SOME PROPERTIES OF ZERO GRADATIONS ON SAMANTA FUZZY TOPOLOGICAL SPACES

Mohammad Abry*
School of Mathematics and Computer Science
University of Damghan
Postal Code 3671641167, Damghan
Iran
mabry@du.ac.ir

Jafar Zanjani
School of Mathematics and Computer Science
University of Damghan
Postal Code 3671641167, Damghan
Iran
j_zanjani@std.du.ac.ir
pjefa@yahoo.com

Abstract. Considering the fuzzy topological spaces in the sense of Samanta, the notion of a zero gradation as a fuzzy topological invariant is introduced that might be the first basic step to develop a theory of dimension on the fuzzy topological spaces. Also, some critical properties and applications are established.
Keywords: fuzzy topology, gradation of openness, zero gradation.

1. Introduction

In a recent paper [3], we investigated the zero dimensionality of fuzzy topological spaces in the sense of Lowen and showed how the concept might be sensitive to the choice of definition of fuzzy topology. In this paper and with almost the same purpose, we consider an important modification of definition of a fuzzy topology that has many applications in many branches of science and technology. It has been formulated with respect to the concept of the gradation of openness that originally appeared in a paper by Samanta et al. [12]. Our main goal is to introduce a concept of zero dimensionality so that some required initial properties of a dimension are satisfied while being compatible with the previous relative notions.

The notion of fuzzy topological space was first introduced by Chang [11] over the system of fuzzy sets proposed by Zadeh [22]. In Chang's definition, constant functions between fuzzy topological spaces were not necessarily continuous. Lowen [18] and Hutton [17] have given different ideas for the definition of fuzzy topology. Samanta et al. [12] introduced the concept of fuzzy topology

*. Corresponding author

by the notion of gradation of openness as a function $\tau : \mathbb{I}^X \to \mathbb{I}$ satisfying certain axioms. Dimension of fuzzy topological spaces has been studied by several researchers [4], [5], [6], [7], [9], [13]. The subject, however, is still dealing with major obstacles such as lack of a satisfactory concept of a boundary as well as comprehensive definition of a dimension.

In this paper, we introduce the concept of zero gradation which in addition to a natural candida for zero dimensionality of a gradation might be helpful for the characterization of some kind of disconnectedness of fuzzy topological spaces. We prove that the property of being zero gradation is an invariant by strongly gradation preserving maps, and so it is a fuzzy topological invariant. Some examples of zero and nonzero gradations are also presented.

2. Preliminaries

We give below some basic preliminaries required for this paper.

Definition 2.1 ([22])**.** A *fuzzy set* μ in a nonempty set X is defined as a map $\mu : X \to \mathbb{I}$, where $\mathbb{I} = [0, 1]$. The family of all fuzzy sets on X is denoted by $\mathbb{I}^X$. A fuzzy set μ is said to be *contained in* a fuzzy set η if $\mu(x) \leq \eta(x)$ for each x in X, denoted by $\mu \leq \eta$. The *union* and *intersection* of a family of fuzzy sets is defined by $\bigvee \mu_\alpha = \sup(\mu_\alpha)$ and $\bigwedge \mu_\alpha = \inf(\mu_\alpha)$, respectively.

Definition 2.2. For every $x \in X$ and every $\alpha \in (0, 1)$, the fuzzy set x_α with membership function

$$x_\alpha(y) = \begin{cases} \alpha, & y = x, \\ 0, & y \neq x, \end{cases}$$

is called a *fuzzy point.* x_α is said to be *contained in* a fuzzy set μ, denoted by $x_\alpha \in \mu$, if $\alpha < \mu(x)$ [21].

We omit the case $\alpha = \mu(x)$ in the above definition for some technical reasons. For instance, every second countable fail to be first countable if we allow the equality. Anyway, any fuzzy set is the union of all points which are contained in it and for every two fuzzy sets μ and ν we have $\mu \leq \nu$ if and only if $x_\alpha \in \mu$ implies $x_\alpha \in \nu$, for every fuzzy point x_α. Any point x_1 is called a crisp point. We denote a constant fuzzy set whose unique value is $c \in [0, 1]$ by c_X.

Definition 2.3 ([18])**.** A *Lowen fuzzy topology* is a family δ of fuzzy sets in X which satisfies the following conditions:

(i) $\forall c \in \mathbb{I}$, $c_X \in \delta$,

(ii) $\forall \mu, \nu \in \delta \Rightarrow \mu \wedge \nu \in \delta$,

(iii) $\forall (\mu_j)_{j \in J} \subset \delta \Rightarrow \bigvee_{j \in J} \mu_j \in \delta$.

The pair (X, δ) is called a *Lowen fuzzy topological space*. Open fuzzy sets, closed fuzzy sets and fuzzy clopens are defined as usual. In Chang's definition of fuzzy topology, condition (i) should be replaced by $(i)'$ $0, 1 \in \delta$. A base or subbase for a fuzzy space have the same meaning in the classic sense.

It should be noted that the concept of the boundary of a fuzzy set is essential in the definition of inductive dimension. Cuchillo and Tarres [14] proposed a definition of fuzzy boundary of a fuzzy set and we use their definition throughout this paper.

Definition 2.4 ([14]). Let μ be a fuzzy set in a Lowen fuzzy topological space X. The *fuzzy boundary* of μ, denoted by $\mathrm{Fr}(\mu)$, is defined as the infimum of all closed fuzzy sets σ in X with the property $\sigma(x) \geq \overline{\mu}(x)$ for all $x \in X$ for which $\overline{\mu}(x) - \mu^\circ(x) > 0$.

It is ready to see that a fuzzy set μ is clopen if and only if $\mathrm{Fr}(\mu) = 0_{\mathrm{X}}$. The concept of zero gradation is based on the notion of zero dimensionality of Lowen fuzzy topological spaces. Adnadjevic [5], [6] defined two dimension functions $F - \mathrm{ind}$, $F - \mathrm{Ind}$ for the generalized fuzzy spaces. If the definition of the Adnadjevic's dimension function is particularized, in the case of zero dimensionality, the following definition is obtained.

Definition 2.5 ([14]). A Lowen fuzzy topological space (X, δ) is called *zero-dimensional* and it is denoted by $\mathrm{ind(X)} = 0$ if for each fuzzy point x_α in X and every open fuzzy set μ containing x_α, there exists an open fuzzy set σ in X with $\mathrm{Fr}(\sigma) = 0_{\mathrm{X}}$ such that $x_\alpha \in \sigma \leq \mu$.

Example 2.6 ([13]). Let δ be a Lowen fuzzy topology on $X = [0, 1]$ with subbase

$$\{c_X : c \in \mathbb{I}\} \cup \{\mu\},$$

where

$$\mu(x) = \begin{cases} \frac{1}{2}, & 0 \leq x \leq \frac{1}{3}, \\ 0, & \frac{1}{3} < x \leq 1. \end{cases}$$

Clearly any non-constant open fuzzy sets has the form

$$\nu(x) = \begin{cases} a, & 0 \leq x \leq \frac{1}{3}, \\ b, & \frac{1}{3} < x \leq 1, \end{cases} \quad \frac{1}{2} \geq a > b \geq 0.$$

There exists no clopen fuzzy set σ such that $\sigma \leq \mu$ because the constant fuzzy sets are the only clopen fuzzy sets. Thus $\mathrm{ind(X)} \neq 0$.

It is ready to see that for every non empty set X the fuzzy topological space (X, δ) is zero-dimensional, where $\delta = \{c_X : c \in \mathbb{I}\}$.

Remark 2.7 ([20])**.** Let (X, δ) be a Lowen fuzzy topological space and $Y \subset X$, then the family $\delta_1 = \{\mu|_Y : \mu \in \delta\}$ is a Lowen fuzzy topology for Y and (Y, δ_1) is called a subspace of (X, δ). If ind(X) = 0, then ind(Y) = 0. Note that restriction of every clopen fuzzy subset of X on subspace Y is a clopen fuzzy set in Y.

3. Zero gradations

In Lowen and Chang's definition of fuzzy topology, fuzziness in the concept of openness of a fuzzy subset has not been considered. The initial request is that the topology be a fuzzy subset of a power set of fuzzy subsets. For this purpose, Samanta et al. gave an axiomatic definition in [12] so called a gradation of openness. Here we use the following modified definition.

Definition 3.1. A fuzzy topology is a mapping $\tau : \mathbb{I}^X \to \mathbb{I}$ which satisfies the following conditions:

(i) $\tau(c_X) = 1$, for all $c \in \mathbb{I}$,

(ii) $\tau(\mu_1 \wedge \mu_2) \geq \tau(\mu_1) \wedge \tau(\mu_2)$,

(iii) $\tau(\bigvee_i \mu_i) \geq \bigwedge_i \tau(\mu_i)$.

The mapping τ is called a *gradation of openness* on X. The real number $\tau(\mu)$ is the degree of openness of the fuzzy subset $\mu \in \mathbb{I}^X$ and may range from 0 "completely non-open set" to 1 "completely open set". The pair (X, τ) is called a *fuzzy topological space*. In Samanta's definition of fuzzy topology the condition (i) should be replaced by $\tau(\bar{0}) = \tau(\bar{1}) = 1$. Suppose that τ_1 and τ_2 are two fuzzy topologies on a given set X. If $\tau_1 \supset \tau_2$ i.e., $\tau_1(\mu) \geq \tau_2(\mu)$ for every $\mu \in \mathbb{I}^X$, we say that τ_1 is *finer* than τ_2.

Example 3.2. Let $X = \mathbb{I}$ with the Lowen fuzzy topology generated by the subbasis $\{\eta, c_X\}$, where η is the fuzzy set defined by

$$\eta(x) = \begin{cases} 0, & 0 \leq x \leq \frac{1}{2}, \\ \frac{1}{2}, & \frac{1}{2} < x \leq 1. \end{cases}$$

It is easy to see that the non-constant open fuzzy sets are

$$\eta_{a,b}(x) = \begin{cases} a, & 0 \leq x \leq \frac{1}{2}, \\ b, & \frac{1}{2} < x \leq 1. \end{cases}$$

where $0 \leq a \leq b \leq \frac{1}{2}$. define $\tau : \mathbb{I}^X \to \mathbb{I}$ by

$$\tau(\mu) = \begin{cases} 1, & \mu = c_X, \\ \frac{1}{2}, & \mu = \eta_{a,b}, \\ 0, & \text{otherwise}. \end{cases}$$

Then τ is a gradation of openness on X.

Definition 3.3. The mapping $\xi : \mathbb{I}^X \to \mathbb{I}$ is called a *gradation of closedness* on X if it satisfies:

(i) $\xi(c_X) = 1$, for all $c \in \mathbb{I}$,

(ii) $\xi(\mu_1 \vee \mu_2) \geq \xi(\mu_1) \wedge \xi(\mu_2)$,

(iii) $\xi(\bigwedge_i \mu_i) \geq \bigwedge_i \xi(\mu_i)$.

The mapping $\tau : \mathbb{I}^X \to \mathbb{I}$ is called a *gradation of clopenness* on X if it is a gradation of openness and a gradation of closedness on X. Indeed, the mapping τ is a gradation of clopenness on X if and only if it satisfies:

(i) $\tau(c_X) = 1$, for all $c \in \mathbb{I}$,

(ii) $\tau(\mu_i) \geq r \Rightarrow \tau(\bigwedge_i \mu_i) \geq r$, $\tau(\bigvee_i \mu_i) \geq r$. where $r \in (0, 1]$

For example, the mapping $\tau : \mathbb{I}^X \to \mathbb{I}$ is defined by

$$\tau(\mu) = \begin{cases} 1, & \mu = c_X, \\ k, & \text{otherwise.} \end{cases} \; ; \; k \in [0, 1].$$

is a gradation of clopenness that is called *constant gradation*. Especially for $k = 0$ the gradation of clopenness τ is denoted by τ_0.

Remark 3.4. If τ is a gradation of openness (closedness) on X, then the mapping $\tau' : \mathbb{I}^X \to \mathbb{I}$ given by $\tau'(\mu) = \tau(\mu^c)$ will be a gradation of closedness (openness) on X that is called *conjugate gradation* of τ. Therefore, if τ is a gradation of clopenness on X then its conjugate is so.

Proposition 3.5. *Let τ be a gradation of openness on X. Then for each $r \in \mathbb{I}$, $\tau_r = \{\mu \in \mathbb{I}^X : \tau(\mu) \geq r\}$ is a Lowen fuzzy topology on X that is called the r-level fuzzy topology on X with respect to the gradation of openness τ.*

Proof. The first condition for a Lowen fuzzy topology is easy: Since $\tau(c_X) = 1 \geq r$, so $c_X \in \tau_r$. To check the second condition, let μ_1 and μ_2 be two fuzzy sets in τ_r. Hence $\tau(\mu_1) \geq r$ and $\tau(\mu_2) \geq r$. Since τ is a gradation of openness, so $\tau(\mu_1 \wedge \mu_2) \geq \tau(\mu_1) \wedge \tau(\mu_2) \geq r$. Therefore, $\mu_1 \wedge \mu_2 \in \tau_r$. To check the third condition, let $(\mu_i)_i$ be a family of fuzzy sets in τ_r such that $\tau(\mu_i) \geq r$. By *(iii)* of definition 3.1 $\tau(\bigvee_i \mu_i) \geq \bigwedge_i \tau(\mu_i) \geq r$. Hence, $\bigvee_i \mu_i \in \tau_r$. □

It is ready to see that the family of all r-level fuzzy topologies with respect to τ is a descending family and for each $r \in (0, 1]$, $\tau_r = \bigcap_{s<r} \tau_s$.

Proposition 3.6. *Let (X, T) be a Lowen fuzzy topological space. Define for each $r \in (0, 1]$, a mapping $T^r : \mathbb{I}^X \to \mathbb{I}$ by*

$$T^r(\mu) = \begin{cases} 1, & \mu = c_X, \\ r, & \mu \in T,\ \mu \neq c_X, \\ 0, & \textit{otherwise.} \end{cases}$$

Then T^r is a gradation of openness on X such that $(T^r)_r = T$.

Proof. First notice that $T^r(c_X) = 1$. Let $\mu_1,\ \mu_2 \in \mathbb{I}^X$. Then we have the following possibilities: (i) $\mu_1,\ \mu_2 \notin T$, (ii) $\mu_1 \in T,\ \mu_2 \notin T$, (iii) $\mu_1,\ \mu_2 \in T$. In the case (i), $T^r(\mu_1) = T^r(\mu_2) = 0$, and hence $T^r(\mu_1 \wedge \mu_2) \geq 0 = T^r(\mu_1) \wedge T^r(\mu_2)$. In the case (ii), $T^r(\mu_1) = r$ and $T^r(\mu_2) = 0$. Then $T^r(\mu_1 \wedge \mu_2) \geq 0 = T^r(\mu_1) \wedge T^r(\mu_2)$. In the case (iii), $T^r(\mu_1) = T^r(\mu_2) = r$ and by hypothesis $\mu_1 \wedge \mu_2 \in T$, and hence $T^r(\mu_1 \wedge \mu_2) \geq r = T^r(\mu_1) \wedge T^r(\mu_2)$. Therefore, $T^r(\mu_1 \wedge \mu_2) \geq T^r(\mu_1) \wedge T^r(\mu_2)$. Now let $(\mu_i)_i$ be a family of fuzzy sets in $\mathbb{I}^X$. Thus, $T^r(\bigvee_i \mu_i) = 1$ *or* r which is greater than or equal to $\bigwedge_i T^r(\mu_i) = r$ *or* 0. □

The mapping T^r is called the r-th gradation on X and (X, T^r) is called the r-th graded fuzzy topological space. It should be noted that the family $\{(T^r)_s :\ 0 < s \leq 1\}$ is a descending family and for each $s \in (0, 1]$, $(T^r)_s = \bigcap_{t<s}(T^r)_t$. Indeed, if $s \geq t$, then obviously, $(T^r)_s \subset (T^r)_t$. Hence, $\{(T^r)_s :\ 0 < s \leq 1\}$ is a descending family. Clearly $(T^r)_s \subset \bigcap_{t<s}(T^r)_t\ \ , 0 < s \leq 1$. Also, if $\mu \notin (T^r)_s$, then $(T^r)(\mu) < s$ which implies $\exists t \in (0, 1]$ such that $(T^r)(\mu) < t < s$. So, $\mu \notin (T^r)_t$ for some $t < s$. Hence, $\mu \notin \bigcap_{t<s}(T^r)_t$. Consequently $\bigcap_{t<s}(T^r)_t \subset (T^r)_s$.

Definition 3.7. Let τ be a gradation of openness on X and $Y \subset X$. Define a mapping $\tau_Y : \mathbb{I}^Y \to \mathbb{I}$ by

$$\tau_Y(\mu) = \bigvee\{\tau(\eta) : \eta \in \mathbb{I}^X, \eta|_Y = \mu\}.$$

Then τ_Y is a gradation of openness on Y. We say that τ_Y is a *subgradation* of τ. It is obvious that $\tau_Y(\mu) \geq \tau(\mu_X)$.

Definition 3.8. Let τ_1 and τ_2 be two gradations of openness on X and Y, respectively, and $f : X \to Y$ be a mapping. Then f is called a *gradation preserving map* (*strongly gradation preserving map*) if for each $\mu \in \mathbb{I}^Y$, $\tau_2(\mu) \leq \tau_1(f^{-1}(\mu))$ $(\tau_2(\mu) = \tau_1(f^{-1}(\mu)))$.

For example, let τ_1 and τ_2 be two fuzzy topologies on a given set X such that τ_1 is finer than τ_2. Thus, the identity map $i : (X, \tau_1) \to (X, \tau_2)$ is a gradation preserving map.

Definition 3.9. Two gradations of openness σ and τ on X are called *equal* and it is denoted by $\sigma \approx \tau$ if the identity map is a strongly gradation preserving map from (X, τ) to (X, σ). In this case $\sigma_r = \tau_r$ for all $r \in \mathbb{I}$.

Proposition 3.10. *Let τ_1 and τ_2 be two gradations of openness on X and Y, respectively, and $f : X \to Y$ be a mapping. Then f is a gradition preserving map if and only if $f : (X, (\tau_1)_r) \to (Y, (\tau_2)_r)$ is continuous with respect to r-level fuzzy topology for all $r \in (0, 1]$.*

Proof. Suppose f is a gradation preserving map and $\mu \in (\tau_2)_r$. Then $\tau_1(f^{-1}(\mu)) \geq \tau_2(\mu) \geq r$. Thus $f^{-1}(\mu) \in (\tau_1)_r$. Conversely, suppose $f : (X, (\tau_1)_r) \to (Y, (\tau_2)_r)$ is continuous with respect to r-level fuzzy topology for all $r \in (0, 1]$. Choose $\mu \in \mathbb{I}^Y$. If $\tau_2(\mu) = 0$, then $\tau_1(f^{-1}(\mu)) \geq \tau_2(\mu)$. If $\tau_2(\mu) = r$, so $\mu \in (\tau_2)_r$. Now by countinuity of f it follows that $f^{-1}(\mu) \in (\tau_1)_r$. Hence $\tau_1(f^{-1}(\mu)) \geq r = \tau_2(\mu)$. □

Applications of fuzzy topology and gradation concepts have already been done by S. K. Samanta and others in knowledge engineering, natural language, neural network etc [8], [10]. In these studies, the r-level fuzzy topologies play a major role. In the sequel, we introduce a concept of zero gradation and prove that the property of being zero gradation is invariant under strongly gradation preserving maps. Some examples of zero and nonzero gradations are also presented.

Definition 3.11. Let τ be a gradation of openness on X. Then τ is called a *zero gradation* if the fuzzy topological space (X, τ_r) is zero-dimensional for all $r \in (0, 1]$.

Consider the fuzzy topological space (X, τ_0). Then the gradation of openness τ_0 is a zero gradation. Note that the r-level fuzzy topology on X with respect to the gradation of openness τ_0 is

$$(\tau_0)_r = \{\mu \in \mathbb{I}^X : \tau_0(\mu) \geq r\} = \{c_X : c \in \mathbb{I}\},$$

for all $r \in (0, 1]$. It is ready to see that the fuzzy topological spaces $(\tau_0)_r$ is zero-dimensional for all $r \in (0, 1]$.

Example 3.12. Let $(\mathbb{I}, \tau)$ be a fuzzy topological space. If the gradation of openness τ be the same as Example 3.2, then τ is not a zero gradation because the fuzzy topological space $(\mathbb{I}, \tau_{\frac{1}{2}})$ is not zero-dimensional. Take the fuzzy point $x_{\frac{1}{3}}$ where $x \in (\frac{1}{2}, 1]$. we have $x_{\frac{1}{3}} \in \eta$ and there exists no clopen fuzzy set $\sigma \in \tau_{\frac{1}{2}}$ such that $x_{\frac{1}{3}} \in \sigma \leq \eta$ since the constant fuzzy sets are the only clopen fuzzy sets. Thus $\mathrm{ind}(\mathbb{I}, \tau_{\frac{1}{2}}) \neq 0$.

It is ready to see that if τ is a zero gradation on X such that $\tau(\mu) = \tau(\mu^c)$ for all $\mu \in \mathbb{I}^X$, then its conjugate τ' is a zero gradation too.

Theorem 3.13. *Let τ be a zero gradation on X. If $Y \subset X$ and τ_Y is a subgradation of τ. Then τ_Y is a zero gradation.*

Proof. It suffices to prove that the Lowen fuzzy topological space $(Y, (\tau_Y)_r)$ is zero-dimensional for all $r \in (0, 1]$. Choose an arbitrary fuzzy point y_α in Y and let μ be an open fuzzy set containing y_α in the Lowen fuzzy topology $(\tau_Y)_r$. Note that $\tau_Y(\mu) \geq r$. Now we consider the fuzzy point y_α as a fuzzy point in X. It is obvious that there exists an open fuzzy set μ' in τ_r such that $\tau(\mu') \geq r$ and $\mu'|_Y = \mu$. Since the Lowen fuzzy topological space (X, τ_r) is zero-dimensional for all $r \in (0, 1]$, there exists an open fuzzy set η' such that $\mathrm{Fr}(\eta') = 0_{\mathrm{X}}$ and $y_\alpha \in \eta' \leq \mu'$. Now we put $\eta = \eta'|_Y$, then $y_\alpha \in \eta \leq \mu$. $\square$

Remark 3.14. Let $\{T_r\}$ be a non-empty descending family of Lowen fuzzy topologies on X such that $T_r = \bigcap_{s<r} T_s$ for all $r \in (0, 1]$. It is easy to verify that the mapping $\tau : \mathbb{I}^X \to \mathbb{I}$ defined as

$$\tau(\mu) = \bigvee \{r \in (0, 1] : \mu \in T_r\}$$

is a fuzzy topology on X. If T_r is a zero-dimensional space for all $r \in (0,1]$, the mapping τ will be a zero gradation since $\tau_r = T_r$ for all $r \in (0,1]$.

Proposition 3.15. *The infimum of any family of zero gradations on a given set X is a zero gradation.*

Proof. Let $\{\tau_i\}_i$ be an arbitrary family of zero gradations on X. One can readily check that $\bigwedge_i \tau_i$ is a fuzzy topology on X. Because the r-level Lowen fuzzy topology $(\bigwedge_i \tau_i)_r = \{\mu \in \mathbb{I}^X : \bigwedge_i \tau_i(\mu) \geq r\}$ is a subspace of $(\tau_i)_r$, for all $r \in (0,1]$, we conclude that $\bigwedge_i \tau_i$ is a zero gradation by Remark 2.7. $\square$

Note that the converse of the Proposition 3.15 is not necessarily hold. Choose two fuzzy topologies τ_0 and τ on $\mathbb{I}$, where τ is the same as Example 3.2. Then $\tau_0 \wedge \tau$ is a zero gradation, while by Example 3.12, τ is not zero gradation.

Theorem 3.16. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a bijective strongly gradation preserving map. If τ_1 is a zero gradation, then τ_2 is a zero gradation.*

Proof. We show that Lowen space $(Y, (\tau_2)_r)$ is zero dimensional for all $r \in (0,1]$. Choose fuzzy subset μ containing fuzzy point y_α with $\tau_2(\mu) \geq r$, for all $r \in (0,1]$. $f^{-1}(y_\alpha)$ is the fuzzy point $(f^{-1}(y))_\alpha$ in X and $f^{-1}(\mu)$ is a open fuzzy set in X respect to $(\tau_1)_r$. Note that $\tau_1(f^{-1}(\mu)) = \tau_2(\mu) \geq r$. According to zero dimensionality of $(X, (\tau_1)_r)$ there is a clopen fuzzy set λ' in $(\tau_1)_r$ such that $(f^{-1}(y))_\alpha \leq \lambda' \leq f^{-1}(\mu)$. Now we put $\lambda = f(\lambda')$, Thus $y_\alpha \in \lambda \leq \mu$. $\square$

The condition of being strongly for the gradation preserving map f in the Theorem 3.16 is essential. Consider the identity map $i : (X, \tau_1) \to (X, \tau_2)$, where $X = \mathbb{I}$, τ_1 is the constant gradation (when $k = 1$) and the gradation of openness τ_2 is the same as Example 3.2. $\tau_2(\eta) \neq \tau_1(\eta)$, then the identity map i is not a strongly gradation preserving map. Note that τ_1 is a zero gradation and τ_2 is not a zero gradation.

Proposition 3.17. *Let τ be a zero gradation on X and $\sigma \approx \tau$, then σ is a zero gradation.*

Proof. The proof is obvious since $\{\sigma_r : r \in [0,1]\} = \{\tau_r : r \in [0,1]\}$. $\square$

Corollary 3.18. *If σ is a zero gradation, then two gradations of openness $\tau.\sigma$ and $\phi \circ \sigma$ are zero gradation because $\tau.\sigma \approx \sigma$ and $\phi \circ \sigma \approx \sigma$, where τ is the constant gradation on X $(k \neq 0)$ and $\phi : \mathbb{I} \to \mathbb{I}$ is any strictly increasing continuous function with $\phi(1) = 1$.*

Let π_1 and π_2 be two projection maps on $X \times Y$. Then the fuzzy product of two fuzzy space (X, τ) and (Y, σ) is denoted by $(X \times Y, \tau \times \sigma)$ where $\nu = \tau \times \sigma$ is defined as follows

$$\nu(C) = \begin{cases} \tau(A)(\sigma(B)), & C \in K_1 (C \in K_2) \\ \nu(C_1) \wedge \nu(C_2), & C = C_1 \wedge C_2,\ (C_1, C_2 \in K_1 \cup K_2) \\ \bigvee \nu(C_i), & C = \bigvee C_i,\ (C_i \in K') \\ 0, & \text{otherwise,} \end{cases}$$

where $K_1 = \{A\pi_1 : \tau(A) > 0\}, K_2 = \{B\pi_2 : \sigma(B) > 0\}$ and K' is the set of finite intersections of the elements in $K_1 \cup K_2$. One may easily verify the following.

Theorem 3.19. *The product of two zero gradation is a zero gradation.*

Proposition 3.20. *Let τ and σ be two zero gradations on two disjoint sets X and Y, recpectively . Then there is a zero gradation on $X \cup Y$ such that τ and σ are two subgradations of it.*

Proof. Let τ and σ be two gradations of openness on X and Y, respectively. Put $Z = X \cup Y$ and for each fuzzy set μ on Z, Define

$$\delta(\mu) = \min\{\tau(\mu|_X),\ \sigma(\mu|_Y)\}.$$

It is ready to see that δ is a gradation of openness on Z. To check out that δ is a zero gradation, We must verify the r-levels of δ

$$\begin{aligned}\delta_r &= \{\mu \in \mathbb{I}^Z :\ \delta(\mu) \geq r\} = \{\mu \in \mathbb{I}^Z :\ \min\{\tau(\mu|_X),\ \sigma(\mu|_Y)\} \geq r\} \\ &= \{\mu \in \mathbb{I}^Z :\ \tau(\mu|_X) \geq r\} \cap \{\mu \in \mathbb{I}^Z :\ \sigma(\mu|_Y) \geq r\}.\end{aligned}$$

Let z_α be an arbitrary fuzzy point in δ_r and, say, $z \in X$. Let μ be an open fuzzy set containing z_α in the Lowen fuzzy topological space δ_r . So $\tau(\mu|_X) \geq r$. τ is a zero gradation and hence there exists a fuzzy clopen set η in τ_r such that $z_\alpha \in \eta \leq \mu|_X$. Now let η' be a fuzzy set on Z such that $\eta'|_X = \eta$ and $\eta'|_Y = 0$. Then it is ready to see that η' is a clopen fuzzy set in δ_r and $z_\alpha \in \eta' \leq \mu$. Thus, δ is a zero gradation. □

Note that there may be a gradation of openness δ on the space $X \cup Y$, such that δ is not a zero gradation, But the restriction of δ on X and Y is a zero gradation. For example, consider the gradation of openness τ in the Example 3.12 on the $X = [0, \frac{1}{2})$ and $Y = [\frac{1}{2}, 1]$.

A Lowen fuzzy topological space is called almost zero-dimensional if it is T_1 and for every fuzzy point x_α and every open fuzzy set μ contains x_α there exists a fuzzy neighborhood ν of x_α with $x_\alpha \in \nu \leq \mu$ such that ν is an intersection of clopen fuzzy sets [3].

Definition 3.21. A gradation of openness τ on X is called *almost zero gradation* if the fuzzy topological space (X, τ_r) is almost zero-dimensional for all $r \in (0, 1]$.

The following proposition is a direct consequence of [3, Proposition 3.19]

Proposition 3.22. *Let τ be an almost zero gradation on X. If τ_r has a countable base for all $r \in (0, 1]$ then τ is a zero gradation.*

We constructed a fuzzy topological space X in [3, Example 3.21] which mainly use the curious properties of the famous Erdős space as most important example of an almost zero-dimensional space, see [1], [2], [15], [16], [19] for the more information and generalization of the original concept. As an application if we define $\tau : \mathbb{I}^X \to \mathbb{I}$ by

$$\tau(\mu) = \begin{cases} 1, & \mu \in \delta, \\ 0, & \mu \in \mathbb{I}^X \setminus \delta \end{cases}$$

then τ is a zero gradation on X.

Remark 3.23. Similar to crisp case there is no relation between being finer and zero gradation. Consider three gradations τ_0, τ_1 and τ, where τ_1 is the constant gradation (when $k = 1$) and the gradation of openness τ is the same as Example 3.2. It is obvious that $\tau_1 \supset \tau \supset \tau_0$. Note that τ_0 and τ_1 are zero gradation and τ is not a zero gradation.

References

[1] M. Abry, J.J. Dijkstra, *On topological Kadec norms*, Math. Ann., 332 (2005), 759–765.

[2] M. Abry, J.J. Dijkstra, *Universal spaces for almost n-dimensionality*, Proc. Amer. Math. Soc., 135 (2007), 2623-2628.

[3] M. Abry, J. Zanjani, *On zero-dimensional fuzzy spaces*, Ann. Fuzzy Math. Inform., 10 (2)(2015), 299-308.

[4] M. Abry, J. Zanjani, *An inductive fuzzy dimension*, Journal of Algebraic Systems, 5 (1)(2017), 15-25.

[5] D. Adnadjevic, *Dimension of fuzzy spaces*, Fuzzy Sets and Systems, 26 (1988), 85-92.

[6] D. Adnadjevic, *Some properties of the dimension F-Ind of GF-Spaces*, Fuzzy Sets and Systems, 54 (1993), 221-224.

[7] N. Ajmal, J.K. Kohli, *Zero dimensional and strongly zero dimensional fuzzy topological spaces*, Fuzzy Sets and Systems, 61 (1994), 231-237.

[8] K. Atanassov, S. Sotirov, V. Kodogiannis, *Intuitionistic fuzzy estimations of the Wi-Fi connections*, In: First Int. Workshop on Generalized Nets, Intuitionistic Fuzzy Sets and Knowledge Engineering, London, September 6-7, (2006), 75-80.

[9] T. Baiju, S.J. John, *Covering dimension and normality in L-topological spaces*, Mathematical Sciences, 6:35 (2012), 1-6.

[10] A. Bassiri, A.A. Alesheikh, M.R. Malek, *Spatio-temporal object modeling in fuzzy topological space*, The International Archives of the Photogrammetry, Remote Sensing and Spatial Information Sciences, Vol. XXXVII. Part B2. Beijing, 2008.

[11] C.L. Chang, *Fuzzy topological spaces*, J. Math. Anal. Appl., 24 (1968), 182–190.

[12] K.C. Chattopadhyay, R.N. Hazra, S.K. Samanta, *Gradation of openness; Fuzzy topology*, Fuzzy Sets and Systems, 49 (1992), 237-242.

[13] E. Cuchillo-Ibanez, J. Tarres, *On zero dimensionality in fuzzy topological spaces*, Fuzzy Sets and Systems, 82 (1996), 361-367.

[14] E. Cuchillo-Ibanez, J. Tarres, *On the boundary of fuzzy sets,* Fuzzy Sets and Systems, 89(1997), 113-119.

[15] J.J. Dijkstra, J.V. Mill, *Characterizing complete Erdős space*, Canad. J. Math., 61 (2009), 124-140.

[16] P. Erdős, *The dimension of the rational points in Hilbert space*, Ann. of Math., 41 (1940), 734-736.

[17] B. Hutton, *Products of fuzzy topological spaces*, Top. Appl., 11 (1980), 59–67.

[18] R. Lowen, *Fuzzy topological spaces and fuzzy compactness*, J. Math. Anal. Appl., 56 (1976), 621–633.

[19] L.G. Oversteegen, E.D. Tymchatyn, *On the dimension of certain totally disconnected spaces*, Proc. Amer. Math. Soc., 122 (1994), 885-891.

[20] N. Palaniappan, *Fuzzy Topology*, Narosa Publishing House, New Delhi, 2002.

[21] Srivastava, R., Lal, S. N., Srivastava, A. K. *Fuzzy Hausdorff topological spaces*, J. Math. Anal. Appl., 81 (1981), 497-506.

[22] L.A. Zadeh, *Fuzzy sets*, Information and control, 8(1965), 338–353.

Accepted: 19.02.2017