BANACH AND KANNAN CONTRACTIONS ON S-METRIC SPACE

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Abstract. Unique fixed points are obtained for Banach and Kannan contractions on an S-metric space. Also, the unique fixed points are shown to be S-contractive fixed points.
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1. Introduction

Let X be a nonempty set. Sedghi et al [10] introduced an S-metric $S : X \times X \times X \to [0, \infty)$ on X satisfying the following conditions:

(S1) $S(x, y, z) = 0$ if and only if $x, y, z \in X$ with $x = y = z$,

(S2) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$ for all $x, y, z, a \in X$.

The pair (X, S) is called an S-metric space. We obtain from Axiom (S2) that

(1.1) $$S(x, x, y) = S(y, y, x) \text{ for all } x, y \in X.$$

Definition 1.1. A sequence $\langle x_n \rangle_{n=1}^{\infty}$ in a S-metric space (X, S) is said to be S-convergent, if there exists a point x in X such that $S(x_n, x_n, x) \to 0$ as $n \to \infty$.

Definition 1.2. A sequence $\langle x_n \rangle_{n=1}^{\infty}$ in a S-metric space (X, S) is said to be S-Cauchy, if $\lim_{n,m\to\infty} S(x_n, x_n, x_m) = 0$.

Definition 1.3. The space (X, S) is said to be S-complete, if every S-Cauchy sequence in X converges in it.

In his introductory work on S-metric space, Sedghi et al [10] proved the follwing Banach's contraction mapping theorem:

Theorem 1.1. *Let f be a self-map on a complete S-metric space (X, S) such that*

(1.2) $$S(fx, fx, fy) \leq \alpha S(x, x, y) \textit{ for all } x, y \in X,$$

where $0 < \alpha < 1$*. Then* f *has a unique fixed point.*

As an interesting application of the infimum property of the real numbers, unique fixed points of Banach and Kannan contractions are obtained in S-metric space. Then, an S-contractive fixed point is introduced. Also, the unique fixed points for these contractions are shown to be S-contractive fixed points.

2. Main results

The well-known infimum property of real numbers states that, a nonempty and bounded set of real numbers has an infimum in $\mathbb{R}$. In particular,

Lemma 2.1. *If* S *is a nonempty subset of nonnegative real numbers, then* $\alpha = \inf S \geq 0$ *and* $\lim_{n\to\infty} p_n = \alpha$ *for some sequence* $\langle p_n\rangle_{n=1}^{\infty}$ *in* S*.*

The following proof of Theorem 1.1 differs from that of [10]. In fact, we empoy Lemma 2.1, without the concern of iterations:

Step 1 – Existence of the infimum

Define $A = \{S(fx, fx, x) : x \in X\}$. Then by Lemma 2.1, A has an infimum $a \geq 0$, by the infimum property.

Step 2 – Vanishing infimum

If $a > 0$, writing $y = fx$ in (1.2) and using (1.1), we get

$$S(f^2x, f^2x, fx) = S(fx, fx, f^2x) \leq \alpha S(x, x, fx) = \alpha S(fx, fx, x)$$

for all $x \in X$. This implies that $0 < a \leq \alpha a < a$, which is again a contradiction. Hence $a = 0$.

Step 3 – Existence of a sequence

Hence, there exists a sequence $\langle x_n\rangle_{n=1}^{\infty}$ in X such that

$$S(fx_n, fx_n, x_n) \in A \text{ for all } n = 1, 2, 3, \ldots \text{ and } \lim_{n\to\infty} S(fx_n, fx_n, x_n) = 0. \tag{2.1}$$

Step 4 – $\langle x_n\rangle_{n=1}^{\infty}$ is S-Cauchy

In fact, by $(S2)$ and (1.1), we have

$$\begin{aligned} S(x_n, x_n, x_m) &\leq S(x_n, x_n, fx_n) + S(x_n, x_n, fx_n) + S(x_m, x_m, fx_n) \\ &= 2S(x_n, x_n, fx_n) + S(x_m, x_m, fx_n) \\ &\leq 2S(x_n, x_n, fx_n) + S(x_m, x_m, fx_m) + S(x_m, x_m, fx_m) \\ &\qquad + S(fx_n, fx_n, fx_m) \\ &= 2[S(x_n, x_n, fx_n) + S(x_m, x_m, fx_m)] + S(fx_n, fx_n, fx_m). \qquad (2.2) \end{aligned}$$

Now, with $x = x_n$ and $y = x_m$, (1.2) gives,

$$S(fx_n, fx_n, fx_m) \leq \alpha S(x_n, x_n, x_m)$$

Inserting this in (2.2), we get

$$S(x_n, x_n, x_m) \leq 2[S(x_n, x_n, fx_n) + S(x_m, x_m, fx_m)] + \alpha S(x_n, x_n, x_m)$$

or

$$(1 - \alpha)S(x_n, x_n, x_m) \leq 2[S(x_n, x_n, fx_n) + S(x_m, x_m, fx_m)]$$

so that

$$S(x_n, x_n, x_m) \leq \left(\frac{2}{1-\alpha}\right)[S(x_n, x_n, fx_n) + S(x_m, x_m, fx_m)].$$

Applying the limit as $m, n \to \infty$ in this and using (2.1) we obtain that $\langle x_n \rangle_{n=1}^{\infty}$ is a S-Cauchy sequence in X.

Step 5 – S-convergence
Since, X is S-complete, we find the point p in X such that

$$\lim_{n\to\infty} x_n = p. \tag{2.3}$$

Step 6 – S-convergent limit as a fixed point
Again repeatedly using ($S2$),

$$\begin{aligned} S(fp, fp, p) &\leq S(fp, fp, fx_n) + S(fp, fp, fx_n) + S(p, p, fx_n). \\ &= 2S(fp, fp, fx_n) + S(p, p, fx_n) \\ &= 2S(fx_n, fx_n, fp) + S(fx_n, fx_n, p) \qquad (2.4) \end{aligned}$$

Now, from (1.2) with $x = x_n$ and $y = p$, it follows that

$$S(fx_n, fx_n, fp) \leq \alpha S(x_n, x_n, p) \tag{2.5}$$

Substituting (2.5) in (2.4), we get

$$S(fp, fp, p) \leq 2\alpha S(x_n, x_n, p) + S(fx_n, fx_n, p)$$

In the limiting case as $n \to \infty$, this in view of (2.1) and (2.3) implies $S(fp, fp, p) = 0$ or $fp = p$. Thus p is a fixed point.

Step 7 – Uniqueness of the fixed point
Let q be another fixed point of f. Then, (1.2) with $x = p$ and $y = q$ gives

$$S(p, p, q) = S(fp, fp, fq) \leq \alpha S(p, p, q) \text{ or } (1 - \alpha)S(p, p, q) \leq 0$$

so that $p = q$. That is, p is the unique fixed point of f.

3. S-contractive fixed point

The notion of 2-metric space was introduced by Gahler [1] and a G-metric space was introduced by Mustafa and Sims in [2], as generalizations of a metric space. In these settings, contractive fixed points were introduced in [3] and [4] respectively. For further study on this idea, one can refer to [5, 6, 7, 8, 9].

Now we introduce a contractive fixed point in an S metric space as follows:

Definition 3.1. Let f be a self-map on an S-metric space (X, S). A fixed point p of f is a contractive fixed point, if for every $x_0 \in X$, the f-orbit $O_f(x_0) = \langle x_0, fx_0, ..., f^n x_0, ...\rangle$ converges to p.

We now show that the unique fixed point p is an S-contractive fixed point. In fact, writing $x = f^{n-1}x_0, y = p$ in (1.2), we get

$$S(f^n x, f^n x, fp) = S(f^n x, f^n x, p) \leq \alpha S(f^{n-1}x, f^{n-1}x, p) \text{ for } n \geq 1. \tag{3.1}$$

Proceeding the limit as $n \to \infty$ in (3.1) , we get $S(f^n x, f^n x, p) \to 0$. Thus $f^n x_0 \to p$ for each $x_0 \in X$. Thus p is a S-contractive fixed point of f.

Our next result is:

Theorem 3.1. *Let f be a self-map on a complete S-metric space (X, S) such that*

$$S(fx, fx, fy) \leq \alpha[S(fx, fx, x) + S(fy, fy, y)] \tag{3.2}$$

for all $x, y \in X$, where $0 \leq \alpha < 1/3$. Then f has a unique fixed point p, which is an S-contractive fixed point.

A unique fixed point p for (3.2) is obtained, similar to the previous proof and is omitted here. Now we show that p is an S-contractive fixed point of (3.2), as follows:

Writing $x = f^{n-1}x_0, y = p$ in (3.2), and then using (1.1), we get

$$\begin{aligned} S(f^n x, f^n x, fp) &= S(f^n x, f^n x, p) \\ &\leq \alpha[S(f^n x, f^n x, f^{n-1}x) + S(fp, fp, p)] \\ &= \alpha S(f^n x, f^n x, f^{n-1}x) \\ &\leq \alpha[S(f^n x, f^n x, p) + S(f^n x, f^n x, p) + S(f^{n-1}x, f^{n-1}x, fp) \\ &= \alpha[2S(f^n x, f^n x, p) + S(f^{n-1}x, f^{n-1}x, fp)] \end{aligned}$$

or

$$S(f^n x, f^n x, p) \leq \left(\tfrac{\alpha}{1-2\alpha}\right) S(f^{n-1}x, f^{n-1}x, p) \to 0, \text{ as } n \to \infty, \tag{3.3}$$

since $\alpha/((1 - 2\alpha)$ is less than 1. Thus $f^n x_0 \to p$ for each $x_0 \in X$ so that p is a S-contractive fixed point of f.

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