STRUCTURE OF (w_1, w_2)-TEMPERED ULTRADISTRIBUTION USING SHORT-TIME FOURIER TRANSFORM

Hamed M. Obiedat*
Ibraheem Abu-falahah
Department of Mathematics
Hashemite University
P.O.Box 150459
Zarqa 13115-Jordan
hobiedat@hu.edu.jo
iabufalahah@hu.edu.jo

Abstract. We characterize the space $\mathfrak{S}_{w_1,w_2}$ of test functions of $(w_1, w_2)-$tempered ultradistribution in terms of their short-time Fourier transform. As a result of this characterization and using Riesz representation theorem, we characterize the space $(w_1, w_2)-$tempered ultradistribution.
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1. Introduction

In mathematical analysis, distributions (generalized functions) are objects which generalize functions. They extend the concept of derivative to all integrable functions and beyond, and used to formulate generalized solutions of partial differential equations. They play a crucial role in physics and engineering where many non-continuous problems naturally lead to differential equations whose solutions are distributions, such as the Dirac delta distribution. In late forties when Laurent Schwartz gave his formulation of distribution theory. This formulation leads to extensive applications in mathematical analysis, mathematical physics, and engineering. Recently, the theory of distributions devised by L. Schwartz is used in micro local analysis, signal processing, image processing and wavelets.

The Schwartz space $\mathfrak{S}$, as defined by Laurent Schwartz (see [13]), consists of all $C^\infty(\mathbb{R}^n)$ functions φ such that $\left\|x^\alpha \partial^\beta \varphi\right\|_\infty < \infty$ for all $\alpha, \beta \in \mathbb{N}^n$. The topological dual space of $\mathfrak{S}$, is a space of generalized functions, called tempered distributions. Tempered distributions have essential connections with the Fourier transform and partial differential equations. Moreover, they fit in many ways to provide a satisfactory framework of mathematical analysis and mathematical physics.

*. Corresponding author

In 1963, A. Beurling presented his generalization of tempered distributions. The aim of this generalization was to find an appropriate context for his work on pseudo-analytic extensions (see [2]).

In 1967 (see [3]), G. Björck studied and expanded the theory of Beurling on ultra distributions to extend the work of Hörmander on existence, nonexistence, and regularity of solutions of differential equations with constant coefficient and also consider equations which have no solutions. The Beurling-Björck space $\mathfrak{S}_w$, as defined by G. Björck, consists all $C^\infty(\mathbb{R}^n)$ functions φ such that $\left\|e^{kw(x)}\partial^\beta\varphi\right\|_\infty < \infty$ and $\left\|e^{kw(x)}\partial^\beta\widehat{\varphi}\right\|_\infty < \infty$ for all $\alpha, \beta \in \mathbb{N}^n$, where w is a subadditive weight function satisfying the classical Beurling conditions. The topological dual $\mathfrak{S}'_w$ of $\mathfrak{S}_w$ is a space of generalized functions, called $w-$tempered ultra distributions. When $w(x) = \log(1 + |x|)$ the Beurling- Björck space $\mathfrak{S}_w$ becomes the Schwartz space $\mathfrak{S}$ (see [1] and [4]). Komatsu (see [10]) proved important structure theorems for these ultradistributions. Some other types of ultradistributions have also been studied by Gel'fand and Shilov (see [9]), called spaces of type S which are well-known in the theory of tempered ultradistributions.

In (see [11]), the authors introduced the space $\mathfrak{S}_{w_1,w_2}$ of all $C^\infty(\mathbb{R}^n)$ functions φ such that $\left\|e^{kw_1(x)}\partial^\beta\varphi\right\|_\infty < \infty$ and $\left\|e^{kw_2(x)}\partial^\beta\widehat{\varphi}\right\|_\infty < \infty$ for all $k \in \mathbb{N}$ and $\beta \in \mathbb{N}^n$, where w_1 and w_2 are two weights satisfying the classical Beurling conditions.

In this paper, we characterize the space $\mathfrak{S}_{w_1,w_2}$ of test functions of (w_1, w_2)-tempered ultradistribution in terms of their short-time Fourier transform. As a result of this characterization and using Riesz representation theorem, we characterize the space (w_1, w_2)-tempered ultradistribution.

The symbols C^∞, C_0^∞, L^p, etc., denote the usual spaces of functions defined on $\mathbb{R}^n$, with complex values. We denote $|\cdot|$ the Euclidean norm on $\mathbb{R}^n$, while $\|\cdot\|_p$ indicates the p-norm in the space L^p, where $1 \le p \le \infty$. In general, we work on the Euclidean space $\mathbb{R}^n$ unless we indicate other than that as appropriate. Partial derivatives will be denoted ∂^α, where α is a multi-index $(\alpha_1, ..., \alpha_n)$ in $\mathbb{N}_0^n$. We will use the standard abbreviations $|\alpha| = \alpha_1 + ... + \alpha_n$, $x^\alpha = x_1^{\alpha_1}...x_n^{\alpha_n}$. The Fourier transform of a function f will be denoted $\mathcal{F}(f)$ or $\widehat{f}$ and it will be defined as $\int_{\mathbb{R}^n} e^{-2\pi i x\xi} f(x)\, dx$. With $\mathcal{C}_0$ we denote the Banach space of continuous functions vanishing at infinity with supremum norm.

2. Preliminary definitions and results

In this section, we start with the definition of the space of admissible functions $\mathcal{M}_c$ as they introduced by Björck.

Definition 1 ([3]). *With $\mathcal{M}_c$ we indicate the space of functions $w : \mathbb{R}^n \to \mathbb{R}$ of the form $w(x) = \Omega(|x|)$, where*

1. $\Omega : [0, \infty) \to [0, \infty)$ *is increasing, continuous and concave,*

2. $\Omega(0) = 0$,

3. $\int_{\mathbb{R}} \frac{\Omega(t)}{(1+t^2)} dt < \infty$,

4. $\Omega(t) \geq a + b \ln(1+t)$ *for some* $a \in \mathbb{R}$ *and some* $b > 0$.

Standard classes of functions w in $\mathcal{M}_c$ are given by

$$w(x) = |x|^d \text{ for } 0 < d < 1, \text{ and } w(x) = p\ln(1+|x|) \text{ for } p > 0.$$

Remark 2. Let us observe for future use that if we take $N > \frac{n}{b}$ is an integer, then

$$C_N = \int_{\mathbb{R}^n} e^{-Nw(x)} dx < \infty, \text{ for all } w \in \mathcal{M}_c,$$

where b is the constant in Condition 4 of Definition 1.

Theorem 3 ([11])**.** *Given* $w_1, w_2 \in \mathcal{M}_c$*, the space* $\mathfrak{S}_{w_1,w_2}$ *can be described as a set as well as topologically by*

$$\mathfrak{S}_{w_1,w_2} = \left\{ \begin{array}{l} \varphi : \mathbb{R}^n \to \mathbb{C} : \varphi \textit{ is continuous and for all} \\ k = 0, 1, 2, ..., \; p_{k,0}(\varphi) < \infty, \pi_{k,0}(\varphi) < \infty \end{array} \right\},$$

where $p_{k,0}(\varphi) = \left\| e^{kw_1}\varphi \right\|_{\infty}$, $\pi_{k,0}(\varphi) = \left\| e^{kw_2}\widehat{\varphi} \right\|_{\infty}$. *The space* $\mathfrak{S}_{w_1,w_2}$*, equipped with the family of semi-norms*

$$\mathcal{N} = \{p_{k,0}, \pi_{k,0} : k \in \mathbb{N}_0\},$$

is a Fréchet space.

Example 4. From Theorem 3, it is clear that the Gaussian $f(x) = e^{-\pi|x|^2}$ belongs to $\mathfrak{S}_{w_1,w_2}$ for all $w_1, w_2 \in \mathcal{M}_c$.

It is well known that Fourier series are a good tool to represent periodic functions. However, they fail to represent non-periodic funtctions. To solve this problem, the short-time Fourier transform was introduced by D. Gabor [6]. The short-time Fourier transform works by first cutting off the function by multiplying it by another function called window then the Fourier transform. This technique maps a function of time x into a function of time x and frequency ξ.

Definition 5 ([7], [8])**.** *The short-time Fourier transform (STFT) of a function or distribution* f *on* $\mathbb{R}^n$ *with respect to a non-zero window function* g *is formally defined as*

$$\nu_g f(x,\xi) = \int_{\mathbb{R}^n} f(t)\overline{g(t-x)} e^{-2\pi i t.\xi} dt = \widehat{(f\overline{T_x g})}(\xi) =< f, M_\xi T_x g >,$$

where $T_x g(t) = g(t-x)$ *is the translation operator and* $M_\xi g(t) = e^{2\pi i t.\xi} g(t)$ *is the modulation operator.*

The composition of T_x and M_ξ is the time-frequency shift

$$(M_\xi T_x g)(t) = e^{2\pi i x.\xi} g(t-x),$$

and its Fourier transform is given by

$$\widehat{M_\xi T_x g} = e^{2\pi i x.\xi} M_{-x} T_\xi \widehat{g}.$$

The main properties of the short-time Fourier transform is given in the following lemma.

Lemma 6 ([7], [8])**.** *For $f, g \in \mathfrak{S}_{w_1,w_2}$, the STFT has the following properties.*

1. (Inversion formula)

$$\int\int_{\mathbb{R}^n\times\mathbb{R}^n} \nu_g f(x,\xi)(M_\xi T_x g)(t)dxd\xi = \|g\|_2^2 f. \tag{1}$$

2. (STFT of the Fourier transforms)

$$\nu_{\widehat{g}} \widehat{f}(x,\xi) = e^{-2\pi i x.\xi} \nu_g f(-\xi, x).$$

3. (Fourier transform of the STFT)

$$\widehat{\nu_g f}(x,\xi) = e^{2\pi i x.\xi} f(-\xi)\overline{\widehat{g}(x).} \tag{2}$$

Remark 7. The space $\nu_g(\mathfrak{S}_{w_1,w_2}) = \{\nu_g f : f \in \mathfrak{S}_{w_1,w_2}\}$ has no functions with compact support.

Now we will introduce two auxiliary results that we will use in the proof of the topological characterization of the space $\mathfrak{S}_{w_1,w_2}$ via the short-time Fourier transform.

Lemma 8 ([8])**.** *Let f and g be two nonnegative measurable functions. If $N > n$, there exists $C > 0$ such that*

$$\left\|e^{kw}(f*g)\right\|_\infty \leq C\left\|e^{2(N+k)w} f\right\|_\infty \left\|e^{2(N+k)w} g\right\|_\infty,$$

for all $k = 0, 1, 2, \ldots$. The constant C does not depend on k.

In the following lemma, we include a proof using the topological characterization of $\mathfrak{S}_{w_1,w_2}$ given in Theorem 3 which imposes no conditions on the derivative. Our proof is an adaptation of the proof of (Proposition 2.6 stated in [8]).

Lemma 9. *Let $g \in \mathfrak{S}_{w_1,w_2}$ be fixed and assume that $F : \mathbb{R}^{2n} \to \mathbb{C}$ is a measurable function that has a subexponential decay, i.e. such that for each $k = 0, 1, 2, \ldots,$ there is a constant $C = C_k > 0$ satisfying*

$$|F(x,\xi)| \leq Ce^{-k(w_1(x)+w_2(\xi))}.$$

Then the integral

$$f(t) = \int\int_{\mathbb{R}^{2n}} F(x,\xi)(M_\xi T_x g)(t)dxd\xi$$

defines a function in $\mathfrak{S}_{w_1,w_2}$.

Proof. To prove that $f \in \mathfrak{S}_{w_1,w_2}$, we start with

$$\begin{aligned}
\left|(e^{kw_1(t)}f)(t)\right| &\leq \int\int_{\mathbb{R}^{2n}} (F(x,\xi)e^{kw_1(t)}(M_\xi T_x g)(t))dxd\xi \\
&\leq \int\int_{\mathbb{R}^{2n}} |F(x,\xi)| \left|M_\xi T_x(e^{kw_1(t+x)}g))(t)\right| dxd\xi \\
&\leq \int\int_{\mathbb{R}^{2n}} |F(x,\xi)| \left|T_x(e^{kw_1(t+x)}g))(t)\right| dxd\xi \\
&\leq \int\int_{\mathbb{R}^{2n}} e^{kw_1(x)}e^{Nw_2(\xi)}e^{-Nw_2(\xi)} |F(x,\xi)| \left\|e^{kw_1}g\right\|_\infty dxd\xi \\
&\leq \int\int_{\mathbb{R}^{2n}} e^{(k+N)(w_1(x)+w_2(\xi))}e^{-N((w_1(x)+w_2(\xi))} |F(x,\xi)| \left\|e^{kw_1}g\right\|_\infty dxd\xi \\
&\leq \left\|e^{kw_1}g\right\|_\infty \left\|e^{(N+k)(w_1(x)+w_2(\xi))}F\right\|_\infty \int\int_{\mathbb{R}^{2n}} e^{-N((w_1(x)+w_2(\xi))}dxd\xi \\
&\leq C\left\|e^{(N+k)(w_1(x)+w_2(\xi))}F\right\|_\infty .
\end{aligned}$$

So,

$$\left\|e^{kw_1}f\right\|_\infty \leq C\left\|e^{(N+k)(w_1(x)+w_2(\xi))}F\right\|_\infty . \tag{3}$$

This implies that $\left\|e^{kw_2}f\right\|_\infty < \infty$.

To show that $\left\|e^{kw_2}\widehat{f}\right\|_\infty < \infty$, we write

$$\widehat{f}(\tau) = \int\int_{\mathbb{R}^{2n}} (F(x,\xi)(M_{-x}T_\xi\widehat{g})(\tau))e^{2\pi i x.\xi}dxd\xi,$$

using

$$(\widehat{M_\xi T_x g})(\tau) = (M_{-x}T_\xi\widehat{g})(\tau)e^{2\pi i x.\xi}.$$

Using an argument similar to the one leading to the proof of (3), we have

$$\left|e^{kw_2(\tau)}\widehat{f}(\tau)\right| \leq C\left\|e^{(N+k)(w_1(x)+w_2(\xi))}F\right\|_\infty .$$

This completes the proof of Lemma 9. $\square$

Remark 10. Given $w \in \mathcal{M}_c$, for the Gaussian $g(x) = e^{-\pi|x|^2}$ and f with $e^{-kw}f \in L^1$ for some $k \in \mathbb{N}_0$, then $\nu_g f$ is well-defined and continuous. In fact,

$$\begin{aligned}
|\nu_g f(x,\xi)| &= \left|\int_{\mathbb{R}^n} f(t)\overline{g(t-x)}e^{-2\pi i t.\xi}dt\right| \\
&\leq \int_{\mathbb{R}^n} \left|f(t)\overline{g(t-x)}e^{-2\pi i t.\xi}\right| dt \\
&= \int_{\mathbb{R}^n} e^{-kw(t)}\left|f(t)\right| e^{kw(t)}\left|\overline{g(t-x)}\right| dt \\
&\leq \int_{\mathbb{R}^n} e^{-kw(t)}\left|f(t)\right| e^{kw(t-x)}\left|\overline{g(t-x)}\right| e^{kw(x)}dt \\
&= \left\|e^{-kw}f\right\|_1 \left\|e^{kw}g\right\|_\infty e^{kw(x)}.
\end{aligned}$$

This shows that $\nu_g f$ is well-defined. Moreover, if we fix $(x_0, \xi_0) \in \mathbb{R}^{2n}$ and let (x_j, ξ_j) be any sequence in $\mathbb{R}^{2n}$ converging to (x_0, ξ_0) as $j \to \infty$, the function $f(t)\overline{g(t-x_j)}e^{-2\pi i t.\xi_j}$ converges to $f(t)\overline{g(t-x_0)}e^{-2\pi i t.\xi_0}$ pointwise as $j \to \infty$ and

$$\begin{aligned}
\left|f(t)\overline{g(t-x_j)}e^{-2\pi i t.\xi_j}\right| &\leq \left|e^{-kw(t)}f(t)e^{kw(t)}\overline{g(t-x_j)}e^{-2\pi i t.\xi_j}\right| \\
&\leq \left|e^{-kw(t)}f(t)e^{kw(t-x_j)}\overline{g(t-x_j)}e^{kw(x_j)}\right| \\
&\leq C\left|e^{-kw(t)}f(t)\right| \left\|e^{kw}g\right\|_\infty \\
&\leq C\left|e^{-kw(t)}f(t)\right|.
\end{aligned}$$

Since the function $\left|e^{-kw(t)}f(t)\right| \in L^1$, we can apply Lebesgue Dominated Convergence Theorem to obtain

$$\nu_g f(x_j, \xi_j) \to \nu_g f(x_0, \xi_0)$$

as $j \to \infty$. This shows the continuity of $\nu_g f$.

3. The short-time Fourier transform over $\mathfrak{S}_{w_1,w_2}$

We use the topological characterization as stated in Theorem 3. Our proof imposes no conditions on the derivative.

Theorem 11. *Let $g(x) = e^{-\pi|x|^2}$ be the Gaussian. Then the space $\mathfrak{S}_{w_1,w_2}$ can be described as a set as well as topologically by*

(4) $\mathfrak{S}_{w_1,w_2} = f : \mathbb{R}^n \to \mathbb{C}$*: $e^{-kw_1}f \in L^1$ for some $k \in \mathbb{N}_0, \forall k \in \mathbb{N}_0, \pi_k(f) < \infty$,*

where $\pi_k(f) = \left\|e^{k(w_1(x)+w_2(\xi))}\nu_g f\right\|_\infty$.

Proof. Let us indicate $\mathfrak{B}_{w_1,w_2}$ the space defined in (4). Observe that the condition $e^{-kw_1}f \in L^1$ for some $k \in \mathbb{N}_0$ implies that $\nu_g f$ is continuous by Remark

10, so the formulation of the condition $\left\|e^{k(w_1(x)+w_2(\xi))}\nu_g f\right\|_\infty$ makes sense. We define in $\mathfrak{B}_{w_1,w_2}$ a structure of Fréchet space by means of the countable family of semi-norms

$$\mathcal{N} = \{\pi_k : k = 0, 1, 2, ...\}.$$

We will show that $\mathfrak{B}_{w_1,w_2} = \mathfrak{S}_{w_1,w_2}$. To do so, we first prove that $\mathfrak{B}_{w_1,w_2} \subseteq \mathfrak{S}_{w_1,w_2}$ continuously. Fix $f \in \mathfrak{B}_{w_1,w_2}$, we need to show that $\|e^{kw_2}\widehat{f}\|_\infty$ and $\|e^{kw_1}f\|_\infty$ are finite. Since $f \in \mathfrak{B}_{w_1,w_2}$, then $\pi_k(f) < \infty$ for all $k \in \mathbb{N}_0$ which implies that $\nu_g f$ has a subexponential decay. Then by Lemma 9 and the inversion formula given in Lemma 6, we can write

$$f(t) = \|g\|_2^{-2} \int\int_{\mathbb{R}^{2n}} (\nu_g f(x,\xi)(M_\xi T_x g)(t))dxd\xi.$$

Using Lemma 9, we have that $\left\|e^{kw_2}\widehat{f}\right\|_\infty$ and $\left\|e^{kw_1}f\right\|_\infty$ are finite for all $k \in \mathbb{N}_0$. Conversely, let $f \in \mathfrak{S}_{w_1,w_2}$, then we know that f is continuous and for all $k \in \mathbb{N}_0$ $p_{k,0}(f) < \infty$, $\pi_{k,0}(f) < \infty$. It is clear that $e^{-kw_1}f \in L^1$ for some $k \in \mathbb{N}_0$ since $f \in \mathfrak{S}_{w_1,w_2}$. To show that $\pi_k(f) < \infty$ for all $k \in \mathbb{N}_0$, we write

$$\begin{aligned} e^{2kw_1(x)}|\nu_g f(x,\xi)| &= e^{2kw_1(x)}\left|\int_{\mathbb{R}^n} f(t)g(x-t)e^{-2\pi i\xi.t}dt\right| \\ &\leq \| e^{2kw_1}(|f| * |g|) \|_\infty . \end{aligned}$$

Using Lemma 8 we get the following estimate

$$\begin{aligned} e^{2kw_1(x)}|\nu_g f(x,\xi)| &\leq \left\|e^{2kw_1}(|f| * |g|)\right\|_\infty \\ &\leq C\left\|e^{2(N+2k)w_1}f\right\|_\infty \left\|e^{2(N+2k)w_1}g\right\|_\infty \\ &\leq C\left\|e^{2(N+2k)w_1}f\right\|_\infty . \end{aligned}$$

Then

$$e^{2kw_1(x)}|\nu_g f(x,\xi)| \leq C\left\|e^{2(N+2k)w_1}f\right\|_\infty . \tag{5}$$

Moreover, since we can write $\nu_g f(x,\xi) = e^{-2\pi i\xi.x}\nu_{\widehat{g}}\widehat{f}(\xi,-x)$, we have the following estimate.

$$\begin{aligned} e^{2kw_2(\xi)}|\nu_g f(x,\xi)| &\leq e^{2kw_2(\xi)}\left|\nu_{\widehat{g}}\widehat{f}(\xi,-x)\right| \\ &\leq \left\|e^{2kw_2}(\left|\widehat{f}\right| * |\widehat{g}|)\right\|_\infty \end{aligned}$$

Once again, using Lemma 8 we obtain

$$\begin{aligned} e^{2kw_2(\xi)}|\nu_g f(x,\xi)| &\leq \left\|e^{2kw_2}(\left|\widehat{f}\right| * |\widehat{g}|)\right\|_\infty \\ &\leq C\left\|e^{2(N+2k)w_2}\widehat{f}\right\|_\infty \left\|e^{2(N+2k)w_2}\widehat{g}\right\|_\infty \\ &\leq C\left\|e^{2(N+2k)w_2}\widehat{f}\right\|_\infty . \end{aligned}$$

Then

$$e^{2kw_2(\xi)} |\nu_g f(x,\xi)| \leq C \left\| e^{2(N+2k)w_2} \widehat{f} \right\|_\infty . \tag{6}$$

Combining (5) and (6), we have that

$$e^{2k(w_1(x)+w_2(\xi))} |\nu_g f(x,\xi)|^2 \leq C(\left\| e^{2(N+2k)w_1} f \right\|_\infty \left\| e^{2(N+2k)w_2} \widehat{f} \right\|_\infty).$$

This implies that

$$\pi_k(f) \leq C(\left\| e^{2(N+2k)w_1} f \right\|_\infty + \left\| e^{2(N+2k)w_2} \widehat{f} \right\|_\infty). \tag{7}$$

So, $f \in \mathfrak{B}_{w_1,w_2}$. Hence $\mathfrak{B}_{w_1,w_2} \subseteq \mathfrak{S}_{w_1,w_2}$ and the inclusion is continuous. This completes the proof of Theorem 11. □

Remark 12. Let $g(x) = e^{-\pi|x|^2}$ be the Gaussian. Then for $f \in \mathfrak{S}_{w_1,w_2}(\mathbb{R}^n)$, we have $\nu_g f \in \mathfrak{S}_{w_1,w_2}(\mathbb{R}^{2n})$.

An argument, similar to the one in Theorem 11, we can show the following characterization.

Corollary 13. *Let $\alpha, \beta > 1$ and $g(x) = e^{-\pi|x|^2}$ be the Gaussian. Then the Gelfand-Shilov space S_α^β can be described as a set as well as topologically by $S_\alpha^\beta = \{\} f : \mathbb{R}^n \to \mathbb{C}$: $e^{-m|x|^{1/\alpha}} f \in L^1$ for some $m \in \mathbb{N}_0$ and $\pi_k(f) < \infty$for all $k \in \mathbb{N}_0$, where $\pi_k(f) = \|e^{k(|x|^{1/\alpha}+|\xi|^{1/\beta})}\nu_g f\|_\infty$.*

4. Representation theorems for functionals in the space $\mathfrak{S}'_{w_1,w_2}$

Theorem 14 ([12]). *Given a functional L in the topological dual of the space $\mathcal{C}_0$, there exists a unique regular complex Borel measure μ so that*

$$L(\varphi) = \int_{\mathbb{R}^n} \varphi d\mu.$$

Moreover, the norm of the functional L is equal to the total variation $|\mu|$ of the measure μ. Conversely, any such measure μ defines a continuous linear functional on $\mathcal{C}_0$.

Theorem 15. *Let $g(x) = e^{-\pi|x|^2}$be the Gaussian. Then if $L : \mathfrak{S}_{w_1,w_2} \to \mathbb{C}$, the following statements are equivalent: (i) $L \in \mathfrak{S}'_{w_1,w_2}$ (ii) There exist a regular complex Borel measure μ of finite total variation and $k \in \mathbb{N}_0$ so that*

$$L = e^{k(w_1(x)+w_2(\xi))}\nu_g d\mu,$$

in the sense of $\mathfrak{S}'_{w_1,w_2}$.

Proof. $(i) \Rightarrow (ii)$. Given $L \in \mathfrak{S}'_{w_1,w_2}$, there exist k, C so that

$$L(\varphi) \leq C \left\| e^{k(w_1(x)+w_2(\xi))} \nu_g \varphi \right\|_\infty$$

for all $\varphi \in \mathfrak{S}_{w_1,w_2}$. Moreover, the map

$$\begin{aligned} \mathfrak{S}_{w_1,w_2}(\mathbb{R}^n) &\to \mathcal{C}_0(\mathbb{R}^{2n}) \\ \varphi &\to e^{k(w_1(x)+w_2(\xi))} \nu_g \varphi \end{aligned}$$

is well-defined, linear, continuous and injective. Let $\mathcal{R}$ be the range of this map. We define on $\mathcal{R}$ the map

$$l_1 \left(e^{k(w_1(x)+w_2(\xi))} \nu_g \varphi \right) = L(\varphi),$$

for a unique $\varphi \in \mathfrak{S}_{w_1,w_2}$. The map $l_1 : \mathcal{R} \to \mathbb{C}$ is linear and continuous. By the Hahn-Banach theorem, there exists a functional L_1 in the topological dual $\mathcal{C}'_0(\mathbb{R}^{2n})$ of $\mathcal{C}_0(\mathbb{R}^{2n})$ such that $\|L_1\| = \|l_1\|$ and the restriction of L_1 to $\mathcal{R}$ is l_1. Using Theorem 14, there exist a regular complex Borel measure μ of finite total variation so that

$$L_1(f) = \int_{\mathbb{R}^{2n}} f d\mu$$

for all $f \in \mathcal{C}_0(\mathbb{R}^{2n})$. If $f \in \mathcal{R}$, we conclude

$$L(\varphi) = \int_{\mathbb{R}^{2n}} e^{k(w_1(x)+w_2(\xi))} \nu_g \varphi d\mu$$

for all $\varphi \in \mathfrak{S}_{w_1,w_2}$. In the sense of $\mathfrak{S}'_{w_1,w_2}$,

$$L = e^{k(w_1(x)+w_2(\xi))} \nu_g d\mu.$$

$(ii) \Rightarrow (i)$. If μ is a regular complex Borel measure satisfying (ii) and $\varphi \in \mathfrak{S}_{w_1,w_2}$, then

$$L(\varphi) = \int_{\mathbb{R}^{2n}} e^{k(w_1(x)+w_2(\xi))} \nu_g \varphi d\mu.$$

This implies that

$$\begin{aligned} |L(\varphi)| &\leq \left| \int_{\mathbb{R}^{2n}} e^{k(w_1(x)+w_2(\xi))} \nu_g \varphi d\mu \right| \\ &\leq |\mu| (\mathbb{R}^{2n}) \left\| e^{k(w_1(x)+w_2(\xi))} \nu_g \varphi \right\|_\infty \\ &\leq C(\left\| e^{k(w_1(x)+w_2(\xi))} \nu_g \varphi \right\|_\infty). \end{aligned}$$

It may be noted that μ, employed to obtain the above inequality, is of finite total variation. This completes the proof of Theorem 15. $\square$

Remark 16. Any $L \in (S_\alpha^\beta)'(\mathbb{R}^n)$, with $\alpha, \beta > 1$, can be written as

$$L = e^{k(e^{k(|x|^{1/\alpha}+|\xi|^{1/\beta})})}\nu_g d\mu$$

which characterizes the dual space of the Gelfand-Shilov space $(S_\alpha^\beta)'(\mathbb{R}^n)$.

Remark 17. Any $L \in \mathfrak{S}'$ can be written as

$$L = (1 + |x|^k + |\xi|^k)\nu_g d\mu$$

which characterizes tempered distributions.

Corollary 18. *If* $L \in \mathfrak{S}'_{w_1,w_2}$ *and* $\varphi \in \mathfrak{S}_{w_1,w_2}$, *then the classical definition of the convolution* $L * \varphi$ *is defined by*

$$(L * \varphi, \psi)_{\mathfrak{S}'_{w_1,w_2},\mathfrak{S}_{w_1,w_2}} = (L_x, (\varphi_z, \psi(x+y)))_{\mathfrak{S}'_{w_1,w_2},\mathfrak{S}_{w_1,w_2}}$$

for all $\psi \in \mathfrak{S}_{w_1,w_2}$. *Moreover, the functional* $L * \varphi$ *coincides with the functional given by the integration against the function*

$$f(y) = (L, \varphi(y - \cdot))_{\mathfrak{S}'_{w_1,w_2},\mathfrak{S}_{w_1,w_2}}.$$

Proof. The Inequality

$$\begin{aligned}
|f(y)| &= \left|\int_{\mathbb{R}^{2n}} e^{k(w_1(x)+w_2(\xi))}\nu_g \varphi(y-x,\xi)d\mu\right| \\
&\le \left|\int_{\mathbb{R}^{2n}} e^{k(w_1(y-x)+w_2(\xi))}e^{kw_1(y)}\nu_g \varphi(y-x,\xi)d\mu\right| \\
&\le C\left\|e^{k(w_1(x)+w_2(\xi))}\nu_g\varphi\right\|_\infty
\end{aligned}$$

implies the continuity of $f(y) = (L, \varphi(y - \cdot))_{\mathfrak{S}'_{w_1,w_2},\mathfrak{S}_{w_1,w_2}}$. Then

$$\begin{aligned}
&(L * \varphi, \psi)_{\mathfrak{S}'_{w_1,w_2},\mathfrak{S}_{w_1,w_2}} = (L_x, (\varphi_z, \psi(x+y)))_{\mathfrak{S}'_{w_1,w_2},\mathfrak{S}_{w_1,w_2}} \\
&= \int_{\mathbb{R}^{2n}} e^{k(w_1(x)+w_2(\xi))}\nu_g(\int_{\mathbb{R}^n} \varphi(y-x)\psi(y)dy)dxd\xi \\
&= \int_{\mathbb{R}^{2n}} e^{k(w_1(x)+w_2(\xi))}\nu_g(\psi * \varphi(x))dxd\xi \\
&= (e^{k(w_1(x)+w_2(\xi))}\nu_g, \psi * \varphi(x))_{\mathfrak{S}'_{w_1,w_2},\mathfrak{S}_{w_1,w_2}} \\
&= (e^{k(w_1(x)+w_2(\xi))}\nu_g\varphi(y-x,\xi), \psi(y))_{\mathfrak{S}'_{w_1,w_2},\mathfrak{S}_{w_1,w_2}} \\
&= ((e^{k(w_1(x)+w_2(\xi))}\nu_g, \varphi(y-x))_{\mathfrak{S}'_{w_1,w_2},\mathfrak{S}_{w_1,w_2}}, \psi(y))_{\mathfrak{S}'_{w_1,w_2},\mathfrak{S}_{w_1,w_2}} \\
&= ((e^{k(w_1(x)+w_2(\xi))}\nu_g, (\varphi(y-x), \psi(y))_{\mathfrak{S}'_{w_1,w_2},\mathfrak{S}_{w_1,w_2}} \\
&= (L_x, (\varphi(y-x), \psi(y)))_{\mathfrak{S}'_{w_1,w_2},\mathfrak{S}_{w_1,w_2}}.
\end{aligned}$$

$\square$

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