ or $L \subseteq P$.*

Proof. Suppose that P is weakly prime ternary. We have $N = (N : M)RM$, $K = (K : M)RM$ and $L = (L : M)RM$, and so $NKL = (N : M)(K : N)(L : N)RM$. Suppose $0 \neq NKL \subseteq P$, but $N \nsubseteq P$, $N \nsubseteq P$ and $L \nsubseteq P$.

Then, $(N : M) \nsubseteq (P : M)$, $(K : M) \nsubseteq (P : M)$ and $(L : M) \nsubseteq (P : M)$. As $(P : M)$ is weakly prime ternary following Lemma 2.13**,** then, either $(N : M)(K : M)(L : M) \nsubseteq (P : M)$ or $(N : M)(K : M)(L : M) = 0$. In first case, we have $NKL = (N : M)(K : N)(L : N)RM \nsubseteq (P : M)RM = P$, a contradiction. If $(N : M)(K : M)(L : M) = 0$, then, $NKL = 0RM = 0$ and also we get a contradiction. Therefore, either $N \subseteq P$ or $K \subseteq P$ or $L \subseteq P$. Conversely, to prove that P is weakly prime ternary in M, it enough by Lemma 2.13 to prove that $(P : M)$ is weakly prime ternary in R. Let r_1, r_2, $r_3 \in R$, such that $0 \neq r_1r_2r_3 \in (P : M)$, but $r_1 \notin (P : M)$, $r_2 \notin (P : M)$ and $r_3 \notin (P : M)$. Let $N = (r_1)RM$, $K = (r_2)RM$ and $L = (r_3)RM$. Then, $0 \neq NKL = (r_1)(r_2)(r_3)RM \subseteq P$. Indeed, if $NKL = (r_1)(r_2)(r_3)RM = 0$, then, $(r_1r_2r_3) \subseteq Ann(M) = 0$, which is a contradiction. By assumption, either $(r_1)RM = N \subseteq P$ or $(r_2)RM = K \subseteq P$ or $(r_1)RM = L \subseteq P$ and so, either $r_1 \in (P : M)$ or $r_2 \in (P : M)$ or $r_3 \in (P : M)$, a contradiction. Therefore, $(P : M)$ is weakly prime ternary in R and so P is weakly prime ternary in M. □

Corollary 2.15. *Let P be a proper ternary subsemimodule of finitely generated faithful multiplication ternary $R-$semimodule M. Then, P is weakly prime ternary if and only if whenever $m_1, m_2, m_3 \in M$, $0 \neq m_1m_2m_3 \in P$ implies $m_1 \in P$ or $m_2 \in P$ or $m_3 \in P$.*

Theorem 2.16. *Let M be a finitely generated faithful multiplication ternary R-semimodule and P be a proper ternary subsemimodules of M. Then, P is almost prime in M if and only if whenever N, K and L are ternary subsemimodules of M such that $NKL \subseteq P - (P : M)^2P$, then, either $N \subseteq P$ or $K \subseteq P$ or $L \subseteq P$.*

Proof. By Theorem 2.12, P is almost prime in M if and only if $(P : M)$ is almost prime in R. As $(P : M)^3 = ((P : M)^2P : M)$ by Lemma 2.11**,** then, the proof is similar to that of Theorem 2.16. □

Recall that, a ternary subsemimodule N of a ternary R-semimodule M is called Q-ternary subsemimodule (= partitioning ternary subsemimodule) if there exists a subset Q of M such that

1) $M = \cup\{q + N : q \in Q\}$.

2) If $q_1, q_2 \in Q$, then $(q_1 + N) \cap (q_2 + N) \neq \emptyset \Leftrightarrow q_1 = q_2$.

Let N be a Q-ternary subsemimodule of a ternary R-semimodule M. Then $M/N_{(Q)} = \{q + N : q \in Q\}$ forms a ternary R-semimodule under the following addition "$\oplus$"and ternary scalar multiplication "$\odot$",$(q_1 + N) \oplus (q_2 + N) = q_3 + N$ where $q_3 \in Q$ is unique such that $q_1 + q_2 + N \subseteq q_3 + N$, and $r \odot s \odot (q_1 + N) = q_4 + N$ where $q_4 \in Q$ is unique such that $rsq_1 + N \subseteq q_4 + N$. This ternary R-semimodule$M/N_{(Q)}$ is called the quotient ternary semimodule of M by N and denoted by $(M/N_{(Q)}, \oplus, \odot)$ or just $M/N_{(Q)}$.

Lemma 2.17 ([21])**.** *Let N be a Q-ternary subsemimodule of a ternary R-semimodule M. If A is a subtractive ternary subsemimodule of M such that $N \subseteq A$, then N is a $Q \cap A$-ternary subsemimodule of A.*

Lemma 2.18 ([18]). *Let N be a Q-ternary subsemimodule of a ternary R-semimodule M. If $r; s \in R$ and $m \in M$, then there exists a unique $q \in Q$ such that $rsm \in r \odot s \odot (q + N)$.*

Theorem 2.19. *Let N be a Q-ternary subsemimodule of a ternary R-semimodule M and P be asubtractive ternary subsemimodule of M with $N \subseteq P$. Then*

(1) If P is almost prime ternary subsemimodule of M, then $P/N_{(Q\cap P)}$ is almost prime ternary subsemimodule of $M/N_{(Q)}$.

(2) If N, $P/N_{(Q\cap P)}$ are almost prime ternary subsemimodule of M, $M/N_{(Q)}$ respectively, then P is almost prime ternary subsemimodule of M.

Proof. Let P be almost prime ternary subsemimodule of M. Let $r, s \in R$ and $q_1 + N \in M/N_{(Q)}$ such that $r \odot s \odot (q_1 + N) \in P/N_{(Q\cap P)} - (P/N_{(Q\cap P)} : M/N_{(Q)})^2 P/N_{(Q\cap P)}$. By Lemma 2.17, N is $Q \cap P$-ternary subsemimodule of P. Hence, there exists a unique $q_2 \in Q \cap P$ such that $r \odot s \odot (q_1 + N) = q_2 + N$ where $rsq_1 + N \subseteq q_2 + N$. Since $N \subseteq P$, then $rsq_1 \in P$ and since $r \odot s \odot (q_1 + N) \notin (P/N_{(Q\cap P)} : M/N_{(Q)})^2 P/N_{(Q\cap P)} = (P : M)^2 P/N_{(Q\cap P)}$, then $rsq_1 \notin (P : M)^2 N$. As P is almost prime ternary subsemimodule, either $r \in (P : M)$ or $s \in (P : M)$ or $q_1 \in P$. If $q_1 \in P$, then $q_1 \in Q \cap P$ and hence $q_1 + N \in P/N_{(Q\cap P)}$. Without loss of generality suppose that $r \in (P : M)$. For $q_1 + N \in M/N_{(Q)}$ and $s' \in R$, let $r \odot s' \odot (q + N) = q_3 + N$ where q_3 is a unique element of Q such that $rs'q = q_3 + n$ for some $n \in N$. Now $r \in (P : M) \Rightarrow rs'q \in P \Rightarrow q_3 + n \in P \Rightarrow q_3 \in P$, as P is a subtractive ternary subsemimodule of M and $n \in N \subseteq P$. Hence $q_3 \in Q \cap P$. Now $r \odot s' \odot (q+N) = q_3 + N \in P/N_{(Q\cap P)}$ for all $s' \in R$ and $q + N \in M/N_{(Q)}$. Therefore $r \in (P/N_{(Q\cap P)} : M/N_{(Q)})$. Thus $P/N_{(Q\cap P)}$ is almost prime ternary subsmimodule of $M/N_{(Q)}$.

2) Suppose that $N, P/N_{(Q\cap P)}$ are almost prime subsemimodule of M, $M/N_{(Q)}$ respectively. Let $rsm \in P - (P : M)^2 P$ where $r,s \in R$, $m \in M$. If $rsm \in N - (N : M)^2 N$, then we are through, since N is almost prime ternary subsemimodule of M. So suppose that $rsm \in P - N$. By using Lemma 2.18, there exists a unique $q_1 \in Q$ such that $m \in q_1 + N$ and $rsm \in r \odot s \odot (q_1 + N) = q_2 + N$ where q_2 is a unique element of Q such that $rsq_1 + N \subseteq q_2 + N$. Now $rsm \in P$, $rsm \in q_2 + N$ implies $q_2 \in P$, as P is a subtractive ternary subseminodule and $N \subseteq P$. Hence $r \odot s \odot (q_1 + N) = q_2 + N \in P/N_{(Q\cap P)} - (P/N_{(Q\cap P)} : M/N_{(Q)})^2 P/N_{(Q\cap P)}$. As $P/N_{(Q\cap P)}$ is almost prime ternary subsemimodule, $r \in (P/N_{(Q\cap P)} : M/N_{(Q)})$ or $s \in (P/N_{(Q\cap P)} : M/N_{(Q)})$ or $q_1 + N \in P/N_{(Q\cap P)}$. If $q_1 + N \in P/N_{(Q\cap P)}$, then $q_1 \in P$. Hence $m \in q_1 + N \subseteq P$. Now without loss of generality assume that $r \in (P/N_{(Q\cap P)} : M/N_{(Q)})$. Let $x \in M$ and $s' \in R$. By using Lemma 2.18, there exists a unique $q_3 \in Q$ such that $x \in q_3 + N$ and $rs'x \in r \odot s \odot (q_3 + N) = q_4 + N$ where q_4 is a unique element of Q such that $rs'q_3 + N \subseteq q_4 + N$. Now $q_4 + N = r \odot s' \odot (q_3 + N) \in P/N_{(Q\cap P)}$ and hence $q_4 \in P$. As $rs'x \in q_4 + N$ and $N \subseteq P$, $rs'x \in P$. So $r \in (P : M)$. Therefore P is almost prime ternary subsemimodule of M. □

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