FIXED POINT THEOREMS FOR ONE AND TWO SELF-MAPS ON A G-METRIC SPACE

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Abstract. The proof of a recent result of Vats et al is presented, by employing the infimum property of nonegative real numbers. Then the unique fixed point is shown to be the G-limit of all the orbits of the form $x, fx, ..., f^n x, ..., x \in X$. That is, the unique fixed point is a G-contractive fixed point. Further, a fixed point for a pair of self-maps is obtained as another application of the infimum property.
Keywords: the infimum property, G-metric space, G-Cauchy sequence, fixed point, G-contractive fixed point.

1. Introduction

Let X be a nonempty set and $G : X \times X \times X \to \mathbb{R}$ such that

(G1) $G(x, y, z) \geq 0$ for all $x, y, z \in X$ with $G(x, y, z) = 0$ if $x = y = z$,

(G2) $G(x, x, y) > 0$ for all $x, y \in X$ with $x \neq y$,

(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$,

(G4) $G(x, y, z) = G(x, z, y) = G(y, x, z) = G(z, x, y) = G(y, z, x) = G(z, y, x)$ for all $x, y, z \in X$

(G5) $G(x, y, z) \leq G(x, w, w) + G(w, y, z)$ for all $x, y, z, w \in X$.

Mustafa and Sims [2] introduced the pair (X, G) as a G-metric space with G-metric G on X. Axioms (G5) is known as the *rectangle inequality* (of G). Note that

$$G(x, y, y) \leq 2G(x, x, y) \text{ for all } x, y \in X. \tag{1.1}$$

Given a G-metric space (X, G), define

$$\rho_G(x, y) = G(x, y, y) + G(x, x, y) \text{ for all } x, y, z \in X. \tag{1.2}$$

Then it is seen in [2] that ρ_G is a metric on X, and that the family of all G-balls $\{B_G(x,r) : x \in X, r > 0\}$ is the base topology, called the G-metric topology $\tau(G)$ on X, where $B_G(x,r) = \{y \in X : G(x,y,y) < r\}$. Further, it was shown that the G-metric topology coincides with the metric topology induced by the metric ρ_G, which allows us to readily transform many concepts from metric spaces into the setting of G-metric space.

Definition 1.1. A sequence $\langle x_n \rangle_{n=1}^{\infty}$ in a G-metric space (X,G) is said to be G-convergent with limit $p \in X$, if it converges to p in the G-metric topology $\tau(G)$.

Lemma 1.1 ([2]). *The following statements are equivalent in a G-metric space (X,G):*

(*a*) $\langle x_n \rangle_{n=1}^{\infty} \subset X$ *is G-convergent with limit* $p \in X$,

(*b*) $\lim_{n\to\infty} G(x_n, x_n, p) = 0$,

(*c*) $\lim_{n\to\infty} G(x_n, p, p) = 0$.

Definition 1.2. A sequence $\langle x_n \rangle_{n=1}^{\infty}$ in a G-metric space (X,G) is said to be G-Cauchy, if $G(x_n, x_m, x_m) \to 0$ as $n, m \to \infty$.

Definition 1.3. A G-metric space (X,G) is said to be G-complete, if every G-Cauchy sequence in X converges in it.

The infimum property of real numbers states that if $S \subset \mathbb{R}$ be nonempty and bounded, then $\alpha = \inf S$ exists in $\mathbb{R}$. In particular, we have

Lemma 1.2. *If S is a nonempty subset of nonnegative real numbers, then $\alpha = \inf S \geq 0$ and $\lim_{n\to\infty} p_n = \alpha$ for some sequence $\langle p_n \rangle_{n=1}^{\infty}$ in S.*

In this paper, two applications of Lemma 1.2 are presented: The first one is to obtain a unique fixed point for a contraction type of Vats et al [7]. This is shown to be a G-contractive fixed point. The second is to obtain a common fixed point for a pair of self-maps.

2. Main result

Vats et al [7] proved the following fixed point theorem:

Theorem 2.1. *Suppose that (X,G) is a complete G-metric space and f be a self-map on X satisfying*

$$\begin{aligned} G(fx,fy,fz) \leq k \max\{&G(x,fx,fx)+G(y,fy,fy)+G(z,fz,fz),\\ &G(x,fy,fy)+G(y,fx,fx)+G(z,fy,fy),\\ &G(x,fz,fz)+G(z,fx,fx)+G(y,fz,fz)\}\\ &\textit{for all } x,y,z \in X, \end{aligned} \tag{2.1}$$

where $0 < k < 1/4$. Then f will have a unique fixed point.

Let $x_0 \in X$ be arbitrary. The proof of Theorem 2.1 in [7] initiates with an orbit $x_0, fx_0, ..., f^n x_0, ...$ In fact, , $\langle f^n x_0 \rangle_{n=1}^{\infty}$ is a G-Cauchy in X. Since X is G-complete, $f^n x_0 \to t$ as $n \to \infty$ for some $t \in X$. Then, t is shown to be a unique fixed point of f.

Applying Lemma 1.2, we obtain below a fixed point of f satisfying (2.1), which is independent of f-iterations.

Proof. Let $S = \{G(x, fx, fx) : x \in X\}$. Inview of Lemma 1.2, S has the infimum, say $a \geq 0$. If $a > 0$, from (2.1) with $y = fx$ and $z = fx$ and the rectangle inequality (G5), we have

$$\begin{aligned} G(fx, f^2x, f^2x) \leq k \max \big\{ & G(x, fx, fx) + G(fx, f^2x, f^2x) + G(fx, f^2x, f^2x), \\ & G(x, fx, fx) + G(fx, fx, fx) + G(fx, f^2x, f^2x), \\ & G(x, f^2x, f^2x) + G(fx, f^2x, f^2x) + G(fx, f^2x, f^2x) \big\} \\ \leq & k \max \big\{ G(x, fx, fx) + 2G(fx, f^2x, f^2x), \\ & G(x, fx, fx) + G(fx, f^2x, f^2x), \\ & [G(x, fx, fx) + G(fx, f^2x, f^2x)] + 2G(fx, f^2x, f^2x) \big\} \\ = & k[G(x, fx, fx) + 3G(fx, f^2x, f^2x)], \end{aligned}$$

or

$$G(fx, f^2x, f^2x) \leq \frac{k}{1-3k} G(x, fx, fx). \tag{2.2}$$

Since $k/(1-3k)$ is less than 1, from (2.2), it would follow that

$$G(fx, f^2x, f^2x) < a \text{ where } G(fx, f^2x, f^2x) \in S.$$

In other words, a cannot be a lower bound of S, which is a contradiction.

Therefore, $a = \inf S = 0$, and hence we can choose the points $x_1, x_2, ..., x_n, ...$ in X such that

$$G(x_n, fx_n, fx_n) \in S \text{ for } n = 1, 2, 3, ... \text{ and } \lim_{n \to \infty} G(x_n, fx_n, fx_n) = 0. \tag{2.3}$$

Now repeated use of (G5) followed by (1.1), we get

$$\begin{aligned} G(x_n, x_m, x_m) & \leq G(x_n, fx_n, fx_n) + G(fx_n, x_m, x_m) \\ & \leq G(x_n, fx_n, fx_n) + [G(fx_n, fx_m, fx_m) + G(fx_m, x_m, x_m)] \\ & \leq G(x_n, fx_n, fx_n) \\ & \quad + [G(fx_n, fx_m, fx_m) + 2G(x_m, fx_m, fx_m)]. \end{aligned} \tag{2.4}$$

Now writing $x = x_n, y = z = x_m$ in (2.1) and using (G5) and (1.1), and then simplifying, we get

$$G(fx_n, fx_m, fx_m) \leq \frac{k}{1-3k} G(x_n, fx_n, fx_n) + \frac{2k}{1-3k} G(x_m, fx_m, fx_m).$$

Inserting this in (2.4), we get

$$\begin{aligned}G(x_n, x_m, x_m) \leq & G(x_n, fx_n, fx_n) + \frac{k}{1-3k} G(x_n, fx_n, fx_n) \\ & + \frac{2k}{1-3k} G(x_m, fx_m, fx_m) + 2G(x_m, fx_m, fx_m).\end{aligned}$$

Employing the limit as $m, n \to \infty$ in this and using (2.3), it follows that $\langle x_n \rangle_{n=1}^{\infty}$ is G-Cauchy. Since X is G-complete,

$$\lim_{n\to\infty} x_n = p \text{ for some } p \in X. \tag{2.5}$$

Again, by repeated application of rectangle inequality (G5), we have

$$\begin{aligned}G(p, fp, fp) &\leq G(p, fx_n, fx_n) + G(fx_n, fp, fp) \\ &\leq [G(p, x_n, x_n) + G(x_n, fx_n, fx_n)] + G(fx_n, fp, fp).\end{aligned} \tag{2.6}$$

Now, (2.1) with $x = x_n, y = z = p$ implies

$$\begin{aligned}G(fx_n, fp, fp) \leq k \max\big\{&G(x_n, fx_n, fx_n) + G(p, fp, fp) + G(p, fp, fp), \\ &G(x_n, fp, fp) + G(p, fx_n, fx_n) + G(p, fp, fp), \\ &G(x_n, fp, fp) + G(p, fp, fp), +G(p, fx_n, fx_n)\big\} \\ = kM&,\end{aligned}$$

where

$$\begin{aligned}M = \max\{&G(x_n, fx_n, fx_n) + 2G(p, fp, fp), \\ &G(x_n, fp, fp) + G(p, fp, fp) + G(p, fx_n, fx_n)\big\}.\end{aligned}$$

With this, (2.6) becomes

$$G(p, fp, fp) \leq G(p, x_n, x_n) + G(x_n, fx_n, fx_n) + kM. \tag{2.7}$$

Case (a). Let $M = G(x_n, fx_n, fx_n) + 2G(p, fp, fp)$. Then (2.7) can be written as

$$G(p, fp, fp) \leq G(p, x_n, x_n) + G(x_n, fx_n, fx_n) + k[G(x_n, fx_n, fx_n) + 2G(p, fp, fp)]$$

or

$$G(p, fp, fp) \leq \left(\frac{1}{1-2k}\right) G(p, x_n, x_n) + \left(\frac{1+k}{1-2k}\right) G(x_n, fx_n, fx_n). \tag{2.8}$$

Proceeding the limit as $n \to \infty$ in (2.8), and then using (2.3), (2.5) and Lemma 1.1, we get $0 \leq G(p, fp, fp) = 0$ or $fp = p$. That is p is a fixed point of f.

Case (b). Let $M = G(x_n, fp, fp) + G(p, fp, fp) + G(p, fx_n, fx_n)$. Then (2.7) can be written as

$$
\begin{aligned}
G(p, fp, fp) &\le G(p, x_n, x_n) + G(x_n, fx_n, fx_n) \\
&\quad + k[G(x_n, fp, fp) + G(p, fp, fp) + G(p, fx_n, fx_n)] \\
&\le G(p, x_n, x_n) + G(x_n, fx_n, fx_n) \\
&\quad + k[G(x_n, p, p) + G(p, fp, fp) \\
&\quad + G(p, fp, fp) + G(x_n, fx_n, fx_n) + G(p, x_n, x_n)]
\end{aligned}
$$

or $G(p, fp, fp) \le \frac{1+k}{1-2k}[G(p, x_n, x_n) + G(x_n, fx_n, fx_n)] + \frac{k}{1-2k} G(x_n, p, p)$,

Proceeding the limit as $n \to \infty$ in this, and using (2.3), (2.5) and Lemma 1.1, we obtain that $0 \le G(p, fp, fp) = 0$ or $fp = p$. That is p is a fixed point of f. The uniqueness of the fixed point follws from (2.1) directly. □

Definition 2.1 ([3]). A fixed point p of f on a G-metric space (X, G) is a G-contractive fixed point, if for each $x \in X$, the orbit $O_f(x) = \{x, fx, ..., f^n x, ...\}$ converges to p.

Example 2.1. Let $X = [0, \infty)$ with $G(x, y, z) = 0$ if $x = y = z$, $\max\{x, y, z\}$ otherwise. Define $fx = 0$ if $0 \le x < 1/2$, qx otherwise, for all $x \in X$, where $0 \le q < 1$. Then, we see that 0 is the unique fixed point of f and for each $x \in X$, the f-orbit $O_f(x) = \{x, qx, q^2 x, ..., q^n x, ...\}$ converges to 0. That is, 0 is a G-contractive fixed point of f.

We now show that the unique fixed point p is a G-contractive fixed point. Indeed, taking $y = z = p$ in (2.1) and using (G5), we get

$$
\begin{aligned}
G(f^n x, p, p) &= G(f^n x, fp, fp) \\
&\le k \max\{G(f^{n-1}x, f^n x, f^n x) + G(p, fp, fp) + G(p, fp, fp), \\
&\quad G(f^{n-1}x, fp, fp) + G(p, f^n x, f^n x) + G(p, fp, fp), \\
&\quad G(f^{n-1}x, fp, fp) + G(p, fp, fp) + G(p, f^n x, f^n x)\} \\
&= k \max\{G(f^{n-1}x, f^n x, f^n x) + 0, \\
&\quad G(f^{n-1}x, p, p) + G(p, f^n x, f^n x) + 0, \\
&\quad G(f^{n-1}x, p, p) + 0 + G(p, f^n x, f^n x)\} \\
&= k \max\{G(f^{n-1}x, f^n x, f^n x), \\
&\quad G(f^{n-1}x, p, p) + G(p, f^n x, f^n x) + 0\} \\
&\le k \max\{G(p, f^n x, f^n x) \\
&\quad + G(f^{n-1}x, p, p), G(f^{n-1}x, p, p) + 2G(p, p, f^n x)\} \\
&\le k[2G(p, p, f^n x) + G(f^{n-1}x, p, p)]
\end{aligned}
$$

or

$$G(f^n x, p, p) \le c \cdot G(f^{n-1}x, p, p).$$

where $c = k/(1-2k)$ is less than 1, by the choice of k. By induction, we have

$$G(f^n x, p, p) \leq c^n G(fx, p, p),$$

which as $n \to \infty$ gives $f^n x \to p$ for each $x \in X$, in view of Lemma 1.1. Thus p is a G-contractive fixed point of f. For the G-contractive fixed points under other contraction type conditions, one may refer to [4], [5] and [6].

The following example illustrates Theorem 2.1:

Example 2.2. Let $X = [0,1]$,with G-metric $G(x,y,z) = \max\{|x-y|, |y-z|, |z-x|\}$. Then$(X,d)$is a complete G-metric space. Define $f : X \to X$ by

$$fx = \begin{cases} \dfrac{x}{12}, & 0 \leq x < \dfrac{1}{2} \\ \dfrac{x}{10}, & \dfrac{1}{2} \leq x \leq 1. \end{cases}$$

We show that f satisfies the inequality (2.1) with $k = 1/5$ in four different cases:

Case (i). Let $x, y, z \in [0, 1/2)$. Then

$$G(fx, fy, fz) = \frac{1}{12} \max\{|x-y|, |y-z|, |z-x|\},$$
$$G(x, fx, fx) = 11x/12, G(y, fy, fy) = 11y/12, G(z, fz, fz) = 11z/12.$$

Suppose that $x < y < z$. Then

$$G(fx, fy, fz) = \frac{1}{12}(z-x) \leq \frac{1}{5}\frac{5z}{12} \leq \frac{1}{5} \cdot \frac{11z}{12} = \frac{1}{5}G(z, fz, fz),$$

By symmetry, the subcases $y < z < x$ and $z < x < y$ can be handled.

Case (ii). Let $x, y, z \in [1/2, 1]$. Then

$$G(fx, fy, fz) = \frac{1}{10} \max\{|x-y|, |y-z|, |z-x|\},$$
$$G(x, fx, fx) = 9x/10, G(y, fy, fy) = 9y/12, G(z, fz, fz) = 9z/12.$$

Therefore,

$$G(fx, fy, fz) = \frac{1}{10}(z-x) \leq \frac{1}{5}\frac{5z}{10} \leq \frac{1}{5} \cdot \frac{9z}{10} = \frac{1}{5}G(z, fz, fz).$$

By symmetry, the subcases $y < z < x$ and $z < x < y$ can be handled.

Case (iii). Let $x, y \in [0, 1/2, z \in [1/2, 1]$. Then

$$\begin{aligned} G(fx, fy, fz) &= \max\left\{\frac{|x-y|}{12}, \frac{z}{10} - \frac{y}{12}, \frac{z}{10} - \frac{x}{12}\right\} \\ &\leq \frac{z}{10} = \frac{1}{5}\frac{5z}{10} \leq \frac{1}{5} \cdot \frac{9z}{10} = \frac{1}{5}G(z, fz, fz). \end{aligned}$$

Case (iv). Let $x, y \in [1/2, 1]$, $z \in [0, 1/2)$. Then

$$\begin{aligned} G(fx, fy, fz) &= \max\left\{\frac{|x-y|}{10}, \frac{y}{10} - \frac{z}{12}, \frac{x}{10} - \frac{z}{12}\right\} \\ &\leq \max\left\{\frac{y}{10}, \frac{x}{10}\right\} \\ &\leq \frac{1}{5}\left(\frac{9x}{10} + \frac{9y}{10}\right) = \frac{1}{5}[G(x, fx, fx) + G(y, fy, fy)]. \end{aligned}$$

From all the above cases, we observe that

$$G(fx, fy, fz) \leq \frac{1}{5}[G(x, fx, fx) + G(y, fy, fy) + G(z, fz, fz)] \text{ for all } x, y, z \in X,$$

showing that f satisfies the inequality (2.1) with $k = 1/5$. Note that 0 is unique fixed point of f.

3. Common fixed point theorem for two self-maps

Mustafa et al [1] proved the following result:

Theorem 3.1. *Suppose that (X, G) is a complete G-metric space and f, a self-map on X satisfying*

$$\begin{aligned} G(fx, fy, fz) \leq{}& aG(x, y, z) + bG(x, fx, fx) + cG(y, fy, fy) \\ &+ eG(z, fz, fz) \text{ for all } x, y, z \in X, \end{aligned} \tag{3.1}$$

where a, b, c and e are nonnegative real numbers with $a + b + c + e < 1\cdot$ Then f will have a unique fixed point.

Writing $a = \alpha$, $b = 0$, $c = \beta$ and $e = \gamma$ in Theorem 3.1, we get

Corollary 3.1. *Suppose that (X, G) is a complete G-metric space and f, a self-map on X satisfying*

$$\begin{aligned} G(fx, fy, fz) \leq{}& \alpha G(x, y, z) + \beta G(y, fy, fy) \\ &+ \gamma G(z, fz, fz), \text{ for all } x, y, z \in X, \end{aligned} \tag{3.2}$$

where $\alpha, \beta, \gamma \geq 0$ with $\alpha + \beta + \gamma < 1$. Then f will have a unique fixed point.

As another application of Lemma 1.2, we extend Corollary 3.1 to a pair of self-maps to obtain a common fixed point, as follows:

Theorem 3.2. *Let (X, G) be a complete G-metric space and f, g be two self maps such that*

$$\begin{aligned} G(fx, gy, gz) \leq{}& \alpha G(x, y, z) + \beta G(y, gy, gy) \\ &+ \gamma G(z, gz, gz), \text{ for all } x, y, z \in X, \end{aligned} \tag{3.3}$$

where $\alpha, \beta, \gamma \geq 0$ with $\alpha + \beta + \gamma < 1$. Then f and g will have a unique common fixed point.

Proof. Let $S(f) = \{G(x, fx, fx) : x \in X\}$ and $S(g) = \{G(y, gy, gy) : y \in X\}$. Suppose that $a \geq 0$ be the infimum of $S(f) \bigcup S(g)$.

Case (a). Suppose that $a = \inf S(f)$. If $a > 0$, writing $y = z = fx$ in (3.2), we get

$$G(fx, gfx, gfx) \leq \alpha G(x, fx, fx) + \beta G(fx, gfx, gfx) + \gamma G(fx, gfx, gfx)$$

or

$$G(fx, gfx, gfx) \leq \frac{\alpha}{1-\beta-\gamma} \cdot G(x, fx, fx). \tag{3.4}$$

Since $\alpha/(1-\beta-\gamma)$ is less than 1, from (3.4), it follows that $G(fx, gfx, gfx) < a$ for some $x \in X$, which contradicts with the choice of a . Therefore $a = 0$.

Case (b). Suppose that $a = \inf S(g)$. Again, if $a > 0$, writing $x = gy$ and $z = y$ in (3.2), we get

$$G(fgy, gy, gy) \leq \alpha G(gy, y, y) + \beta G(y, gy, gy) + \gamma G(y, gy, gy)$$

or

$$G(fgy, gy, gy) \leq (\alpha + \beta + \gamma) G(y, gy, gy) < a, \tag{3.5}$$

which contradicts with the choice of a. Therefore, $a = 0$. Choose the points $x_1, x_2, ..., x_n, ...$ in X such that

$$G(x_n, fx_n, fx_n) \in S(f) \text{ for } n = 1, 2, 3, ...$$

and

$$\lim_{n \to \infty} G(x_n, fx_n, fx_n) = 0. \tag{3.6}$$

Similarly we can choose the points $x_1, x_2, ..., x_m, ...$ in X such that

$$G(x_m, gx_m, gx_m) \in S(g) \text{ for } n = 1, 2, 3, ...$$

and

$$\lim_{n \to \infty} G(x_m, gx_m, gx_m) = 0. \tag{3.7}$$

Repeatedly using the rectangle inequality (G5) and (3.2), we see that

$$\begin{aligned} G(x_n, x_m, x_m) \leq & \; G(x_n, fx_n, fx_n) \\ & + G(fx_n, gx_m, gx_m) + 2G(x_m, gx_m, gx_m). \end{aligned} \tag{3.8}$$

Now

$$G(fx_n, gx_m, gx_m) \le \alpha G(x_n, x_m, x_m) \\ + \beta G(x_m, gx_m, gx_m) + \gamma G(x_m, gx_m, gx_m). \tag{3.9}$$

Substituting (3.9) in (3.8), we obtain

$$G(x_n, x_m, x_m) \le G(x_n, fx_n, fx_n) + \alpha G(x_n, x_m, x_m) + \beta G(x_m, gx_m, gx_m) \\ + \gamma G(x_m, gx_m, gx_m) + 2G(x_m, gx_m, gx_m)$$

or

$$G(x_n, x_m, x_m) \le \frac{1}{1-\alpha} \cdot G(x_n, fx_n, fx_n) + \frac{2+\beta+\gamma}{1-\alpha} \cdot G(x_m, gx_m, gx_m)$$

As $m, n \to \infty$ this in view of (3.6) and (3.7), gives $G(x_n, x_m, x_m) \to 0$, proving that $\langle x_n \rangle_{n=1}^{\infty}$ is G-Cauchy.

Since X is G-complete, we can find a point $p \in X$ such that

$$\lim_{n\to\infty} x_n = p. \tag{3.10}$$

Again by repeated application of rectangle inequality (G5), (G4) and (3.2), we have

$$G(p, fp, fp) \le [G(p, x_m, x_m) + G(x_m, gx_m, gx_m)] + 2G(fp, gx_m, gx_m). \tag{3.11}$$

But, from (3.2), we have

$$G(fp, gx_m, gx_m) \le \alpha G(p, x_m, x_m) \\ + \beta G(x_m, gx_m, gx_m) + \gamma G(x_m, gx_m, gx_m). \tag{3.12}$$

Substituting (3.12) in (3.11), and then simplifying, it follows that

$$G(p, fp, fp) \le (1+2\alpha)G(x_m, x_m, p) + (1+\beta+\gamma)G(x_m, gx_m, gx_m).$$

As $m \to \infty$ this finally yields $d(p, fp, fp) \le 0$ or $fp = p$.

Now, using (G5) and writing $x = y = z = p$ in (3.2), we have

$$G(fp, gp, gp) \le \alpha G(p, p, p) + \beta G(p, gp, gp) + \gamma G(p, gp, gp) \\ \le (\beta+\gamma)[G(p, fp, fp) + G(fp, gp, gp)]$$

or

$$(1-\beta-\gamma)G(fp, gp, gp) \le (\beta+\gamma)G(p, fp, fp). \tag{3.13}$$

This further implies that $(1-\beta-\gamma)G(fp,gp,gp)\leq 0$ or $fp=gp$. Since p is a fixed point of f, p will be a common fixed point of f and g.

Uniqueness: Suppose q is another common fixed of f and g. That is $fq=q$ and $gq=q$. Then from (3.2), we have

$$G(p,q,q)=G(fp,gq,gq)\leq\alpha G(p,q,q)+\beta G(q,gq,gq)+\gamma G(q,gq,gq)=0,$$

which implies that $G(p,q,q)\leq 0$ or $p=q$. That is, p is an unique common fixed point of f and g. □

Remark 3.1. Writing $g=f$, (3.3) reduces to (3.2). Hence Theorem 3.2 is an extension of Corollary 3.1 to two self-maps.

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