MATHEMATICAL ANALYSIS OF AN AGE-STRUCTURED QUARANTINE/ISOLATION MODEL

Mohammad A. Safi
Department of Mathematics
The Hashemite University
Zarqa, Jordan
masafi@hu.edu.jo

Abstract. A new age-structured model for disease transmission, subject to the use of quarantine (of asymptomatic cases) and isolation (of individuals with disease symptoms) is presented and rigorously analyzed. The model is, first of all, shown to be properly-posed mathematically by formulating it as an abstract Cauchy problem. For the case where the contact rate is separable (i.e., $\beta(a,b) = \beta_1(a)\beta_2(b)$), rigorous analysis of the model reveals that it has a globally-asymptotically stable disease-free equilibrium whenever its associated reproduction number is less than unity. The model has a unique endemic equilibrium when the threshold quantity exceeds unity. The endemic equilibrium is shown to be locally stable whenever its associated reproduction number exceeds unity. Furthermore, it is shown that adding age-factor to the basic quarantine-isolation model in M. A. Safi and A. B. Gumel (*Discrete Contin. Dyn. Syst. Ser. B.* : 209-231, 2010) does not alter the qualitative dynamics of the autonomous system (with respect to the elimination or persistence of the disease). Numerical simulations show disease elimination whenever its associated reproduction number is less than unity and the disease will persist in this case whenever its associated reproduction number exceeds unity.

Keywords: age-structure, abstract Cauchy problem, C_0-semigroup, equilibria, stability.

1. Introduction

Quarantine (of individuals suspected of being exposed to the disease) and isolation (of those with clinical symptoms of the disease) have been used in the control of numerous diseases such as leprosy, plague, cholera, typhus, yellow fever, smallpox, diphtheria, tuberculosis, measles, ebola, pandemic influenza, severe acute respiratory syndrome (SARS) and more recently the 2009 swine influenza pandemic [1, 10, 13, 17, 20]. A number of mathematical modeling work have been carried out to assess the impact of quarantine and isolation in combatting the spread of diseases (see, for instance, [1, 4, 10, 13, 17, 18, 23, 24] and some of the references therein). However, many of the models used for assessing the impact of the quarantine and isolation measures tend to be built based on the assumption that the contact rates are age-dependent. However, some studies such as those reported by Elvebac *et al.* [3], suggest that disease trans-

mission models with age-dependent contact rates are more realistic than those that do not consider age-dependent contact rates. Numerous studies, based on the design and analysis of age-structured models, have been presented in the literature (see, for instance, [3, 6, 7, 8, 9, 11, 12, 14, 26, 28, 30] and some of the references therein).

The purpose of this study is to provide a rigorous qualitative analysis of a new age-structured model for disease transmission, subject to the use of quarantine (of asymptomatic cases) and isolation (of individuals with disease symptoms). The model to be designed extends the SEIQHR model given in [21] by considering an age-densities of all individuals (susceptible, latent (exposed), infectious, quarantined, isolated (hospitalized) and recovered).

The paper is organized as follows. The model is formulated in Section 2. The locally and globally asymptotic stability of disease-free equilibrium is established in Section 3. The existence and locally asymptotical stability of endemic equilibrium is also studied in Section 4.

2. Model formulation

The total population, at time t and age a (where $0 < a \leq A$, with A being the maximum possible age attained by individuals in the population), is subdivided into six mutually-exclusive compartments of susceptible ($S(a,t)$), latent (those who have been infected but do not show clinical symptoms of the disease yet) ($E(a,t)$), infectious ($I(a,t)$), quarantined ($Q(a,t)$), isolated (hospitalized) ($H(a,t)$) and recovered ($R(a,t)$) individuals, so that

$$N(a,t) = S(a,t) + E(a,t) + I(a,t) + Q(a,t) + H(a,t) + R(a,t).$$

The population density evolves according to the following equation:

$$\frac{\partial N(a,t)}{\partial a} + \frac{\partial N(a,t)}{\partial t} = -\mu(a)N(a,t),$$

where $\mu(a)$ represents the instantaneous age-dependent death rate (the probability that an individual of age a dies before reaching the age $a + \Delta a$ is $\mu(a)\Delta a + O(\Delta a)$).

It is assumed that the net reproductive rate of the population equals to unity, and the total population is at an equilibrium. Thus, the following relations hold [15]:

$$\int_0^A v(a)e^{-\int_0^a \mu(\tau)d\tau}da = 1; \;\; N(a,t) = N_\infty(a) = v_0 e^{-\int_0^a \mu(\tau)d\tau}, \tag{1}$$

where $v(a)$ is the age-dependent fertility rate, $N_\infty(a)$ is the steady-state population distribution and v_0 is the steady-state number of newborns.

Let $\beta(a,b)$ be the age-dependent contact rate. In other words, the probability of contact between a susceptible of age a and an infectious individual of age b,

during the time interval $[t, t+\Delta t]$, is given by $\beta(a,b)\Delta t + O(\Delta t)$. Hence, the associated *force of infection* (i.e., the rate at which new cases of infection are generated) at time t, denoted by $\lambda(a,t)$, is given by (this form of the force of infection has been used in [11, 14]):

$$\lambda(a,t) = \int_0^A \beta(a,b)[I(b,t) + \eta H(b,t)]db,$$

where $0 < \eta < 1$ is the modification parameter accounts for the assumed reduction in disease transmission by hospitalized individuals in comparison to non-hospitalized infectious individuals in the I class. Thus, η measures the efficacy of isolation or treatment given to hospitalized individuals (isolation is perfect if $\eta = 0$; leaky if $0 < \eta < 1$ and completely ineffective if $\eta = 1$).

The age-structured dynamics of an infectious disease in the presence of quarantine of exposed individuals and isolation of infectious individuals is given by the following system of partial differential equations :

$$\begin{aligned}
\frac{\partial S}{\partial a} + \frac{\partial S}{\partial t} &= -[\mu(a) + \lambda(a,t)]S(a,t),\\
\frac{\partial E}{\partial a} + \frac{\partial E}{\partial t} &= \lambda(a,t)S(a,t) - [\mu(a) + \kappa + \sigma]E(a,t),\\
\frac{\partial I}{\partial a} + \frac{\partial I}{\partial t} &= \kappa E(a,t) - [\mu(a) + \psi + \gamma_1]I(a,t),\\
\frac{\partial Q}{\partial a} + \frac{\partial Q}{\partial t} &= \sigma E(a,t) - [\mu(a) + \alpha]Q(a,t),\\
\frac{\partial H}{\partial a} + \frac{\partial H}{\partial t} &= \psi I(a,t) + \alpha Q(a,t) - [\mu(a) + \gamma_2]H(a,t),\\
\frac{\partial R}{\partial a} + \frac{\partial R}{\partial t} &= \gamma_1 I(a,t) + \gamma_2 H(a,t) - \mu(a)R(a,t),
\end{aligned} \tag{2}$$

subject to the boundary and initial conditions:

$$\begin{aligned}
&S(0,t) = v_0, \quad E(0,t) = 0, \quad I(0,t) = 0, \quad Q(0,t) = 0,\\
&H(0,t) = 0, \quad R(0,t) = 0, \quad S(a,0) = S_0(a), \quad E(a,0) = E_0(a),\\
&I(a,0) = I_0(a), \quad Q(a,0) = Q_0(a), \quad H(a,0) = H_0(a), \quad R(a,0) = R_0(a).
\end{aligned} \tag{3}$$

In (2), $\mu(a)$ is the age-dependent natural death rate, κ is the rate at which latent individuals develop symptoms, σ is the quarantine rate of exposed individuals, α is the hospitalization rate of quarantined individuals, ψ is the hospitalization rate of infectious individuals and γ_i $(i = 1, 2)$ is the recovery rate for infectious and hospitalized individuals, respectively. Furthermore, it is assumed that disease-induced mortality is negligible (so that infected individuals do not die of the disease) and that recovery confers permanent immunity against reinfection. The parameters of the model are described in Table 1. The model extends the model in [21] by considering an age-densities of all individuals. In line with [11, 14] the model extends the models in [2, 6, 7] by assuming that the infection

rate (λ) is a function of age. Furthermore, the model (2) is an extension of the SIR models in [6, 11], by including classes for latent (E), quarantined (Q) and Hospitalized (H) individuals.

2.1 Basic properties

To show the existence and uniqueness of the solution for the system (2) with the initial-boundary conditions (3) the natural death rate, $\mu(a)$, is eliminated from the model (2) by the following change of variables [12, 25]:

$$\begin{aligned} &x_1(a,t) = \frac{S(a,t)}{N_\infty(a)},\ x_2(a,t) = \frac{E(a,t)}{N_\infty(a)},\ x_3(a,t) = \frac{I(a,t)}{N_\infty(a)},\\ &x_4(a,t) = \frac{Q(a,t)}{N_\infty(a)},\ x_5(a,t) = \frac{H(a,t)}{N_\infty(a)},\ x_6(a,t) = \frac{R(a,t)}{N_\infty(a)}. \end{aligned} \tag{4}$$

Using (4) in (2) gives the following transformed system:

$$\begin{aligned} &\frac{\partial x_1}{\partial a} + \frac{\partial x_1}{\partial t} = -\lambda_1(a,t)x_1(a,t),\\ &\frac{\partial x_2}{\partial a} + \frac{\partial x_2}{\partial t} = \lambda_1(a,t)x_1(a,t) - (\kappa+\sigma)x_2(a,t),\\ &\frac{\partial x_3}{\partial a} + \frac{\partial x_3}{\partial t} = \kappa x_2(a,t) - (\psi+\gamma_1)x_3(a,t),\\ &\frac{\partial x_4}{\partial a} + \frac{\partial x_4}{\partial t} = \sigma x_2(a,t) - \alpha x_4(a,t),\\ &\frac{\partial x_5}{\partial a} + \frac{\partial x_5}{\partial t} = \psi x_3(a,t) + \alpha x_4(a,t) - \gamma_2 x_5(a,t),\\ &\frac{\partial x_6}{\partial a} + \frac{\partial x_6}{\partial t} = \gamma_1 x_3(a,t) + \gamma_2 x_5(a,t), \end{aligned} \tag{5}$$

and associated boundary and initial conditions:

$$\begin{aligned} &x_1(0,t) = 1, \quad x_2(0,t) = 0, \quad x_3(0,t) = 0, \quad x_4(0,t) = 0,\\ &x_5(0,t) = 0, \quad x_1(a,0) = f_1(a), \quad x_2(a,0) = f_2(a),\\ &x_3(a,0) = f_3(a), \quad x_4(a,0) = f_4(a), \quad x_5(a,0) = f_5(a), \quad x_6(a,0) = f_6(a). \end{aligned} \tag{6}$$

Furthermore, the force of infection, $\lambda(a,t)$, now becomes

$$\lambda_1(a,t) = \int_0^A \beta(a,b)[x_3(b,t) + \eta x_5(b,t)]N_\infty(b)db. \tag{7}$$

Using the relation $x_1(a,t) + x_2(a,t) + x_3(a,t) + x_4(a,t) + x_5(a,t) + x_6(a,t) = 1$ and noting that the variable $x_6(a,t)$ does not feature in the other equations of the model. Furthermore, introducing a new variable $\bar{x}_1(a,t) = x_1(a,t) - 1$ into

the transformed system (5) gives

$$
\begin{aligned}
&\frac{\partial \bar{x}_1}{\partial a} + \frac{\partial \bar{x}_1}{\partial t} = -\lambda_1(a,t)\bar{x}_1(a,t),\\
&\frac{\partial x_2}{\partial a} + \frac{\partial x_2}{\partial t} = \lambda_1(a,t)\bar{x}_1(a,t) - (\kappa+\sigma)x_2(a,t),\\
&\frac{\partial x_3}{\partial a} + \frac{\partial x_3}{\partial t} = \kappa x_2(a,t) - (\psi+\gamma_1)x_3(a,t),\\
&\frac{\partial x_4}{\partial a} + \frac{\partial x_4}{\partial t} = \sigma x_2(a,t) - \alpha x_4(a,t),\\
&\frac{\partial x_5}{\partial a} + \frac{\partial x_5}{\partial t} = \psi x_3(a,t) + \alpha x_4(a,t) - \gamma_2 x_5(a,t),
\end{aligned} \tag{8}
$$

and associated boundary and initial conditions:

$$
\begin{aligned}
&\bar{x}_1(0,t) = 0, \quad x_2(0,t) = 0, \quad x_3(0,t) = 0, \quad x_4(0,t) = 0,\\
&x_5(0,t) = 0, \quad x_6(0,t) = 0, \quad \bar{x}_1(a,0) = f_1(a) - 1, \quad x_2(a,0) = f_2(a),\\
&x_3(a,0) = f_3(a), \quad x_4(a,0) = f_4(a), \quad x_5(a,0) = f_5(a).
\end{aligned} \tag{9}
$$

Following [11], the initial-boundary-value problem $\{(8),(9)\}$ is written as an abstract Cauchy problem on the Banach space

$$X = L^1(0,A) \times L^1(0,A) \times L^1(0,A) \times L^1(0,A) \times L^1(0,A)$$

(with the norm $\|\phi\| = \sum_{j=1}^{5} \|\phi_j\|_1$, where $\phi(a) = (\phi_1(a), \phi_2(a), \phi_3(a), \phi_4(a), \phi_5(a))^T \in X$ and $\|\phi_j\|_1$ is the norm of $L^1(0,A)$) by defining a linear operator, $\mathfrak{A}$, as follows (by considering the derivatives with respect to age and the negative coefficients of $\phi_i (i = 1, 2. \cdots, 5)$ in (8)):

$$
(\mathfrak{A}\phi)(a) = \begin{pmatrix}
-\dfrac{d\phi_1(a)}{da}\\
-\dfrac{d\phi_2(a)}{da} - (\kappa+\sigma)\phi_2\\
-\dfrac{d\phi_3(a)}{da} - (\psi+\gamma_1)\phi_3\\
-\dfrac{d\phi_4(a)}{da} - \alpha\phi_4\\
-\dfrac{d\phi_5(a)}{da} - \gamma_2\phi_5
\end{pmatrix}, \tag{10}
$$

with $\phi(a) \in D(\mathfrak{A}) = \{\phi \in X : \phi_j \text{ is absolutely continuous on } [0,A], \phi(0) = 0\}$.

Let $F : X \to X$ be the non-linear operator given by (consisting of the remaining terms, excluding those used in the definition of the operator $\mathfrak{A}$, in (8))

$$
(F\phi)(a) = \begin{pmatrix}
-(P(\phi_3,\phi_5))(a)(1+\phi_1)\\
(P(\phi_3,\phi_5))(a)(1+\phi_1)\\
\kappa\phi_2(a)\\
\sigma\phi_2(a)\\
\psi\phi_3 + \alpha\phi_4(a)
\end{pmatrix}, \ \phi \in X \tag{11}
$$

where $P : L^1(0,A) \times L^1(0,A) \to L^\infty(0,A)$ is a linear bounded operator, given by

$$(P(h,k))(a) = \int_0^A \beta(a,b)N_\infty(b)[h(b) + \eta k(b)]db.$$

Thus, it follows from the above definitions that the initial-boundary-value problem $\{(8),(9)\}$ can be re-written as the following abstract Cauchy problem in X:

$$\frac{du(t)}{dt} = \mathfrak{A}u(t) + F(u(t)), \quad u(0) = u_0 \in X, \tag{12}$$

where $u(t) = (\bar{x}_1(\cdot,t), x_2(\cdot,t), x_3(\cdot,t), x_4(\cdot,t), x_5(\cdot,t))^T$ and $u_0 = (f_1(a)-1, f_2(a), f_3(a), f_4(a), f_5(a))^T$. Furthermore, it follows from (10) and(11) that the operator $\mathfrak{A}$ is the infinitesimal generator of C_0-semigroup $T(t), t \geq 0$ and F is continuously Frechet differentiable on X [14]. Thus, using Proposition 4.16 of [30], the results below are established

Theorem 1. *(i) for each $u_0 \in X$, there exist a maximal interval of existence $[0,b)$ and unique continuous mild solution $t \to u(t,u_0)$ such that:*

$$u(t,u_0) = T(t)u_0 + \int_0^t T(t-s)F(u(s,u_0))ds \ \text{ for all } t \in [0,b),$$

with either $b = \infty$ or $\lim_{t\to b^-} \|u(t,u_0)\| = \infty$;

(ii) if $u_0 \in D(\mathfrak{A})$, then $u(t,u_0) \in D(\mathfrak{A})$ for $t \in [0,b)$ and the mild solution $u(t,u_0)$ is continuously differentiable and satisfies (12) on the interval $[0,b)$.

Let,

$$\mathcal{D} = \{(\bar{x}_1, x_2, x_3, x_4, x_5) \in X : \bar{x}_1 \geq -1, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0\},$$

and,

$$\mathcal{D}_0 = \{(\bar{x}_1, x_2, x_3, x_4, x_5) \in X : -1 \leq \bar{x}_1 \leq 0, 0 \leq x_2 \leq 1, \\ 0 \leq x_3 \leq 1, 0 \leq x_4 \leq 1, 0 \leq x_5 \leq 1\}.$$

The following result is needed to show that the abstract Cauchy problem (12) has a unique global classical solution.

Lemma 1. *The mild solution of the abstract Cauchy problem (12), given by $u(t,u_0)$ with $u_0 \in \mathcal{D}$, enters $\mathcal{D}_0$ in finite time, and the set $\mathcal{D}_0$ is positively-invariant.*

Proof. The proof of Lemma 1 is based on the approach given in [11, 14]. First of all, the solution of the system (5), along the characteristic lines, can be represented [9, 29, 30] by fixing arbitrary time (t_1) and age (a_1) and considering the functions $\tilde{x}_1(h) = x_1(a_1 + h, t_1 + h)$ and $\tilde{\lambda}(h) = \lambda_1(a_1 + h, t_1 + h)$. Thus, the first equation of the system (5), along this line, is equivalent to

$$\frac{d\tilde{x}_1}{dh} + \tilde{\lambda}\tilde{x}_1 = 0,$$

which has the following solution

$$\tilde{x}_1(h) = \tilde{x}_1(0) \exp\left(-\int_0^h \tilde{\lambda}(\tau)d\tau\right). \tag{13}$$

Hence,

$$x_1(a_1 + h, t_1 + h) = x_1(a_1, t_1) \exp\left(-\int_0^h \lambda_1(a_1 + \tau, t_1 + \tau)d\tau\right).$$

By setting $(a_1, t_1) = (a - t, 0)$ and $h = t$ for $a > t$ in (13), it follows that

$$x_1(a, t) = f_1(a - t) \exp\left(-\int_0^t \lambda_1(a - t + \tau, \tau)d\tau\right).$$

Similarly, for the case when $a < t$, setting $(a_1, t_1) = (0, t - a)$ and $h = a$ in (13) gives

$$x_1(a, t) = \exp\left(-\int_0^a \lambda_1(\tau, t - a + \tau)d\tau\right).$$

Thus,

$$x_1(a, t) = \begin{cases} f_1(a - t) \exp\left(-\int_0^t \lambda_1(a - t + \tau, \tau)d\tau\right), & a > t \\ \exp\left(-\int_0^a \lambda_1(\tau, t - a + \tau)d\tau\right), & t > a \end{cases} \tag{14}$$

It is clear that $x_1(a, t) \geq 0$ (or, equivalently, $\bar{x} + 1(a, t) \geq -1$) whenever $f_1(a) \geq 0$. Similarly, from the third, fourth and fifth equations of the system (5), the representation for x_3, x_4 and x_5, can, respectively, be found as follows:

$$x_3(a, t) = \begin{cases} f_3(a - t) \exp(-k_1 t) \\ +\kappa \int_0^t x_2(a - t + \tau, \tau) \exp(-k_1(t - \tau))d\tau, & a > t \\ \kappa \int_0^a x_2(\tau, t - a + \tau) \exp(-k_1(a - \tau))d\tau, & t > a, \end{cases} \tag{15}$$

$$x_4(a,t) = \begin{cases} f_4(a-t)\exp(-k_2 t) \\ +\sigma \int_0^t x_2(a-t+\tau,\tau)\exp(-k_2(t-\tau))d\tau, & a > t \\ \sigma \int_0^a x_2(\tau, t-a+\tau)\exp(-k_2(a-\tau))d\tau, & t > a, \end{cases} \tag{16}$$

$$x_5(a,t) = \begin{cases} f_5(a-t)\exp(-k_3 t) + \int_0^t [\psi x_3(a-t+\tau,\tau) \\ +\alpha x_4(a-t+\tau,\tau)]\exp(-k_3(t-\tau))d\tau, & a > t \\ \int_0^t [\psi x_3(a-t+\tau,\tau) \\ +\alpha x_4(a-t+\tau,\tau)]\exp(-k_3(t-\tau))d\tau, & t > a, \end{cases} \tag{17}$$

where, $k_1 = \psi + \gamma_1$, $k_2 = \alpha$, and $k_3 = \gamma_2$.

Substituting (15), (16) and (17)) into (7) gives

$$\lambda_1(t) = (Gx_2)(a,t), \tag{18}$$

where G is the transformation from $x_2(a,t)$ to $\lambda_1(t)$. The second equation of the system (5) can be written as an abstract Cauchy problem:

$$\frac{dx_2}{dt} = Bx_2(a,t) + (Gx_2(a,t))(x_1), \quad x_2(0) = f_2,$$

where B is an operator given by

$$B = -\frac{d}{da} - (\kappa + \sigma), \quad \text{with } D(B) = D(\mathfrak{A}).$$

Hence, x_2 can be found as follows:

$$x_2(t) = Q(t)f_2(0) + \int_0^t Q(t-\tau)(Gx_2(\tau))(\tau)d\tau, \tag{19}$$

where, $Q(t) = \exp tB$ is the positive C_0-semigroup generated by the closed operator B [11]. If $f_2 \geq 0, f_3 \geq 0, f_4 \geq 0$ and $f_5 \geq 0$, then G and $Q(t), t \geq 0$ are positive. Thus, (19) shows that x_2 is positive. Since x_2 is positive, it follows that x_3, x_4 and x_5 are positive. Hence, $u(t,u_0) \in \mathcal{D}$ for all $t \geq 0$ whenever $u_0 \in \mathcal{D}_0$. □

It follows from Lemma 1 that the norm of the local solution, $u(t,u_0)$, $u_0 \in D(\mathfrak{A}) \cap \mathcal{D}$, of the abstract Cauchy problem (12) is finite as long as it is defined. Hence, the following result is established:

Theorem 2. *The abstract Cauchy problem* (12) *has a unique global classical solution on* X *with respect to initial data* $u_0 \in D(\mathfrak{A}) \cap \mathcal{D}$.

Theorem 2 shows that the initial-boundary-value problem $\{(5), (6)\}$ (or, equivalently, $\{(2), (3)\}$) has a unique positive global solution with respect to the positive initial data.

The analysis of the initial-boundary-value problem $\{(5),\ (6)\}$ will be explored for a special form of the contact rate $\beta(a, b)$, given by $\beta(a, b) = \beta_1(a)\beta_2(b)$ (this form of the contact rate, known as separable contact rate, has been used in [5, 15, 16]). Using $\beta(a, b) = \beta_1(a)\beta_2(b)$ in (7) gives:

$$\lambda_2(a,t) = \beta_1(a) \int_0^A \beta_2(b)[x_3(b,t) + \eta x_5(b,t)]N_\infty(b)db. \tag{20}$$

3. Stability of the Disease-Free Equilibrium (DFE)

3.1 Local stability of DFE

The DFE of the system (5) is given by

$$\mathcal{E}_0 = (x_1, x_2, x_3, x_4, x_5, x_6) = (1, 0, 0, 0, 0, 0).$$

The local stability of the DFE is studied by considering exponential solutions of the form ([15, 16]):

$$\begin{array}{l} x_1(a,t) = 1 + \bar{x}_1(a)e^{\zeta t},\ \ x_2(a,t) = \bar{x}_2(a)e^{\zeta t},\ \ x_3(a,t) = \bar{x}_3(a)e^{\zeta t}, \\ x_4(a,t) = \bar{x}_4(a)e^{\zeta t},\ \ x_5(a,t) = \bar{x}_5(a)e^{\zeta t},\ \ x_6(a,t) = \bar{x}_6(a)e^{\zeta t}. \end{array} \tag{21}$$

Linearizing (5) about the DFE ($\mathcal{E}_0$) gives (noting that $\lambda_1(a, t)$ is now replaced by (20)),

$$\begin{aligned} \zeta\bar{x}_2(a) + \frac{d\bar{x}_2(a)}{da} &= \lambda_2(a) - (\kappa + \sigma)\bar{x}_2(a), \\ \zeta\bar{x}_3(a) + \frac{d\bar{x}_3(a)}{da} &= \kappa\bar{x}_2(a) - (\psi + \gamma_1)\bar{x}_3(a), \\ \zeta\bar{x}_4(a) + \frac{d\bar{x}_4(a)}{da} &= \sigma\bar{x}_2(a) - \alpha\bar{x}_4(a), \\ \zeta\bar{x}_5(a) + \frac{d\bar{x}_5(a)}{da} &= \psi\bar{x}_3(a) + \alpha\bar{x}_4 - \gamma_2\bar{x}_5(a), \\ \zeta\bar{x}_6(a) + \frac{d\bar{x}_6(a)}{da} &= \gamma_1\bar{x}_3(a) + \gamma_2\bar{x}_5(a), \end{aligned} \tag{22}$$

where,

$$\lambda_2(a) = \beta_1(a) \int_0^A \beta_2(b)[\bar{x}_3(b) + \eta\bar{x}_5(b)]N_\infty(b)db = \beta_1(a)V_0,$$

with

$$V_0 = \int_0^A \beta_2(b)[\bar{x}_3(b) + \eta\bar{x}_5(b)]N_\infty(b)db; \quad V_0 \neq 0. \tag{23}$$

Solving system (22) gives,

$$\begin{aligned}
\bar{x}_2(a) &= V_0 \int_0^a \beta_1(u)e^{-(\zeta+\kappa+\sigma)(a-u)}du,\\
\bar{x}_3(a) &= V_0\kappa \int_0^a \int_0^u \beta_1(u_1)e^{-(\zeta+\kappa+\sigma)(u-u_1)}e^{-(\zeta+\psi+\gamma_1)(a-u)}du_1du\\
&= V_0Q_1(a,\zeta),\\
\bar{x}_4(a) &= V_0\sigma \int_0^a \int_0^u \beta_1(u_1)e^{-(\zeta+\kappa+\sigma)(u-u_1)}e^{-(\zeta+\alpha)(a-u)}du_1du \qquad (24)\\
&= V_0Q_2(a,\zeta)\\
\bar{x}_5(a) &= \int_0^a [\psi\bar{x}_3(u) + \alpha\bar{x}_4(u)]e^{-(\zeta+\gamma_2)(a-u)}du\\
&= V_0 \int_0^a \psi Q_1(u,\zeta) + \alpha Q_2(u,\zeta)e^{-(\zeta+\gamma_2)(a-u)}du, = V_0Q_3(a,\zeta).
\end{aligned}$$

where,

$$\begin{aligned}
Q_1(a,\zeta) &= \kappa \int_0^a \int_0^u \beta_1(u_1)e^{-(\zeta+\kappa+\sigma)(u-u_1)}e^{-(\zeta+\psi+\gamma_1)(a-u)}du_1du,\\
Q_2(a,\zeta) &= \sigma \int_0^a \int_0^u \beta_1(u_1)e^{-(\zeta+\kappa+\sigma)(u-u_1)}e^{-(\zeta+\alpha)(a-u)}du_1du,\\
Q_3(a,\zeta) &= \int_0^a (\psi Q_1(u,\zeta) + \alpha Q_2(u,\zeta))e^{-(\zeta+\gamma_2)(a-u)}du.
\end{aligned}$$

Substituting the expressions for $\bar{x}_3(a)$ and $\bar{x}_5(a)$ from (24) into (23) gives,

$$V_0 = V_0 \int_0^A \beta_2(b)[Q_1(b,\zeta) + \eta Q_3(b,\zeta)]N_\infty(b)db, \tag{25}$$

so that (by dividing both sides of the equation (25) by V_0)

$$1 = \int_0^A \beta_2(b)[Q_1(b,\zeta) + \eta Q_3(b,\zeta)]N_\infty(b)db. \tag{26}$$

Let,

$$\mathcal{G}(\zeta) = \int_0^A \beta_2(b)[Q_1(b,\zeta) + \eta Q_3(b,\zeta)]N_\infty(b)db. \tag{27}$$

Define the reproduction number as follows:

$$\mathcal{R}_0 = \int_0^A \beta_2(b)[Q(b,0) + \eta Q_3(b,0)]N_\infty(b)db.$$

It is clear that the function $\mathcal{G}$ in (27) satisfies the following properties:

(i) $\mathcal{G}(0) = \mathcal{R}_0$;

(ii) $\mathcal{G}$ is a monotone decreasing function in ζ;

(iii) $\lim_{\zeta\to\infty} \mathcal{G}(\zeta) = 0$.

Hence, the result below can be established, based on the properties of the function $\mathcal{G}$ and equation (26):

Theorem 3. *The DFE of the model* (5) *with* (20)*, given by* $\mathcal{E}_0$*, is locally-asymptotically stable (LAS) if* $\mathcal{R}_0 < 1$*, and unstable if* $\mathcal{R}_0 > 1$.

Theorem 3 shows that the spread of the disease can be effectively controlled if the initial sizes of the sub-populations of the model are in the basin of attraction of the DFE ($\mathcal{E}_0$). For effective disease elimination to be independent of such initial sizes, a global asymptotic stability result has to be proved for the DFE. This is explored below.

3.2 Global stability of DFE

Theorem 4. *The DFE of the model* (5) *with* (20) *is globally-asymptotically stable (GAS) in* $\mathcal{D}_0$ *if* $\mathcal{R}_0 < 1$.

Proof. Following [15, 16], let

$$f(a,t) = \lambda_2(a,t)x_1(a,t). \tag{28}$$

Noting that $x_1(a,t) \leq 1$ in $\mathcal{D}_0$, it follows that

$$f(a,t) \leq \lambda_2(a,t). \tag{29}$$

Integrating (5), with (20), along the characteristic lines whenever $a < t$, gives,

$$\begin{aligned}
x_2(a,t) &= \int_0^a e^{-(\kappa+\sigma)(a-u)} f(u,t)du,\\
x_3(a,t) &= \kappa \int_0^a e^{-(\psi+\gamma_1)(a-u)} x_2(u,t)du\\
&= \kappa \int_0^a e^{-(\psi+\gamma_1)(a-u)} \int_0^u e^{-(\kappa+\sigma)(u-u_1)} f(u_1,t)du_1du,\\
x_4(a,t) &= \sigma \int_0^a e^{-\alpha(a-u)} x_2(u,t)du\\
&= \sigma \int_0^a e^{-\alpha(a-u)} \int_0^u e^{-(\kappa+\sigma)(u-u_1)} f(u_1,t)du_1du,\\
x_5(a,t) &= \int_0^a e^{-\gamma_2(a-u)} [\psi x_3(u,t) + \alpha x_4(u,t)]du\\
&= \psi \int_0^a e^{-\gamma_2(a-u)} \kappa \int_0^u e^{-(\psi+\gamma_1)(u-u_1)} \int_0^{u_1} e^{-(\kappa+\sigma)(u_1-u_2)} f(u_2,t)du_2du_1,\\
&+ \alpha \int_0^a e^{-\gamma_2(a-u)} \kappa \int_0^u e^{-\alpha(u-u_1)} \int_0^{u_1} e^{-(\kappa+\sigma)(u_1-u_2)} f(u_2,t)du_2du_1,
\end{aligned} \tag{30}$$

Substituting the expressions for $x_3(a)$ and $x_5(a)$ in (30) into (29) gives,

$$
\begin{aligned}
f(a,t) \le \beta_1(a)V(t) = \beta_1(a)\int_0^A N_\infty(b)\beta_2(b)\Bigg[\kappa\int_0^a e^{-(\psi+\gamma_1)(a-u)} \\
\cdot\int_0^u e^{-(\kappa+\sigma)(u-u_1)}f(u_1,t)du_1du \\
+\eta\psi\int_0^a e^{-\gamma_2(a-u)}\kappa\int_0^u e^{-(\psi+\gamma_1)(u-u_1)} \\
\cdot\int_0^{u_1} e^{-(\kappa+\sigma)(u_1-u_2)}f(u_2,t)du_2du_1 \\
+\eta\alpha\int_0^a e^{-\gamma_2(a-u)}\kappa\int_0^u e^{-\alpha(u-u_1)} \\
\cdot\int_0^{u_1} e^{-(\kappa+\sigma)(u_1-u_2)}f(u_2,t)du_2du_1\Bigg]db.
\end{aligned} \tag{31}
$$

Let,

$$F(a) = \limsup_{t\to\infty} f(a,t).$$

Taking the lim sup (when $t \to \infty$) of both sides of (31), and using Fatou's Lemma [19] gives,

$$F(a) \le \beta_1(a)L, \tag{32}$$

where,

$$
\begin{aligned}
L = \int_0^A N_\infty(b)\beta_2(b)\Bigg[\kappa\int_0^a e^{-(\psi+\gamma_1)(a-u)} \\
\cdot\int_0^u e^{-(\kappa+\sigma)(u-u_1)}F(u_1)du_1du \\
+\eta\psi\int_0^a e^{-\gamma_2(a-u)}\kappa\int_0^u e^{-(\psi+\gamma_1)(u-u_1)} \\
\cdot\int_0^{u_1} e^{-(\kappa+\sigma)(u_1-u_2)}F(u_2)du_2du_1 \\
+\eta\alpha\int_0^a e^{-\gamma_2(a-u)}\kappa\int_0^u e^{-\alpha(u-u_1)} \\
\cdot\int_0^{u_1} e^{-(\kappa+\sigma)(u_1-u_2)}F(u_2)du_2du_1\Bigg]db.
\end{aligned} \tag{33}
$$

Using inequality (32) in (34) gives

$$
\begin{aligned}
L \le \int_0^A N_\infty(b)\beta_2(b)\Bigg[\kappa\int_0^a e^{-(\psi+\gamma_1)(a-u)} \\
\cdot\int_0^u e^{-(\kappa+\sigma)(u-u_1)}\beta_1(u_1)Ldu_1du \\
+\eta\psi\int_0^a e^{-\gamma_2(a-u)}\kappa\int_0^u e^{-(\psi+\gamma_1)(u-u_1)}
\end{aligned}
$$

$$
\begin{aligned}
&\cdot \int_0^{u_1} e^{-(\kappa+\sigma)(u_1-u_2)}\beta_1(u_2)L du_2 du_1 \\
&+ \eta\alpha \int_0^a e^{-\gamma_2(a-u)}\kappa \int_0^u e^{-\alpha(u-u_1)} \\
&\cdot \int_0^{u_1} e^{-(\kappa+\sigma)(u_1-u_2)}\beta_1(u_2)L du_2 du_1 \Big] db = L\mathcal{R}_0.
\end{aligned} \tag{34}
$$

It follows that whenever $\mathcal{R}_0 < 1$, then $L = 0$. Thus,

$$\limsup_{t\to\infty} f(a,t) = F(a) = 0.$$

Hence (from (28)),

$$\limsup_{t\to\infty} \lambda_2(a,t) = 0. \tag{35}$$

It follows then (by using (35) in (20)) that

$$\lim_{t\to\infty} x_3(a,t) = 0, \quad \lim_{t\to\infty} x_4(a,t) = 0 \quad \text{and} \quad \lim_{t\to\infty} x_5(a,t) = 0.$$

By using comparison theorem [27] it can be shown that

$$\lim_{t\to\infty} x_2(a,t) = 0, \quad \text{and} \quad \lim_{t\to\infty} x_6(a,t) = 0.$$

Finally, using the relation $x_1(a,t) = 1 - x_2(a,t) - x_3(a,t) - x_4(a,t) - x_5(a,t) - x_6(a,t)$, gives

$$\lim_{t\to\infty} x_1(a,t) = 1.$$

Hence, $\lim_{t\to\infty}(x_1(a,t), x_2(a,t), x_3(a,t), x_4(a,t), x_5(a,t), x_6(a,t)) = (1,0,0,0,0,0)$, as required. □

4. Existence and stability of endemic equilibria

In this section, the existence of endemic equilibria (i.e., equilibria where the infected components of the model are non-zero) of the initial-boundary-value problem {(5), (6)} will be explored. Let,

$$\mathcal{E} = (S^*(a), E^*(a), I^*(a), Q^*(a), H^*(a), R^*(a))$$

represents any arbitrary equilibrium point of the model {(5), (6)} with (20). The associated force of infection (20) can be expressed at steady-state as:

$$\lambda_2^*(a) = \beta_1(a) \int_0^A \beta_2(b)[I^*(b) + \eta H^*(b)]N_\infty(b)db = \beta_1(a)V^*, \tag{36}$$

where,

$$V^* = \int_0^A \beta_2(b)[I^*(b) + \eta H^*(b)]N_\infty(b)db. \tag{37}$$

Solving the equations of the system (5) at steady-state, and noting (36), gives

$$
\begin{aligned}
&S^*(a) = e^{-V^* \int_0^a \beta_1(\tau)d\tau},\\
&E^*(a) = V^* e^{-(\kappa+\sigma)a} \int_0^a \beta_1(u_1) e^{-V^* \int_0^{u_1} \beta_1(\tau)d\tau} e^{(\kappa+\sigma)u_1} du_1,\\
&I^*(a) = V^* \kappa e^{-k_1 a} \int_0^a e^{(k_1-\kappa-\sigma)u_1}\\
&\cdot \int_0^{u_1} \beta_1(u_2) e^{-V^* \int_0^{u_2} \beta_1(\tau)d\tau} e^{(\kappa+\sigma)u_2} du_2 du_1 = V^* M_1(a, V^*),\\
&Q^*(a) = V^* \sigma e^{-\alpha a} \int_0^a e^{(\alpha-\kappa-\sigma)u_1}\\
&\cdot \int_0^{u_1} \beta_1(u_2) e^{-V^* \int_0^{u_2} \beta_1(\tau)d\tau} e^{(\kappa+\sigma)u_2} du_2 du_1 = V^* M_2(a, V^*),\\
&H^*(a) = e^{-\gamma_2 a} \int_0^a e^{\gamma_2 u_1} [\psi I^*(u_1) + \alpha Q^*(u_1)] du_1\\
&= V^* e^{-\gamma_2 a} \int_0^a e^{\gamma_2 u_1} [\psi M_1^*(u_1, V^*) + \alpha M_2(u_1, V^*)] du_1\\
&= V^* M_3(a, V^*),\\
&R^*(a) = \int_0^a \gamma_1 I^*(u_1) + \gamma_2 H^*(u_1) du_1,
\end{aligned}
\tag{38}
$$

where,

$$
\begin{aligned}
M_1(a, V^*) &= \kappa e^{-k_1 a} \int_0^a e^{(k_1-\kappa-\sigma)u_1} \int_0^{u_1} \beta_1(u_2) e^{-V^* \int_0^{u_2} \beta_1(\tau)d\tau} e^{(\kappa+\sigma)u_2} du_2 du_1,\\
M_2(a, V^*) &= \sigma e^{-\alpha a} \int_0^a e^{(\alpha-\kappa-\sigma)u_1} \int_0^{u_1} \beta_1(u_2) e^{-V^* \int_0^{u_2} \beta_1(\tau)d\tau} e^{(\kappa+\sigma)u_2} du_2 du_1,\\
M_3(a, V^*) &= e^{-\gamma_2 a} \int_0^a e^{\gamma_2 u_1} [\psi M_1^*(u_1, V^*) + \alpha M_2(u_1, V^*)] du_1.
\end{aligned}
$$

Substituting the expressions for $I^*(a), Y^*(a)$ and $W^*(a)$ from (38) into (37) gives,

$$
V^* = V^* \int_0^A \beta_2(a)[M_1(a, V^*) + \eta M_3(a, \lambda^*)] N_\infty(a) da. \tag{39}
$$

It follows from equation (39) that the quantity V^* is either 0 (which corresponds to the case where there is no disease in the population) or satisfies the following equation:

$$
1 = \int_0^A \beta_2(a)[M_1(a, V^*) + \eta M_3(a, V^*)] N_\infty(a) da. \tag{40}
$$

Let,

$$
\mathcal{H}(V^*) = \int_0^A \beta_2(a)[M_1(a, V^*) + \eta M_3(a, V^*)] N_\infty(a) da.
$$

It is clear that the function $\mathcal{H}$ satisfies the following:

(i) $\mathcal{H}(0) = \mathcal{R}_0$;

(ii) $\mathcal{H}$ is a monotone decreasing function in V^*;

(iii) $\lim_{V^* \to \infty} \mathcal{H}(V^*) = 0$.

The result below can be established, based on the properties of $\mathcal{H}$ and equation (40):

Theorem 5. *The model* (5) *with* (20) *has a unique endemic equilibrium point (EEP), given by* (38)*, whenever* $\mathcal{R}_0 > 1$.

Following [25], the local stability of the unique endemic equilibrium (38) is now explored as follows.

Let $\hat{S}, \hat{E}, \hat{I}, \hat{Q}, \hat{H}$ and $\hat{V}$ be the perturbations of S^*, E^*, I^*, Y^*, W^* and V^*, respectively. Consider, as before, the exponential solutions:

$$\begin{aligned} &\hat{S}(a,t) = \bar{S}(a)e^{\omega t}, \quad \hat{E}(a,t) = \bar{E}(a)e^{\omega t}, \quad \hat{I}(a,t) = \bar{I}(a)e^{\omega t}, \\ &\hat{Q}(a,t) = \bar{Q}(a)e^{\omega t}, \quad \hat{H}(a,t) = \bar{H}(a)e^{\omega t}, \quad \hat{V} = \bar{V}e^{\omega t} \end{aligned} \tag{41}$$

Hence, using (41) in (5) with (36) gives the following system:

$$\begin{aligned} \frac{d\bar{S}(a)}{da} &= -\omega\bar{S}(a) - \beta_1(a)V^*(a)\bar{S}(a) - \beta_1(a)S^*(a)\bar{V}(a) \\ \frac{d\bar{E}(a)}{da} &= -\omega\bar{E}(a) + \beta_1(a)V^*(a)\bar{S}(a) + \beta_1(a)S^*(a)\bar{V}(a) - (\kappa+\sigma)\bar{E}(a) \\ \frac{d\bar{I}(a)}{da} &= -\omega\bar{I}(a) + \kappa\bar{E}(a) - (\psi+\gamma_1)\bar{I}(a) \\ \frac{d\bar{Q}(a)}{da} &= -\omega\bar{Q}(a) + \sigma\bar{E}(a) - \alpha\bar{Q}(a) \\ \frac{d\bar{H}(a)}{da} &= -\omega\bar{H}(a) + \psi\bar{I}(a) + \alpha\bar{Q}(a) - \gamma_2\bar{H}(a) \end{aligned} \tag{42}$$

with initial conditions $\hat{S}(0) = 0, \hat{E}(0) = 0, \hat{I}(0) = 0, \hat{Y}(0) = 0, \hat{W}(0) = 0$. In (42),

$$\bar{V}(a) = \int_0^A \beta_2(b)[\bar{I}(b) + \eta\bar{H}(b)]N_\infty(b)db. \tag{43}$$

It is convenient to introduce the new variables:

$$y_1(a) = \frac{\bar{S}(a)}{\bar{V}(a)},\ y_2(a) = \frac{\bar{E}(a)}{\bar{V}(a)},\ y_3(a) = \frac{\bar{I}(a)}{\bar{V}(a)},\ y_4(a) = \frac{\bar{Q}(a)}{\bar{V}(a)},\ y_5(a) = \frac{\bar{H}(a)}{\bar{V}(a)}.$$

Using the above variables into the system (42) gives,

$$
\begin{aligned}
\frac{dy_1(a)}{da} &= -\omega y_1 - \beta_1(a)V^* y_1 - \beta_1(a)S^*,\\
\frac{dy_2(a)}{da} &= -\omega y_2 + \beta_1(a)V^* y_1 + \beta_1(a)S^* - (\kappa+\sigma)y_2,\\
\frac{dy_3(a)}{da} &= -\omega y_3 + \kappa y_2 - (\psi+\gamma_1)y_3,\\
\frac{dy_4(a)}{da} &= -\omega y_4 + \sigma y_2 - \alpha y_4,\\
\frac{dy_5(a)}{da} &= -\omega y_5 + \psi y_3 + \alpha y_4 - \gamma_2 y_5
\end{aligned} \tag{44}
$$

and,

$$
1 = \int_0^A \beta_2(a)[y_3(a) + \eta y_5(a)]N_\infty(a)da. \tag{45}
$$

Solving system (44) gives

$$
\begin{aligned}
y_1(a) &= \int_0^a -\beta_1(u)S^*(u)e^{-\omega(a-u)}e^{\int_u^a \beta_1(u_1)V^*(u_1)du_1}du,\\
y_2(a) &= \int_0^a [\beta_1(u)S^*(u) + \beta_1(u)V^*(u)y_1(u)]e^{-(\omega+\kappa+\sigma)(a-u)}du,\\
y_3(a) &= \int_0^a \kappa y_2(u)e^{-(\omega+\psi+\gamma_1)(a-u)}du,\\
y_4(a) &= \int_0^a \sigma y_2(u)e^{-(\omega+\alpha)(a-u)}du,\\
y_5(a) &= \int_0^a [\psi y_3(u) + \alpha y_4(u)]e^{-(\omega+\gamma_2)(a-u)}du.
\end{aligned} \tag{46}
$$

Using the equations in (46) into (45) gives

$$
1 = \int_0^A \beta_2(a)[y_3(a) + \eta y_5(a)]N_\infty(a)da. \tag{47}
$$

Define,

$$
\mathcal{Q}(\omega) = \int_0^A \beta_2(a)[y_3(a) + \eta y_5(a)]N_\infty(a)da. \tag{48}
$$

It should be noted that (from system (46) with (47)) the function $\mathcal{Q}(\omega)$ satisfies the following:

(i) $\mathcal{Q}(\omega)$ is a monotone decreasing function in ω;

(ii) $\lim_{\omega\to\infty} \mathcal{Q}(\omega) = 0$.

Now consider $\mathcal{P}(\omega) = \mathcal{Q}(\omega) - 1$, it is clear that if $\mathcal{Q}(0) < 1$, then $\mathcal{P}(\omega)$ has a negative root. Thus the following result is established.

Theorem 6. *The unique endemic equilibrium of the model* (5) *with* (20) *is LAS whenever* $\mathcal{R}_0 > 1$ *and* $\mathcal{Q}(0) < 1$.

The consequence of Theorem 6 is that the disease will persist (become endemic) in the community whenever $\mathcal{R}_0 > 1$ and $\mathcal{Q}(0) < 1$. In summary, it is shown that the model (5) with (20) has a globally-asymptotically stable DFE in $\mathcal{D}_0$ whenever the associated reproduction number $\mathcal{R}_0$ is less than unity. Furthermore, the model has a unique endemic equilibrium whenever the reproduction number exceeds unity. This equilibrium is shown to be locally-asymptotically stable if another condition holds ($\mathcal{Q}(0) < 1$).

5. Numerical simulation

The age-structured model (2) is now analysed by considering a special case of the model where the natural death rate is constant (i.e., μ is independent of age) and the contact rate ($\beta(a, b)$) is constant (denoted by $\beta(a, b) = \beta$; see also [2, 6, 7]). Consider, now, the model (2) with $\mu(a)$ replaced by the constant μ and $\beta(a, b)$ by a constant β, then it can be shown by following [25] that the model becomes

$$\begin{aligned}
\frac{d\widetilde{S}}{dt} &= \mu N_0 - [\mu + \widetilde{\lambda}(t)]\widetilde{S}(t),\\
\frac{d\widetilde{E}}{dt} &= \widetilde{\lambda}(t)\widetilde{S}(t) - (\mu + \kappa + \sigma)\widetilde{E}(t),\\
\frac{d\widetilde{I}}{dt} &= \kappa\widetilde{E}(t) - (\mu + \psi + \gamma_1)\widetilde{I}(t),\\
\frac{d\widetilde{Q}}{dt} &= \sigma\widetilde{I}(t) - (\mu + \alpha)\widetilde{Q}(t),\\
\frac{d\widetilde{H}}{dt} &= \psi\widetilde{I}(t) + \alpha\widetilde{Q}(t) - (\mu + \gamma_2)\widetilde{H}(t),\\
\frac{d\widetilde{R}}{dt} &= \gamma_1\widetilde{I}(t) + \gamma_2\widetilde{H}(t) - \mu\widetilde{R}(t),
\end{aligned} \tag{49}$$

where,

$$\widetilde{\lambda}(t) = \beta(\widetilde{I} + \eta\widetilde{H}). \tag{50}$$

Define the following positively-invariant region for the model (49):

$$\mathcal{D}_c = \left\{(\widetilde{S}, \widetilde{E}, \widetilde{I}, \widetilde{Q}, \widetilde{H}, \widetilde{R}) \in \mathbb{R}^6_+ : \widetilde{S} + \widetilde{E}, \widetilde{I} + \widetilde{Q} + \widetilde{H} + \widetilde{R} \leq N_0\right\}.$$

The DFE of the model (49) is given by

$$\mathcal{E}_{0c} = (N_0, 0, 0, 0, 0, 0), \tag{51}$$

and the associated reproduction number of the model (49) is given by

$$\mathcal{R}_c = \frac{\beta[\kappa k_3 k_4 + \eta(\psi\kappa k_3 + \alpha\sigma k_2)]}{k_1 k_2 k_3 k_4}, \tag{52}$$

where, $k_1 = \mu + \kappa + \sigma$, $k_2 = \mu + \gamma_1 + \psi$, $k_3 = \mu + \alpha$ and $k_4 = \mu + \gamma_2$. It should be mentioned that since the model (49) is a special case of the model (2), it follows that its DFE, $\mathcal{E}_{0c}$, is globally-asymptotically stable whenever $\mathcal{R}_c < 1$ (in line with Theorem 4).

The model (49) is simulated using the parameter values tabulated in Table 2 (until otherwise stated) to gain insight into its quantitative features.

Numerical simulations for the case $Rc < 1$ depicted in Figure 1 show that the combined use of quarantine and isolation can lead to disease elimination (All solutions converged to the DFE) in line with Theorem (4).

Figure 2 depicts the solution profile of the model for the case $Rc > 1$ showing convergence to the EEP in line with Theorem (6), this mean the disease will persist in this case.

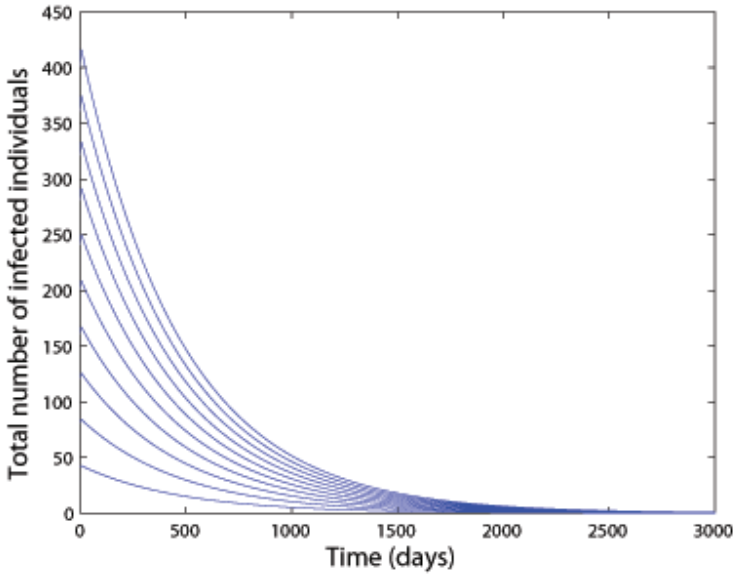

Figure 1: Simulation of the model (49) showing the total number of infected individuals as a function of time for $\mathcal{R}_c < 1$. Parameter values used are as in Table 2 with $\beta = 0.1$ and $\eta = 0.5$ (so that, $\mathcal{R}_c = 0.8065$.)

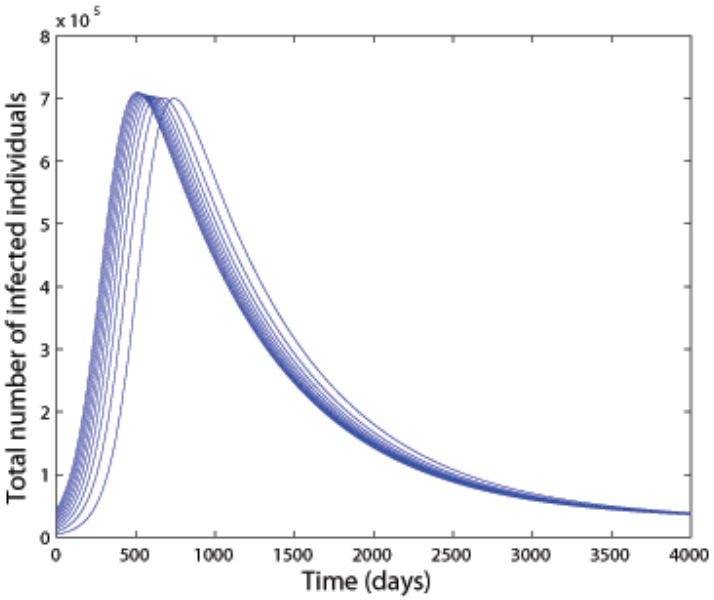

Figure 2: Simulation of the model (49) showing the total number of infected individuals as a function of time for $\mathcal{R}_c > 1$. Parameter values used are as in Table 2 with $\beta = 0.15$ and $\eta = 0.5$ (so that, $\mathcal{R}_c = 1.2097$).

Conclusions

A new age-structured model for disease transmission, subject to the use of quarantine (of asymptomatic cases) and isolation (of individuals with disease symptoms), is presented and rigorously analyzed. The study shows the following:

(i) The model is shown to be properly-posed mathematically, by formulating it as an abstract Cauchy problem;

(ii) It is shown, for the case where the effective contact rate is separable (i.e., $\beta(a, b) = \beta_1(a)\beta_2(b)$), that the disease-free equilibrium of the model is locally- and globally-asymptotically stable whenever a certain epidemiological threshold is less than unity.

(iii) The model has a unique endemic equilibrium when the threshold exceeds unity (this equilibrium is locally-asymptotically stable when another condition holds).

(iv) Numerical simulations show disease elimination whenever its associated reproduction number is less than unity and the disease will persist in this case whenever its associated reproduction number exceeds unity.

Variable	Description
$S(t)$	Population of susceptible individuals
$E(t)$	Population of exposed individuals
$I(t)$	Population of infectious (symptomatic) individuals
$Q(t)$	Population of quarantined individuals
$H(t)$	Population of hospitalized individuals
$R(t)$	Population of recovered individuals

Parameter	Description
$\mu(a)$	Age-dependent natural death rate
$\beta(a,b)$	Age-dependent Effective contact rate
η	Modification parameter for reduction in infectiousness of hospitalized individuals
κ	Progression rate from exposed to infectious class
σ	Quarantine rate of exposed individuals
α	Hospitalization rate of quarantined individuals
ϕ	Hospitalization rate of infectious individuals
γ_1	Recovery rate of infectious individuals
γ_2	Recovery rate of hospitalized individuals

Table 1: Description of variables and parameters of the model (2).

Parameters	Values (per day)
β	[0.1, 0.2]
μ	0.0000351
γ_1	0.03521
γ_2	0.042553
δ_1	0.04227
δ_2	0.027855
κ	0.156986
α	0.156986
ϕ	0.20619
Π	136
σ	0.1
ψ	0.5
η	(0,1]

Table 2: Estimated values for the parameters of the model (49).

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