A STUDY ON PSEUDOORDERS IN ORDERED $*$-SEMIHYPERGROUPS

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Abstract. In this paper, we study the pseudoorders on ordered $*$-semihypergroups in detail. To begin with, we introduce the concept of pseudoorders on an ordered $*$-semihypergroup, and investigate its related properties. Furthermore, the relationship between strongly regular equivalence relations and pseudoorders on an ordered $*$-semihypergroup is established, and some homomorphism theorems of ordered $*$-semihypergroups by pseudoorders are given. Finally, we investigate the direct product of ordered $*$-semihypergroups, and study the pseudoorders on direct product of ordered $*$-semihypergroups.

Keywords: ordered $*$-semihypergroup, pseudoorder, strongly regular equivalence relation, homomorphism, direct product.

1. Introduction

As we know, hyperstructure theory was investigated in 1934, when Marty [17] defined hypergroups, began to analyze their properties and applied them to groups. In the past several decades, a number of different hyperstructures have been widely studied from the theoretical point of view and for their applications to many subjects of pure and applied mathematics by many authors, for example, see [3, 4, 7, 18, 21]. In particular, a semihypergroup is a classic ex-

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ample of algebraic hyperstructure which is a generalization of semigroups and hypergroups. At present, many authors have studied different aspects of semihypergroups, for instance, Bonansinga and Corsini [1], Davvaz [5], Fasino and Freni [8], Hasankhani [10], Hila et al. [12], Leoreanu [16], Naz and Shabir [19], Salvo et al. [22], and many others.

Recall that an ordered semigroup is a semigroup together with a partial order that is compatible with the semigroup operation. Many authors, especially Kehayopulu [13], Kehayopulu and Tsingelis [14, 15], Satyanarayana [23] and Xie [26], studied such semigroups with some restrictions. Furthermore, as a generalization of ordered semigroups, Heidari and Davvaz [11] applied the theory of hyperstructures to ordered semigroups and introduced the concept of ordered semihypergroups, also see [2, 6]. It is well known that regular and strongly regular equivalence relations of ordered semihypergroups always play important roles in the study of ordered semihypergroups structure. For more details, the reader is referred to [6, 9]. On the other hand, Nordahl and Scheiblich [20] considered a unary operation $*$ on semigroups and introduced the concept of regularity on $*$-semigroups. In [25], Wu imposed the $*$-operation on ordered semigroups under the assumption of order preserving. In the present paper, we shall generalize the concept of ordered $*$-semigroups to the hyper version and construct a strongly regular equivalence relation of ordered $*$-semihypergroups by using the notion of pseudoorders on an ordered $*$-semihypergroup such that the corresponding quotient structure is an ordered $*$-semigroup.

The rest of this paper is organized as follows. After an introduction, in Section 2 we recall some basic definitions and results of ordered semihypergroups which will be used throughout this paper and introduce the concept of ordered $*$-semihypergroups. In Section 3, the concept of a pseudoorder on ordered $*$-semihypergroups is introduced, and the related properties are investigated. In addition, the relationship between ordered regular equivalence relations and pseudoorders on an ordered $*$-semihypergroup is established, and several homomorphism theorems of ordered $*$-semihypergroups by pseudoorders are given. In Section 4, we investigate the direct product $S \times T$ of ordered $*$-semihypergroups S and T, and show that $S \times T$ is also an ordered $*$-semihypergroup under a suitable hyperoperation and a unary operation. Moreover, the pseudoorders on $S \times T$ are studied.

2. Preliminaries and some notations

For convenience, let us first give some necessary definitions. A mapping $\circ : S \times S \to \mathcal{P}^{\star}(S)$ is called a hyperoperation on S, where $\mathcal{P}^{\star}(S)$ denotes the family of all nonempty subsets of S. The system $(S, \circ)$ is called a *hypergroupoid*. If A and B are two nonempty subsets of S, then we denote $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$. In particular, for any $x \in S$, we write $x \circ A = \{x\} \circ A$ and $A \circ x = A \circ \{x\}$. Recall that a *semihypergroup* is a hypergroupoid $(S, \circ)$ such that for every $x, y, z \in S$, $x \circ (y \circ z) = (x \circ y) \circ z$ (see [3]).

Let $(S, \circ)$ be a semihypergroup and ρ an equivalence relation on S. If A and B are nonempty subsets of S, then we put

$$A \ \overline{\rho}\ B \Leftrightarrow \begin{cases} (\forall a \in A)(\exists b \in B) \quad a\ \rho\ b, \\ (\forall b' \in B)(\exists a' \in A) \quad a'\ \rho\ b', \end{cases}$$

and

$$A \ \overline{\overline{\rho}}\ B \Leftrightarrow (\forall a \in A)(\forall b \in B)\ a\ \rho\ b.$$

An equivalence relation ρ on a semihypergroup S is called *regular* [6, 9] if

$$(\forall a, b, x \in S)\ a\ \rho\ b \Rightarrow a \circ x\ \overline{\rho}\ b \circ x \text{ and } x \circ a\ \overline{\rho}\ x \circ b;$$

ρ is said to be *strongly regular* [6, 9] if

$$(\forall a, b, x \in S)\ a\ \rho\ b \Rightarrow a \circ x\ \overline{\overline{\rho}}\ b \circ x \text{ and } x \circ a\ \overline{\overline{\rho}}\ x \circ b.$$

Let $(S, \circ)$ be a semihypergroup and ρ an equivalence relation on S. We denote by $a\rho$ the equivalence ρ-class containing a. From [3], we have the following two theorems.

Theorem 2.1. ([3]) *Let $(S, \circ)$ be a semihypergroup and ρ an equivalence relation on S.*

(1) *If ρ is regular, then S/ρ is a semihypergroup with respect to the following hyperoperation: $x\rho \odot y\rho = \{z\rho \mid z \in x \circ y\}$.*

(2) *If the above hyperoperation is well defined on S/ρ, then ρ is regular.*

Theorem 2.2. ([3]) *Let $(S, \circ)$ be a semihypergroup and ρ an equivalence relation on S.*

(1) *If ρ is strongly regular, then S/ρ is a semigroup with respect to the following operation: $x\rho \odot y\rho = z\rho$, for all $z \in x \circ y$.*

(2) *If the above operation is well defined on S/ρ, then ρ is strongly regular.*

As we know, an ordered semigroup $(S, \cdot, \leq)$ is a semigroup $(S, \cdot)$ with an order relation "$\leq$" such that $a \leq b$ implies $xa \leq xb$ and $ax \leq bx$ for any $x \in S$. Furthermore, an order semigroup S with a unary operation $* : S \to S$ is called an *ordered $*$-semigroup* if it satisfies $(x^*)^* = x$ and $(xy)^* = y^* x^*$ for any $x, y \in S$ (see [25]).

We now recall the notion of ordered semihypergroups from [11].

Definition 2.3. An algebraic hyperstructure $(S, \circ, \leq)$ is called an *ordered semihypergroup* (also called *po-semihypergroup* in [11]) if $(S, \circ)$ is a semihypergroup and $(S, \leq)$ is a partially ordered set such that: for any $x, y, a \in S$, $x \leq y$ implies $a \circ x \preceq a \circ y$ and $x \circ a \preceq y \circ a$. Here, if $A, B \in \mathcal{P}^{\star}(S)$, then we say that $A \preceq B$ if for every $a \in A$ there exists $b \in B$ such that $a \leq b$. In particular, if $A = \{a\}$, then we write $a \preceq B$ instead of $\{a\} \preceq B$.

In the following, we shall generalize the concept of ordered $*$-semigroups to the hyper version.

Definition 2.4. An ordered semihypergroup S with a unary operation $* : S \to S$ is called an *ordered $*$-semihypergroup* if it satisfies:

(1) $(\forall x \in S)\ (x^*)^* = x$.

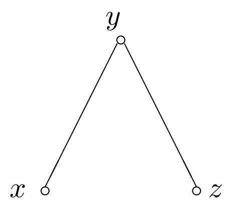
(2) $(\forall x, y \in S)\ (x \circ y)^* = y^* \circ x^*$, where, for any $A \in \mathcal{P}^\star(S)$, the notation A^* is defined by $A^* := \{a^* \in S \mid a \in A\}$.

Such a unary operation $*$ is called an *involution*. If $a \le b$ implies $a^* \le b^*$ for any $a, b \in S$, then $*$ is called an *order preserving involution*.

Example 2.5. Let $S = \{x, y, z\}$ be an ordered semihypergroup. The hyperoperation " $\circ$ ", the order " $\le$ " and the corresponding Hasse diagram are given below. Define the involution $*$ by $x^* = z,\ y^* = y$ and $z^* = x$. It is not difficult to check that $(S, \circ, \le)$ is an ordered $*$-semihypergroup with order preserving involution $*$.

$\circ$	x	y	z
x	$\{x\}$	$\{x,z\}$	$\{x,z\}$
y	$\{x,z\}$	$\{y\}$	$\{x,z\}$
z	$\{x,z\}$	$\{x,z\}$	$\{z\}$

$$\le := \{(x,x),\ (x,y),\ (y,y),\ (z,y),\ (z,z)\}.$$



Lemma 2.6. *Let A, B be nonempty subsets of an ordered $*$-semihypergroup S. Then the following statements hold* :

(1) $A \subseteq B$ *implies* $A^* \subseteq B^*$.

(2) $(A \circ B)^* = B^* \circ A^*$.

(3) $(A \cup B)^* = A^* \cup B^*$.

(4) $(A \cap B)^* = A^* \cap B^*$.

Proof. Straightforward. □

Throughout this paper, unless otherwise mentioned, S will denote an ordered $*$-semihypergroup with order preserving involution $*$. The reader is referred to [4, 24] for notation and terminology not defined in this paper.

3. Pseudoorders on ordered $*$-semihypergroups

In this section we define and characterize the pseudoorders on ordered $*$-semihypergroups, and investigate its related properties. Furthermore, the relationship between strongly regular equivalence relations and pseudoorders on an ordered $*$-semihypergroup is established, and some homomorphism theorems of ordered $*$-semihypergroups by pseudoorders are given.

Definition 3.1. Let $(S; \circ, *; \leq)$ be an ordered $*$-semihypergroup. A relation ρ on S is called *pseudoorder* if

(1) $\leq \subseteq \rho$;

(2) $a\rho b$ implies $a^*\rho b^*$;

(3) $a\rho b$ and $b\rho c$ imply $a\rho c$

(4) $(\forall c \in S)$ $a\rho b$ implies $a \circ c \overline{\overline{\rho}} b \circ c$ and $c \circ a \overline{\overline{\rho}} c \circ b$.

Definition 3.2. An equivalence relation ρ on an ordered $*$-semihypergroup S is called *strongly regular*, if it satisfies

(1) $(\forall a, b, x \in S)$ $a\rho b \Rightarrow a \circ x \overline{\overline{\rho}} b \circ x$ and $x \circ a \overline{\overline{\rho}} x \circ b$;

(2) $(\forall a, b \in S)$ $a\rho b \Rightarrow a^*\rho b^*$.

In the following we shall construct a strongly regular relation ρ° on an ordered $*$-semihypergroup S for which S/ρ° is an ordered $*$-semigroup.

Theorem 3.3. *Let $(S; \circ, *; \leq)$ be an ordered $*$-semihypergroup and ρ be a pseudoorder on S. Then there exists a strongly regular relation ρ° on S such that S/ρ° is an ordered $*$-semigroup.*

Proof. Assume that ρ° is the relation on S defined as follows:

$$(\forall a, b \in S) \quad (a, b) \in \rho^\circ \Leftrightarrow (a, b) \in \rho \cap \rho^{-1}.$$

Then, it is easy to see that ρ° is an equivalence relation. Now, we prove that ρ° is a strongly regular relation on S. Let $a\rho^\circ b$ and $c \in S$. Then $a\rho b$ and $b\rho a$. By condition (4) of Definition 3.1, we have

$$a \circ c \overline{\overline{\rho}} b \circ c, \; b \circ c \overline{\overline{\rho}} a \circ c \; ; \quad c \circ a \overline{\overline{\rho}} c \circ b, \; c \circ b \overline{\overline{\rho}} c \circ a.$$

Thus, for every $x \in a \circ c$ and $y \in b \circ c$, by $a \circ c \overline{\overline{\rho}} b \circ c$, we have $x\rho y$. Also, by $b \circ c \overline{\overline{\rho}} a \circ c$, we have $y\rho x$. This implies that $x\rho^\circ y$. So, $a \circ c \overline{\overline{\rho^\circ}} b \circ c$. By using a similar argument, we can deduce that $c \circ a \overline{\overline{\rho^\circ}} c \circ b$. On the other hand, for any $a, b \in S$, let $a\rho^\circ b$. Then $a\rho b$ and $b\rho a$. Since ρ is a pseudoorder on S, we obtain $a^*\rho b^*$ and $b^*\rho a^*$ which imply that $a^*\rho^\circ b^*$. Hence, ρ° is a strongly regular relation on S. Hence, by Theorem 2.2, S/ρ° with the following operation is a semigroup:

$$x\rho^\circ \odot y\rho^\circ = z\rho^\circ, \text{ for all } z \in x \circ y.$$

Next, we define a relation $\preceq$ on S/ρ° as follows:

$$\preceq := \{(x\rho^\circ, y\rho^\circ) \in S/\rho^\circ \times S/\rho^\circ \mid (x,y) \in \rho\}.$$

We deduce that $(S/\rho^\circ; \odot; \preceq)$ is an ordered semigroup. In fact, for $x\rho^\circ \in S/\rho^\circ$, where $x \in S$, since $(x,x) \in \preceq \subseteq \rho$, it can be obtained that $x\rho^\circ \preceq x\rho^\circ$. If $x\rho^\circ \preceq y\rho^\circ$ and $y\rho^\circ \preceq x\rho^\circ$, then $x\rho y$ and $y\rho x$. Hence, $x\rho^\circ y$ means that $x\rho^\circ = y\rho^\circ$. Suppose that $x\rho^\circ \preceq y\rho^\circ$ and $y\rho^\circ \preceq z\rho^\circ$. Then $x\rho y$ and $y\rho z$. Hence $x\rho z$, and we obtain $x\rho^\circ \preceq z\rho^\circ$. Now, let $x\rho^\circ \preceq \rho^\circ$ and $z\rho^\circ \in S/\rho^\circ$. Then $x\rho y$ and $z \in S$. Since ρ is a pseudoorder on S, we have $x \circ z \overline{\overline{\rho}} y \circ z$ and $z \circ x \overline{\overline{\rho}} z \circ y$. Hence, for every $a \in x \circ z$ and $b \in y \circ z$, we have $a\rho b$. It implies that $a\rho^\circ \preceq b\rho^\circ$. Thus, $x\rho^\circ \odot z\rho^\circ \preceq y\rho^\circ \odot z\rho^\circ$. Similarly, it can be shown that $z\rho^\circ \odot x\rho^\circ \preceq z\rho^\circ \odot y\rho^\circ$.

Finally, we define a unary operation on S/ρ° by:

$$(a\rho^\circ)^\star = a^*\rho^\circ, \text{ for all } a \in S.$$

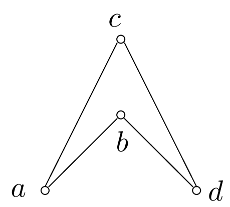
The unary operation $\star$ is well defined. Indeed, let $a\rho^\circ = b\rho^\circ$, i.e., $a\rho^\circ b$. Then $a\rho b$ and $b\rho a$. Note that ρ is a pseudoorder on S, we get that $a^*\rho b^*$ and $b^*\rho a^*$. So, $a^*\rho^\circ b^*$, that is, $a^*\rho^\circ = b^*\rho^\circ$. Furthermore, it is easy to check that S/ρ° is an ordered $*$-semigroup with the unary operation $\star$. Let $*$ be an order preserving involution on S, and $x\rho^\circ \preceq y\rho^\circ$. Then $x\rho y$, by the definition of pseudoorders, we have $x^*\rho y^*$. This means that $x^*\rho^\circ \preceq y^*\rho^\circ$, i.e., $(x\rho^\circ)^\star \preceq (y\rho^\circ)^\star$. Therefore, we conclude that the operation $\star$ is also an order preserving involution on S/ρ°. □

Example 3.4. Let $S = \{a,b,c,d\}$ be an ordered $*$-semihypergroup. The hyperoperation " $\circ$ ", the unary operation " $*$ " and the order " $\leq$ " are given below.

$\circ$	a	b	c	d
a	$\{a,d\}$	$\{a,d\}$	$\{a,d\}$	$\{a\}$
b	$\{a,d\}$	$\{b\}$	$\{a,d\}$	$\{a,d\}$
c	$\{a,d\}$	$\{a,d\}$	$\{c\}$	$\{a,d\}$
d	$\{a\}$	$\{a,d\}$	$\{a,d\}$	$\{d\}$

$$a^* = a,\ b^* = c,\ c^* = b,\ d^* = d.$$

$$\leq := \{(a,a),\ (a,b),\ (a,c),\ (b,b),\ (c,c),\ (d,b),\ (d,c),\ (d,d)\}.$$



Let ρ be a pseudoorder on S defined as follows:

$$\rho = \{(a,a),(b,b),(c,c),(d,d),(a,b),(a,c),(a,d),(d,a),(d,b),(d,c)\}.$$

Then, by the definition of ρ°, we have

$$\rho^\circ = \{(a,a),(b,b),(c,c),(d,d),(a,d)\}.$$

Thus, $S/\rho^\circ = \{\alpha,\beta,\gamma\}$, where $\alpha = \{a,d\}, \beta = \{b\}, \gamma = \{c\}$. Immediately, $(S/\rho^\circ; \odot, \star; \preceq)$ is an ordered $*$-semigroup, where the multiplication $\odot$, the unary operation $\star$ and the order $\preceq$ are defined in the following:

$\odot$	α	β	γ
α	α	α	α
β	α	β	α
γ	α	α	γ

$$\alpha^\star = \alpha,\ \beta^\star = \gamma,\ \gamma^\star = \beta,$$

and

$$\preceq = \{(\alpha,\alpha),(\alpha,\beta),(\beta,\beta),(\alpha,\gamma),(\gamma,\gamma)\}.$$

Theorem 3.5. *Let $(S; \circ, *; \leq)$ be an ordered $*$-semihypergroup and ρ a pseudoorder on S. Let*

$$\mathcal{A} := \{\ \theta \mid \theta \text{ is a pseudoorder on } S \text{ such that } \rho \subseteq \theta\}.$$

Let $\mathcal{B}$ be the set of all pseudoorders on S/ρ°. For $\theta \in \mathcal{A}$, define a relation θ' on S/ρ° as follows:

$$\theta' := \{(x\rho^\circ, y\rho^\circ) \in S/\rho^\circ \times S/\rho^\circ \mid (x,y) \in \theta\}.$$

Then the mapping

$$\varphi : \mathcal{A} \to \mathcal{B},\ \theta \mapsto \theta'$$

is a bijection, and $\theta_1 \subseteq \theta_2$ if and only if $\theta_1' \subseteq \theta_2'$, where $\theta_1, \theta_2 \in \mathcal{A}$.

Proof. Suppose that $\theta \in \mathcal{A}$. We show that θ' is a pseudoorder on S/ρ°. Let $(x\rho^\circ, y\rho^\circ) \in \preceq$. By the proof of Theorem 3.3, we have $(x,y) \in \rho \subseteq \theta$ which implies that $(x\rho^\circ, y\rho^\circ) \in \theta'$. Hence $\preceq \subseteq \theta'$. Also, let $(x\rho^\circ, y\rho^\circ) \in \theta'$ and $z\rho^\circ \in S/\rho^\circ$. Then $(x,y) \in \theta$ and $z \in S$. Thus $x \circ z \overline{\overline{\theta}}$ and $z \circ x \overline{\overline{\theta}} z \circ y$. So, for all $a \in x \circ z$ and $b \in y \circ z$, we have $a\theta b$. Therefore, $(x\rho^\circ \odot z\rho^\circ)\ \theta'\ (y\rho^\circ \odot z\rho^\circ)$. By a similar argument, we deduce that $(z\rho^\circ \odot x\rho^\circ)\ \theta'\ (z\rho^\circ \odot y\rho^\circ)$. Moreover, if $(x\rho^\circ, y\rho^\circ) \in \theta'$, then $(x,y) \in \theta$. So, we get $(x^*, y^*) \in \theta$, and we have $(x^*\rho^\circ, y^*\rho^\circ) \in \theta'$, that is, $((x\rho^\circ)^\star, (y\rho^\circ)^\star) \in \theta'$. Therefore, if $\theta \in \mathcal{A}$, then θ' is a pseudoorder on S/ρ°.

Next, we deduce that the map $\varphi : \mathcal{A} \to \mathcal{B}$ defined by $\varphi(\theta) = \theta'$ is well defined and one-to-one. In fact, let $\theta_1, \theta_2 \in \mathcal{A}$ and $\theta_1 = \theta_2$. Assume that $(x\rho^\circ, y\rho^\circ) \in \theta'$. Then $(x,y) \in \theta_1 = \theta_2$. This implies that $(x\rho^\circ, y\rho^\circ) \in \theta_2'$. Thus

$\theta_1' \subseteq \theta_2'$. Similarly, we obtain that $\theta_2' \subseteq \theta_1'$. On the other hand, let $\theta_1, \theta_2 \in \mathcal{A}$ and $\theta_1' = \theta_2'$. Suppose that $(x, y) \in \theta_1$. Then $(x\rho^\circ, y\rho^\circ) \in \theta_1' = \theta_2'$. Thus $(x, y) \in \theta_2$ which implies that $\theta_1 \subseteq \theta_2$. By using a similar argument, it is easy to see that $\theta_2 \subseteq \theta_1$.

Finally, we prove that φ is onto. Take $\delta \in \mathcal{B}$, we define a relation θ on S as follows:

$$\theta = \{(x, y) \in S \times S \mid (x\rho^\circ, y\rho^\circ) \in \delta\}.$$

We claim that θ is a pseudoorder on S and $\rho \subseteq \theta$. To prove our claim, let $(x, y) \in \rho$, by the proof of Theorem 3.3, $(x\rho^\circ, y\rho^\circ) \in \preceq \subseteq \delta$, and so $(x, y) \in \theta$. Thus, $\rho \subseteq \theta$. If $(x, y) \in \leq$, then $(x, y) \in \rho \subseteq \theta$. Hence, $\leq \subseteq \theta$. Suppose that $(x, y) \in \theta$ and $(y, z) \in \theta$. Then $(x\rho^\circ, y\rho^\circ) \in \delta$ and $(y\rho^\circ, z\rho^\circ) \in \delta$. Thus, $(x\rho^\circ, z\rho^\circ) \in \delta$, and we have $(x, z) \in \theta$. Also, let $(x, y) \in \theta$ and $z \in S$. Then $(x\rho^\circ, y\rho^\circ) \in \delta$ and $z\rho^\circ \in S/\rho^\circ$. So, we obtain $(x\rho^\circ \odot z\rho^\circ) \overline{\overline{\delta}} (y\rho^\circ \odot z\rho^\circ)$ and $(a\rho^\circ, b\rho^\circ) \in \delta$ for all $a \in x \circ z$, $b \in y \circ z$. Thus, $(a, b) \in \theta$, which implies that $x \circ z \; \overline{\overline{\theta}} \; y \circ z$. Similarly, we can deduce that $z \circ x \; \overline{\overline{\theta}} \; z \circ y$. Furthermore, let $(x, y) \in \theta$. Then $(x\rho^\circ, y\rho^\circ) \in \delta$. This implies that $((x\rho^\circ)^\star, (y\rho^\circ)^\star)$, i.e., $(x^*\rho^\circ, y^*\rho^\circ) \in \delta$. Therefore, $(x^*, y^*) \in \theta$. $\square$

Let $(S; \diamond, *; \leq_S)$ and $(T; \circ, \star; \leq_T)$ be two ordered $*$-semihypergroups, $f : S \to T$ a mapping from S into T. f is said to be *isotone* if $x \leq_S y$ implies that $f(x) \leq_T f(y)$ for all $x, y \in S$. f is called *reverse isotone* if $f(x) \leq_T f(y)$ implies that $x \leq_S y$. f is called a *strong homomorphism* if (1) f is isotone; (2) for all $x, y \in S, f(x) \circ f(y) = f(z)$, where z is an arbitrary element of $x \diamond y$; and (3) $f(x^*) = f^\star(x)$ for any $x \in S$. f is said to be *strong isomorphism* if it is strong homomorphism, surjection and reverse isotone. The ordered $*$-semihypergroups S and T are called *strongly isomorphic*, in symbol $S \cong T$, if there exists a strong isomorphism between them.

Proposition 3.6. *Let $(S; \diamond, *; \leq_S)$ and $(T; \circ, \star; \leq_T)$ be two ordered $*$-semihypergroups with order preserving involutions $*$ and $\star$, respectively. Let $f : S \to T$ be a strong homomorphism. The relation ρ on S defined by*

$$\rho := \{(x, y) \in S \times S \mid f(x) \leq_T f(y)\}$$

is a pseudoorder on S.

Proof. Suppose that $(x, y) \in \leq_S$. Since $x \leq_S y$ and f is isotone, we obtain $f(x) \leq_T f(y)$. Thus $(x, y) \in \rho$. If $(x, y) \in \rho$ and $(y, z) \in \rho$. Then $f(x) \leq_T f(y), f(y) \leq_T f(z)$, and thus $f(x) \leq_T f(z)$. This means that $(x, z) \in \rho$. Let $(x, y) \in \rho, z \in S$. For any $a \in x \diamond z$ and $b \in y \diamond z$, we have

$$f(a) = f(x) \circ f(z) \leq_T f(y) \circ f(z) = f(b).$$

Therefore, $(a, b) \in \rho$ and we have $x \circ z \ \overline{\overline{\rho}}\ y \circ z$. Similarly, it can be easily obtained that $z \circ x \ \overline{\overline{\rho}}\ z \circ y$. Furthermore, let $(x, y) \in \rho$. Then $f(x) \leq_T f(y)$. Since $\star$ is an order preserving involution on T, we have $f^\star(x) \leq_T f^\star(y)$. Note that f is a strong homomorphism, we obtain $f(x^*) \leq_T f(y^*)$, that is, $(x^*, y^*) \in \rho$. □

Lemma 3.7. *Let $(S; \diamond, *; \leq_S)$ and $(T; \circ, \star; \leq_T)$ be ordered $*$-semihypergroups with order preserving involution $*$ and $\star$, respectively. Let $f : S \to T$ be a strong homomorphism. Then*

$$ker\varphi := \{(a, b) \in S \times S \mid \varphi(a) = \varphi(b)\}$$

is a strongly regular relation on S and

$$\underline{\varphi}^\circ := \{(a, b) \in S \times S \mid a \ \underline{\varphi}\ b \ \text{ and } \ b \ \underline{\varphi}\ a\} = ker\varphi,$$

where $\underline{\varphi} := \{(a, b) \in S \times S \mid \varphi(a) \leq_T \varphi(b)\}$ is the pseudoorder on S defined by Proposition 3.6.

Proof. (1). $ker\varphi$ is a strong regular relation on S. In fact,

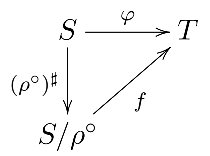
Let $a \ ker\varphi\ b$ and $c \in S$. Then $\varphi(a) = \varphi(b)$, which implies that $\varphi(a) \circ \varphi(c) = \varphi(b) \circ \varphi(c)$. Hence, for all $x \in a \circ c$ and all $y \in b \circ c$, we have $\varphi(x) = \varphi(y)$, i.e., $x \ ker\varphi\ y$. Thus $a \circ c \ \overline{\overline{ker\varphi}}\ b \circ c$. Similarly, we can show that $c \circ a \ \overline{\overline{ker\varphi}}\ c \circ b$.

Suppose that $a \ ker\varphi\ b$ for any $a, b \in S$. Then $\varphi(a) = \varphi(b)$. This means that $\varphi^\star(a) = \varphi^\star(b)$. Since φ is a strong homomorphism, we have $\varphi(a^*) = \varphi(b^*)$. That is $a^* \ ker\varphi\ b^*$.

(2). $ker\varphi = \underline{\varphi}^\circ$. Indeed, let $(a, b) \in \underline{\varphi}^\circ$. Then $a \ \underline{\varphi}\ b$ and $b \ \underline{\varphi}\ a$. That is $\varphi(a) \leq_T \varphi(b)$ and $\varphi(b) \leq_T \varphi(a)$. Hence, $\varphi(a) = \varphi(b)$, i.e., $(a, b) \in ker\varphi$. On the other hand, let $(a, b) \in ker\varphi$ for any $a, b \in S$. Then $\varphi(a) = \varphi(b)$. So, $\varphi(a) \leq_T \varphi(b)$, which implies that $a \ \underline{\varphi}\ b$ and $\varphi(b) \leq_T \varphi(a)$, and we have $b \ \underline{\varphi}\ a$. Thus, $a \ \underline{\varphi}^\circ\ b$. □

Let ρ be a pseudoorder on an ordered $*$-semihypergroup $(S; \circ, *; \leq)$. Then, by Theorem 3.3, $\rho^\circ = \rho \cap \rho^{-1}$ is a strongly regular equivalence relation on S. We denote by $\rho^\sharp$ the mapping from S onto S/ρ°, i.e., $\rho^\sharp : S \to S/\rho^\circ \mid x \mapsto x\rho^\circ$, which is a strong homomorphism. In the following, we give out a homomorphism theorem of ordered $*$-semihypergroups by pseudoorders, which is a generalization of [6] and [15].

Theorem 3.8. *Let $(S; \diamond, *; \leq_S)$ and $(T; \circ, \star; \leq_T)$ be ordered $*$-semihypergroups with order preserving involution $*$ and $\star$ respectively, $\varphi : S \to T$ a strong homomorphism. If ρ is a pseudoorder on S such that $\rho \subseteq \underline{\varphi}$, then the mapping $f : S/\rho^\circ \mid a\rho^\circ \mapsto \varphi(a)$ is the unique strong homomorphism of S/ρ° into T such that the diagram*



commutes. Furthermore, $Imf = Im\varphi$*. Conversely, if* ρ *is a pseudoorder on* S *for which there exists a strong homomorphism* $f : S/\rho^\circ \to T$ *such that the above diagram commutes, then* $\rho \subseteq \underline{\varphi}$*.*

Proof. Let ρ be a pseudoorder on S, $\rho \subseteq \underline{\varphi}$, $f : S/\rho^\circ \to T$ is defined by $f(a\rho^\circ) = \varphi(a)$. Then f is well defined. In fact,

$$\begin{aligned} a\rho^\circ = b\rho^\circ \Rightarrow (a,b) \in \rho^\circ &\Rightarrow (a,b) \in \rho \ \text{ and } \ (b,a) \in \rho \\ &\Rightarrow (a,b) \in \underline{\varphi} \ \text{ and } \ (b,a) \in \underline{\varphi} \\ &\Rightarrow \varphi(a) \leq_T \varphi(b) \ \text{ and } \ \varphi(b) \leq_T \varphi(a) \\ &\Rightarrow \varphi(a) = \varphi(b). \end{aligned}$$

f is a strong homomorphism and $f \cdot \rho^\sharp = \varphi$. If $a, b \in S$, then

$$a\rho^\circ \leq_{S/\rho^\circ} b\rho^\circ \Rightarrow (a,b) \in \rho \subseteq \underline{\varphi} \Rightarrow \varphi(a) \leq_T \varphi(b),$$

that is, $f(a\rho^\circ) \leq_T f(b\rho^\circ)$ and f is isotone. Also,

$$f(a\rho^\circ \odot b\rho^\circ) = f(z\rho^\circ) = \varphi(z) = \varphi(a) \circ \varphi(b) = f(a\rho^\circ) \circ f(b\rho^\circ),$$

for all $z \in a \diamond b$. Furthermore, for any $(a\rho^\circ)^* \in S/\rho^\circ$, we have

$$f((a\rho^\circ)^*) = f(a^*\rho^\circ) = \varphi(a^*) = \varphi^\star(a) = f^\star(a\rho^\circ).$$

Finally, for each $a \in S$, $(f \cdot \rho^\sharp)a = f(\rho^\sharp(a)) = f(a\rho^\circ) = \varphi(a)$. That is, $f \cdot \rho^\sharp = \varphi$. We claim that f is the unique strong homomorphism from S/ρ° to T. To prove our claim, let $g : S/\rho^\circ$ be a strong homomorphism such that $g \cdot \rho^\sharp = \varphi$. Then $f = g$. In fact, for any $a \in S$, we have

$$f(a\rho^\circ) = \varphi(a) = (g \cdot \rho^\sharp)(a) = g(\rho^\sharp(a)) = g(a\rho^\circ).$$

Moreover, $Imf = \{f(a\rho^\circ) \mid a \in S\} = \{\varphi(a) \mid a \in S\} = Im\varphi$.

Conversely, let ρ be a pseudoorder on S, $f : S/\rho^\circ \to T$ a strong homomorphism, $f \cdot \rho^\sharp = \varphi$. Then $\rho \subseteq \underline{\varphi}$. Indeed,

$$\begin{aligned} (a,b) \in \rho \Rightarrow a\rho^\circ \leq_{S/\rho^\circ} b\rho^\circ &\Rightarrow f(a\rho^\circ) \leq_T f(b\rho^\circ) \\ &\Rightarrow f(\rho^\sharp(a)) \leq_T f(\rho^\sharp(b)) \\ &\Rightarrow (f \cdot \rho^\sharp)(a) \leq_T (f \cdot \rho^\sharp)(b) \\ &\Rightarrow \varphi(a) \leq_T \varphi(b) \Rightarrow (a,b) \in \underline{\varphi}. \end{aligned}$$

Hence, the proof is completed. □

Remark 3.9. If S, T are two ordered $*$-semihypergroups, f a homomorphism and reverse isotone mapping of S into T, then it is easy to check that $S \cong Imf$, since every reverse isotone mapping is injection.

Corollary 3.10. *Let* $(S; \circ, *; \leq_S)$ *and* $(T; \diamond, \star; \leq_T)$ *be two ordered* $*$*-semihypergroups,* $\varphi : S \to T$ *a strong homomorphism. Then* $S/ker\varphi \cong Im\varphi$*.*

Proof. By applying the first part of Theorem 3.8 for $\rho = \underline{\varphi}$. The mapping $f : S/\underline{\varphi}^\circ \to T \mid a\underline{\varphi}^\circ \to \varphi(a)$ is a strong homomorphism. In fact, f is reverse isotone. Let $a, b \in S$, $\varphi(a) \leq_T \varphi(b)$. Since $(a, b) \in \underline{\varphi}$, $\underline{\varphi}$ is a pseudoorder on S, we have $(a\underline{\varphi}^\circ, b\underline{\varphi}^\circ) \in \leq_{S/\underline{\varphi}^\circ}$, that is, $a\underline{\varphi}^\circ \leq_{S/\underline{\varphi}^\circ} b\underline{\varphi}^\circ$. Using Remark 3.9, we obtain $S/\underline{\varphi}^\circ \cong Imf$. Since $Imf = Im\varphi$ from the results of Theorem 3.8 and $\underline{\varphi}^\circ = ker\varphi$, by Lemma 3.7, we conclude $S/ker\varphi \cong Im\varphi$. □

Lemma 3.11. *Let $(S; \circ, *; \leq)$ be an ordered $*$-semihypergroup and ρ, σ be pseudoorders on S such that $\rho \subseteq \sigma$. We define a relation σ/ρ on S/ρ° as follows:*

$$\sigma/\rho := \{(x\rho^\circ, y\rho^\circ) \in S/\rho^\circ \times S/\rho^\circ \mid (x, y) \in \sigma\}.$$

Then σ/ρ is a pseudoorder on S/ρ°.

Proof. In fact, for any $a, b \in S$, if $a\rho^\circ \leq_{S/\rho^\circ} b\rho^\circ$, then $(a, b) \in \rho \subseteq \sigma$ and $(a\rho^\circ, b\rho^\circ) \in \sigma/\rho$, that is $\leq_{S/\rho^\circ} \subseteq \sigma/\rho$. Let $(a\rho^\circ, b\rho^\circ) \in \sigma/\rho$ and $(b\rho^\circ, c\rho^\circ) \in \sigma/\rho$. Then $(a, b) \in \sigma$ and $(b, c) \in \sigma$, so $(a, c) \in \sigma$, which means that $(a\rho^\circ, c\rho^\circ) \in \sigma/\rho$. Suppose that $(a\rho^\circ, b\rho^\circ) \in \sigma/\rho$, $c\rho^\circ \in S/\rho^\circ$. Then $(a, b) \in \sigma$ and $c \in S$. Thus, $a \circ c \overline{\overline{\sigma}} b \circ c$ and $c \circ a \overline{\overline{\sigma}} c \circ b$. Thus, for any $x \in a \circ c$, $y \in b \circ c$, we have $(x, y) \in \sigma$. This implies that

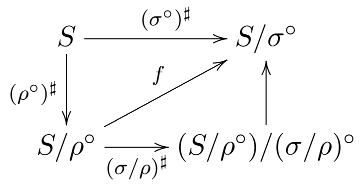
$$a\rho^\circ \odot c\rho^\circ = (x\rho^\circ)\ \sigma/\rho\ (y\rho^\circ) = b\rho^\circ \odot c\rho^\circ.$$

By using a similar argument, we can deduce that $(c\rho^\circ \odot a\rho^\circ)\ \sigma/\rho\ (c\rho^\circ \odot b\rho^\circ)$. Furthermore, for each $a\rho^\circ$, $b\rho^\circ \in S/\rho^\circ$, $(a\rho^\circ)\ \sigma/\rho\ (b\rho^\circ)$, which implies that $(a, b) \in \sigma$. Hence, $(a^*, b^*) \in \sigma$ and $(a^*\rho^\circ)\ \sigma/\rho\ (b^*\rho^\circ)$. That is $(a\rho^\circ)^*\ \sigma/\rho\ (b\rho^\circ)^*$. □

Moreover, by applying Corollary 3.10 and Lemma 3.11, we have the following theorem.

Theorem 3.12. *Let $(S; \circ, *; \leq)$ be an ordered $*$-semihypergroup, ρ, σ be pseudoorders on S such that $\rho \subseteq \sigma$. Then $(S/\rho^\circ)/(\sigma/\rho)^\circ \cong S/\sigma^\circ$.*

Proof. Now, we consider the diagram



The mapping $f : S/\rho^\circ \to S/\sigma^\circ$ is defined by $f(a\rho^\circ) = a\sigma^\circ$. Then

(1) f is well defined. In fact,

$$\begin{aligned} a\rho^\circ = b\rho^\circ &\Rightarrow (a, b) \in \rho \subseteq \sigma \text{ and } (b, a) \in \rho \subseteq \sigma \\ &\Rightarrow (a, b) \in \sigma^\circ \Rightarrow a\sigma^\circ = b\sigma^\circ. \end{aligned}$$

(2) f is a strong homomorphism. Indeed, let $\odot_\rho, \odot_\sigma$ be the multiplications on S/ρ° and S/σ°, respectively. Then we claim that f is isotone. In fact,

$$a\rho^\circ \leq_{S/\rho^\circ} b\rho^\circ \Rightarrow (a,b) \in \rho \subseteq \sigma \Rightarrow a\sigma^\circ \leq_{S/\sigma^\circ} b\sigma^\circ.$$

Furthermore, we have

$$f(a\rho^\circ \odot_\rho b\rho^\circ) = f(c\rho^\circ) = c\sigma^\circ = a\sigma^\circ \odot_\sigma b\sigma^\circ = f(a\rho^\circ) \odot_\sigma f(b\rho^\circ)$$

and

$$f((a\rho^\circ)^*) = f(a^*\rho^\circ) = a^*\sigma^\circ = (a\sigma^\circ)^* = f^\star(a\rho^\circ).$$

Therefore, by Corollary 3.10, $(S/\rho^\circ)/kerf \cong Imf$. Define

$$\underline{f} = \{(a\rho^\circ, b\rho^\circ) \in S/\rho^\circ \times S/\rho^\circ \mid f(a\rho^\circ) \leq_{S/\sigma^\circ} f(b\rho^\circ)\}.$$

Then

$$\begin{aligned}(a\rho^\circ, b\rho^\circ) \in \underline{f} &\Leftrightarrow f(a\rho^\circ) \leq_{S/\sigma^\circ} f(b\rho^\circ)\\ &\Leftrightarrow a\sigma^\circ \leq_{S/\sigma^\circ} b\sigma^\circ\\ &\Leftrightarrow (a,b) \in \sigma\\ &\Leftrightarrow (a\rho^\circ, b\rho^\circ) \in \sigma/\rho.\end{aligned}$$

Hence, $\underline{f} = \sigma/\rho$ and $kerf = \underline{f}^\circ = (\sigma/\rho)^\circ$. Moreover,

$$Imf = \{f(a\rho^\circ) \mid a \in S\} = \{a\sigma^\circ \mid a \in S\} = S/\sigma^\circ.$$

From the above argument, we conclude that $(S/\rho^\circ)/(\sigma/\rho)^\circ \cong S/\sigma^\circ$. □

4. Direct products of ordered $*$-semihypergroups

In this section, we investigate the direct product $S \times T$ of ordered $*$-semihypergroups S and T. Furthermore, the pseudoorders on $S \times T$ are studied.

Let $(S; \circ, *; \leq_S)$ and $(T; \diamond, \dagger; \leq_T)$ be two ordered $*$-semihypergroups with the order preserving involutions $*$ and $\dagger$, respectively. Define the coordinatewise operation on $S \times T$ as follows:

$$\begin{aligned}(\forall (s_1,t_1),(s_2,t_2) \in S \times T)\ (s_1,t_1) \circledcirc (s_2,t_2) &= (s_1 \circ s_2) \times (t_1 \diamond t_2)\\ &= \bigcup_{\substack{k \in s_1 \circ s_2\\ l \in t_1 \diamond t_2}} (k,l),\end{aligned}$$

Obviously, the Cartesian product $S \times T$ of S and T forms a semihypergroup. Define a partial order $\leq$ on $S \times T$ by $(s_1,t_1) \leq (s_2,t_2)$ if and only if $s_1 \leq_S s_2$ and $t_1 \leq_T t_2$. Then $(S \times T; \circledcirc; \leq)$ is an ordered semihypergroup. Furthermore, put a unary operation $\star$ on $S \times T$ as follows:

$$(s,t)^\star = (s^*, t^\dagger).$$

Then, we can deduce that $(S \times T; \circledcirc, \star; \leq)$ is an ordered $*$-semihypergroup. In fact, it is easy to see that the unary operation $\star$ is well defined and $[(s,t)^\star]^\star = (s^*, t^\dagger)^\star = (s,t)$. Moreover,

$$\begin{aligned}[(s_1,t_1) \circledcirc (s_2,t_2)]^\star &= [\bigcup_{\substack{k\in s_1\circ s_2\\ l\in t_1\diamond t_2}} (k,l)\ ]^\star \\ &= \bigcup_{\substack{k\in s_1\circ s_2\\ l\in t_1\diamond t_2}} (k^*, l^\dagger).\end{aligned}$$

On the other hand,

$$\begin{aligned}(s_2,t_2)^\star \circledcirc (s_1,t_1)^\star &= (s_2^*, t_2^\dagger) \circledcirc (s_1^*, t_1^\dagger) \\ &= (s_2^* \circ s_1^*) \times (t_2^\dagger \diamond t_1^\dagger) \\ &= \bigcup_{\substack{u\in s_2^*\circ s_1^*\\ v\in t_2^\dagger\diamond t_1^\dagger}} (u,v) \\ &= \bigcup_{\substack{u^*\in s_1\circ s_2\\ v^\dagger\in t_1\diamond t_2}} (u,v) \\ &= \bigcup_{\substack{u\in s_1\circ s_2\\ v\in t_1\diamond t_2}} (u^*, v^\dagger).\end{aligned}$$

Therefore, $(S \times T; \circledcirc, \star; \leq)$ forms an ordered $*$-semihypergroup.

Lemma 4.1. *Let $(S; \circ, *; \leq_S)$ and $(T; \diamond, \dagger; \leq_T)$ be ordered $*$-semihypergroups, ρ, σ be two pseudoorders on S and T, respectively. Define a relation on $S \times T$ by*

$$(s_1,t_1)\ \delta\ (s_2,t_2) \Leftrightarrow s_1\ \rho\ s_2 \text{ and } t_1\ \sigma\ t_2.$$

Then δ is a pseudoorder on $S \times T$.

Proof. Suppose that $(s_1,t_1) \leq (s_2,t_2)$. Then $s_1 \leq_S s_2$ and $t_1 \leq_T t_2$. So, $s_1\ \rho\ s_2$ and $t_1\ \sigma\ t_2$. This implies that $(s_1,t_1)\ \delta\ (s_2,t_2)$, that is, $\leq \subseteq \delta$. Also, let $(s_1,t_1)\ \delta\ (s_2,t_2)$ and $(s_2,t_2)\ \delta\ (s_3,t_3)$. Then $s_1\ \rho\ s_2$, $s_2\ \rho\ s_3$, $t_1\ \sigma\ t_2$ and $t_2\ \sigma\ t_3$. Hence $s_1\ \rho\ s_3$ and $t_1\ \sigma\ t_3$. Thus, $(s_1,t_1)\ \delta\ (s_3,t_3)$. Furthermore, let $(s_1,t_1)\ \delta\ (s_2,t_2)$ and $(k,l) \in S \times T$. For any $u \in s_1 \circ k$, $v \in t_1 \diamond l$, $m \in s_2 \circ k$ and $n \in t_2 \diamond l$. Since $s_1\ \rho\ s_2$, we have $s_1 \circ k\ \overline{\overline{\rho}}\ s_2 \circ k$, that is, $u\ \rho\ m$. Also, by $t_1\ \sigma\ t_2$, we deduce that $v\ \sigma\ u$. Hence, $(u,v)\ \delta\ (m,n)$. This means that $\bigcup_{\substack{u\in s_1\circ k\\ v\in t_1\diamond l}} (u,v)\ \overline{\overline{\delta}} \bigcup_{\substack{m\in s_2\circ k\\ n\in t_2\diamond l}} (m,n)$. We conclude that $[(s_1,t_1) \circledcirc (k,l)]\ \overline{\overline{\delta}}\ [(s_2,t_2) \circledcirc (k,l)]$. By using a similar argument, we deduce that $[(k,l) \circledcirc (s_1,t_1)]\ \overline{\overline{\delta}}\ [(k,l) \circledcirc (s_2,t_2)]$. Finally, assume that $(s_1,t_1)\ \delta\ (s_2,t_2)$. Then $s_1\ \rho\ s_2$ and $t_1\ \sigma\ t_2$. Hence,

$s_1^* \ \rho \ s_2^*$ and $t_1^\dagger \ \sigma \ t_2^\dagger$ since ρ, σ are two pseudoorders on S and T, respectively. Thus, $(s_1^*, t_1^*) \ \delta \ (s_2^*, t_2^*)$. That is, $(s_1, t_1)^\star \ \delta \ (s_2, t_2)^\star$. □

Theorem 4.2. *Let $(S; \circ, *; \leq_S)$ and $(T; \diamond, \dagger; \leq_T)$ be two ordered $*$-semihypergroups, ρ, σ be two pseudoorders on S and T, respectively. Then*

$$(S \times T)/\delta^\circ \cong S/\rho^\circ \times T/\sigma^\circ.$$

Proof. By Theorem 3.3 and Lemma 4.1, it is easy to check that $(S \times T)/\delta^\circ$ and $S/\rho^\circ \times T/\sigma^\circ$ are both ordered $*$-semigroups with the unary operations $\star$ and $\circledast$ defined as follows:

$$[(s,t)\delta^\circ]^\star = (s,t)^\star \delta^\circ = (s^*, t^\dagger)\delta^\circ,$$
$$(s\rho^\circ, t\sigma^\circ)^\circledast = (s^*\rho^\circ, t^\dagger\sigma^\circ).$$

Now, we consider the mapping $\varphi : (S \times T)/\delta^\circ \to S/\rho^\circ \times T/\sigma^\circ$ defined by $\varphi((s,t)\delta^\circ) = (s\rho^\circ, t\sigma^\circ)$. Then, for any $(s_1,t_1)\delta^\circ$, $(s_2,t_2)\delta^\circ \in (S \times T)/\delta^\circ$, we have

$$\begin{aligned}
(s_1,t_1)\delta^\circ = (s_2,t_2)\delta^\circ &\Leftrightarrow (s_1,t_1) \ \delta^\circ \ (s_2,t_2) \\
&\Leftrightarrow (s_1,t_1) \ \delta \ (s_2,t_2) \ \text{ and } \ (s_2,t_2) \ \delta \ (s_1,t_1) \\
&\Leftrightarrow s_1 \ \rho \ s_2, s_2 \ \rho \ s_1 \ \text{ and } \ t_1 \ \sigma \ t_2, t_2 \ \sigma \ t_1 \\
&\Leftrightarrow s_1 \ \rho^\circ \ s_2 \ \text{ and } \ t_1 \ \sigma^\circ \ t_2 \\
&\Leftrightarrow (s_1\rho^\circ, t_1\sigma^\circ) = (s_2\rho^\circ, t_2\sigma^\circ).
\end{aligned}$$

Thus, φ is well defined. Next, we prove that φ is a strong homomorphism. Let $(s_1,t_1)\delta^\circ$ and $(s_2,t_2)\delta^\circ$ be two arbitrary elements of $(S \times T)/\delta^\circ$. Then

$$\begin{aligned}
(s_1,t_1)\delta^\circ \leq (s_2,t_2)\delta^\circ &\Leftrightarrow (s_1,t_1) \ \delta \ (s_2,t_2) \\
&\Leftrightarrow s_1 \ \rho \ s_2 \ \text{ and } \ t_1 \ \sigma \ t_2 \\
&\Leftrightarrow s_1\rho^\circ \leq_{S/\rho^\circ} s_2\rho^\circ \ \text{ and } \ t_1\sigma^\circ \leq_{T/\sigma^\circ} t_2\sigma^\circ \\
&\Leftrightarrow (s_1\rho^\circ, t_1\sigma^\circ) \leq_{(S/\rho^\circ \times T/\sigma^\circ)} (s_2\rho^\circ, t_2\sigma^\circ) \\
&\Leftrightarrow \varphi((s_1,t_1)\delta^\circ) \leq_{(S/\rho^\circ \times T/\sigma^\circ)} \varphi((s_2,t_2)\delta^\circ).
\end{aligned}$$

Hence, φ is isotone and reverse isotone. Also,

$$\begin{aligned}
\varphi((s_1,t_1)\delta^\circ \bullet (s_2,t_2)\delta^\circ) &= \varphi((s,t)\delta^\circ), \quad \text{for all} \quad (s,t) \in (s_1,t_1) \circledcirc (s_2,t_2) \\
&= (s\rho^\circ, t\sigma^\circ), \quad \text{for all} \quad s \in s_1 \circ s_2, \ t \in t_1 \diamond t_2 \\
&= (s_1\rho^\circ \odot s_2\rho^\circ, t_1\sigma^\circ \otimes t_2\sigma^\circ) \\
&= (s_1\rho^\circ, t_1\sigma^\circ) \times (s_2\rho^\circ, t_2\sigma^\circ) \\
&= \varphi((s_1,t_1)\delta^\circ) \times ((s_2,t_2)\delta^\circ).
\end{aligned}$$

Furthermore, $\varphi[((s,t)\delta^\circ)^\star] = \varphi[(s^*, t^\dagger)\delta^\circ] = (s^*\rho^\circ, t^\dagger\sigma^\circ)$. On the other hand, we have

$$\varphi^\circledast[(s,t)\delta^\circ] = [\varphi((s,t)\delta^\circ)]^\circledast = (s\rho^\circ, t\sigma^\circ)^\circledast = (s^*\rho^\circ, t^\dagger\sigma^\circ).$$

Therefore, $\varphi[((s,t)\delta^{\circ})^{\star}] = \varphi^{\circledast}[(s,t)\delta^{\circ}]$ and φ is a strong homomorphism. It is obvious that φ is surjection. Hence, φ is a strong isomorphism and the proof is completed. □

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