THE FIXED POINT OF MEROMORPHIC SOLUTIONS FOR DIFFERENCE RICCATI EQUATION

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Abstract. In this paper, we mainly investigate some properties of the transcendental meromorphic solution $f(z)$ for the difference Riccati equation

$$f(z+1) = \frac{P_1(z)f(z) + P_2(z)}{f(z) + P_3(z)},$$

where $P_i(z)(i = 1, 2, 3)$ are polynomials. And we obtain some estimates of exponents of convergence of fixed points and $c-$points of $f(z)$ and its shift $f(z + n)$.
Keywords: difference Riccati equation, fixed point, admissible solution.

1. Introduction and main results

Early results for difference equations were largely motivated by the work of Kimura [13] on the iteration of analytic functions. Shimomura [19] and Yanagihara [20] proved the following theorems, respectively.

Theorem A ([19]). *For any polynomial $P(y)$, the difference equation*

$$y(z + 1) = P(y(z))$$

has a non-trivial entire solution.

Theorem B ([20]). *For any rational function $R(y)$, the difference equation*

$$y(z + 1) = R(y(z))$$

has a non-trivial meromorphic solution.

In this paper, we assume that the reader is familiar with the basic notions of Nevanlinna value distribution theory (see, e.g., [11, 14, 21]). In addition, we use the notations $\sigma(f)$ to denote the order of growth of the meromorphic function $f(z)$, $\lambda(f)$ and $\lambda(\frac{1}{f})$ to denote the exponents of convergence of zeros and poles of $f(z)$, respectively. Moreover, we say that a meromorphic function $g(z)$ is small with respect to $f(z)$ if $T(r, g) = S(r, f)$, where $S(r, f) = o(T(r, f))$ outside of a possible exceptional set of finite logarithmic measure. And we denote by $S(f)$ the family of all meromorphic functions which are small compared to $f(z)$. We say that a meromorphic solution $f(z)$ of a difference equation is admissible if all coefficients of the equation are in $S(f)$. We also use the notation $\tau(f)$ to denote the exponent of convergence of fixed points of $f(z)$ that is defined as

$$\tau(f) = \overline{\lim_{r\to\infty}} \frac{\log N\left(r, \frac{1}{f-z}\right)}{\log r}.$$

Recently, a number of papers (including [1-6, 8-10, 12,15-18]) focused on complex difference equations and difference analogues of Nevanlinna theory. As the difference analogues of Nevanlinna theory were being investigated, many results on the complex difference equations have been got rapidly. Many papers (including [1, 6, 10, 12]) mainly dealt with the growth of meromorphic solutions of difference equations.

In [10], Halburd and Korhonen used value distribution theory to obtain the following Theorem C.

Theorem C ([10]). *Let $f(z)$ be an admissible finite order meromorphic solution of the equation*

$$f(z+1)f(z-1) = \frac{c_2(f(z)-c_+)(f(z)-c_-)}{(f(z)-a_+)(f(z)-a_-)} =: R(z, f(z)), \tag{1.1}$$

where the coefficients are meromorphic functions, $c_2 \not\equiv 0$ and $\deg_f(R) = 2$. If the order of the poles of $f(z)$ is bounded, then either $f(z)$ satisfies a difference Riccati equation

$$f(z+1) = \frac{p(z)f(z)+q(z)}{f(z)+s(z)},$$

where $p,\ q,\ s \in S(f)$, or equation (1.1) can be transformed by a bilinear change in $f(z)$ to one of the equations

$$f(z+1)f(z-1) = \frac{\gamma f^2(z) + \delta\lambda^z f(z) + \gamma\mu\lambda^{2z}}{(f(z)-1)(f(z)-\gamma)},$$

$$f(z+1)f(z-1) = \frac{f^2(z) + \delta e^{i\pi z/2}\lambda^z f(z) + \mu\lambda^{2z}}{f^2(z)-1},$$

where $\lambda \in \mathbb{C}$, and $\delta,\ \mu,\ \gamma,\ \underline{\gamma} = \gamma(z-1) \in S(f)$ are arbitrary finite order periodic functions such that δ and γ have period 2 and μ has period 1.

Chen [4] obtained the following Theorem D.

Theorem D ([4]). *Let $P_j(z),\ j = 1, 2, 3$ be nonzero polynomials, such that*

$$\deg P_3(z) > \max\{\deg P_j(z) : j = 1, 2\}.$$

If $c \in \mathbb{C} \setminus \{0\}$, then every finite order transcendental meromorphic solution $f(z)$ of equation

$$f(z+1) = \frac{P_1(z)f(z)}{P_2(z)f(z) + P_3(z)}$$

satisfies:

(*i*) $\lambda(f(z+n) - c) = \sigma(f) \geq 1, \qquad n = 0, 1, 2, \ldots;$

(*ii*) *if* $\deg P_1 \neq \deg P_2$, *then* $\lambda\left(\frac{\Delta f(z)}{f(z)} - c\right) = \sigma(f)$;

(*iii*) *if there is a polynomial $h(z)$ satisfying*

$$(P_2(z) - P_1(z) + cP_3(z))^2 - 4cP_2(z)P_3(z) = h(z)^2,$$

then $\lambda\left(\Delta f(z) - c\right) = \sigma(f)$.

From the above, we see that the difference Riccati equation is an important class of difference equations, it will play an important role in research of difference Painlevé equations. Some papers [2-4] dealt with complex difference Riccati equations.

In this research, we investigate some properties of finite order transcendental meromorphic solution for certain difference Riccati equation, and obtain the following theorems.

Theorem 1.1. *Let $P_j(z)(j = 1, 2, 3)$ be nonzero polynomials, such that there exists an integer $l(1 \leq l \leq 3)$ with*

$$\deg P_l(z) > \max_{j \neq l}\{\deg P_j(z)\}. \tag{1.2}$$

If $c \in \mathbb{C}$, then every finite order transcendental meromorphic solution $f(z)$ of the difference equation

$$f(z+1) = \frac{P_1(z)f(z) + P_2(z)}{f(z) + P_3(z)}, \tag{1.3}$$

where $P_1(z)f(z) + P_2(z)$ and $f(z) + P_3(z)$ are relatively prime polynomials in f, satisfies $\lambda\left(f(z+n) - c\right) = \sigma(f)$, $n = 0, 1, 2, \ldots$.

Theorem 1.2. *Let $P_j(z)(j = 1, 2, 3)$ be nonzero polynomials, such that*

$$\deg P_1(z) > \max\{\deg P_2(z), \deg P_3(z)\} \qquad \textit{and} \qquad \deg P_1(z) \geq 2, \tag{1.4}$$

or

$$\deg P_3(z) > \max\{\deg P_2(z), \deg P_1(z)\} \qquad \textit{and} \qquad \deg P_3(z) \geq 2. \tag{1.5}$$

Then every finite order transcendental meromorphic solution $f(z)$ of the difference equation (1.3) *satisfies $\tau(f(z+n)) = \sigma(f)$, $n = 0, 1, 2, \ldots$.*

2. Proof of Theorem 1.1

We need the following lemmas to prove Theorem 1.1.

Lemma 2.1 ([8,15]). *Let f be a transcendental meromorphic solution of finite order σ of the difference equation*

$$P(z, f) = 0,$$

where $P(z, f)$ is a difference polynomial in $f(z)$ and its shifts. If $P(z, a) \not\equiv 0$ for a slowly moving target function a, i.e. $T(r, a) = S(r, f)$, then

$$m\left(r, \frac{1}{f-a}\right) = S(r, f)$$

outside of a possible exceptional set of finite logarithmic measure.

Remark 2.1. Using the same method as in the proof of Lemma 2.1 (see [8]), we can prove that in Lemma 2.1, if all coefficients $b_\lambda(z)$ of $P(z, f(z))$ satisfy $\sigma(b_\lambda(z)) = \sigma_1 < \sigma(f(z)) = \sigma$, and if $P(z, a) \not\equiv 0$ for a meromorphic function $a(z)$ satisfying $T(r, a) = S(r, f)$, then for a given $\varepsilon(0 < \varepsilon < \sigma - \sigma_1)$

$$m\left(r, \frac{1}{f(z)-a}\right) = S(r, f(z)) + O(r^{\sigma_1+\varepsilon})$$

holds for all r outside of a possible exceptional set of finite logarithmic measure.

Lemma 2.2 ([15]). *Let f be a transcendental meromorphic solution of finite order σ of a difference equation of the form*

$$H(z, f)P(z, f) = Q(z, f),$$

where $H(z, f)$ is a difference product of total degree n in $f(z)$ and its shifts, and where $P(z, f)$, $Q(z, f)$ are difference polynomials such that the total degree $degQ(z, f) \leq n$. Then for each $\varepsilon > 0$,

$$m(r, P(z, f)) = O(r^{\sigma-1+\varepsilon}) + S(r, f)$$

possibly outside of an exceptional set of finite logarithmic measure.

Lemma 2.3 (Valiron-Mohon'ko, see [14]). *Let $f(z)$ be a meromorphic function. Then for all irreducible rational function in $f(z)$,*

$$R(z, f(z)) = \frac{a_0(z) + a_1(z)f(z) + \ldots + a_m(z)f(z)^m}{b_0(z) + b_1(z)f(z) + \ldots + b_n(z)f(z)^n}$$

with meromorphic coefficients $a_i(z)(i = 0, 1, ..., m)$, $b_j(z)(j = 0, 1, ..., n)$, the characteristic function of $R(z, f(z))$ satisfies

$$T(r, R(z, f(z))) = dT(r, f) + O(\Psi(r)),$$

where $d = deg_f R = \max\{m, n\}$ *and* $\Psi(r) = \max_{i,j}\{T(r, a_i), T(r, b_j)\}$.

In the remark of [10, p.15], it is pointed out that the following Lemma 2.4 holds.

Lemma 2.4 ([2]). *Let f be a nonconstant finite order meromorphic function. Then*

$$N(r+1, f) = N(r, f) + S(r, f), \quad T(r+1, f) = T(r, f) + S(r, f)$$

outside of a possible exceptional set of finite logarithmic measure.

Remark 2.2. In [6], Chiang and Feng proved that let f be a meromorphic function with exponent of convergence of poles $\lambda(\frac{1}{f}) = \lambda < \infty$, $\eta \neq 0$ be fixed, then for each $\varepsilon > 0$,

$$N(r, f(z+\eta)) = N(r, f) + O(r^{\lambda-1+\varepsilon}) + O(\log r).$$

Lemma 2.5 ([6]). *Let $f(z)$ be a meromorphic function with order $\sigma = \sigma(f) < +\infty$, and let η be a fixed nonzero complex number, then for each $\varepsilon > 0$, we have*

$$T(r, f(z+\eta)) = T(r, f) + O(r^{\sigma-1+\varepsilon}) + O(\log r).$$

Lemma 2.6. ([7]). *Let $g : (0, +\infty) \to R$, $h : (0, +\infty) \to R$ be non-decreasing functions. If $(i) g(r) \leq h(r)$ outside of an exceptional set of finite linear measure, or $(ii) g(r) \leq h(r)$, $r \notin H \cup (0, 1]$, where $H \subset (1, \infty)$ is a set of finite logarithmic measure, then for any $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) \leq h(\alpha r)$ for all $r > r_0$.*

Lemma 2.7. *Suppose that $P_j(z)$ $(j = 1, 2, 3)$ satisfy conditions in Theorem 1.1, $c \in \mathbb{C}$ is a constant, and $f(z)$ is a nonconstant meromorphic function. Then*

$$f_1(z) = (P_1(z) - c)f(z) + P_2(z) - cP_3(z)$$

and

$$f_2(z) = f(z) + P_3(z)$$

have at most finitely many common zeros.

Proof. Suppose that z_0 is a common zero of $f_1(z)$ and $f_2(z)$. Then $f_2(z_0) = f(z_0) + P_3(z_0) = 0$. Thus, $f(z_0) = -P_3(z_0)$. Substituting $f(z_0) = -P_3(z_0)$ into $f_1(z)$, we obtain

$$f_1(z_0) = -(P_1(z_0) - c)P_3(z_0) + P_2(z_0) - cP_3(z_0) = -P_1(z_0)P_3(z_0) + P_2(z_0) = 0.$$

Since $P_1(z)f(z) + P_2(z)$ and $f(z) + P_3(z)$ are relatively prime polynomials in f, we get $-P_1(z)P_3(z) + P_2(z) \not\equiv 0$. And since $-P_1(z)P_3(z) + P_2(z)$ has only finitely many zeros, we see that $f_1(z)$ and $f_2(z)$ have at most finitely many common zeros.

Using the similar method as in the proof of Lemma 2.7, we can prove the following Lemma 2.8.

Lemma 2.8. *Suppose that $P_j(z)$ $(j = 1, 2, 3)$ satisfy conditions in Theorem* 1.2, *and $f(z)$ is a nonconstant meromorphic function. Then*

$$f_1(z) = (P_1(z) - z)f(z) + P_2(z) - zP_3(z)$$

and

$$f_2(z) = f(z) + P_3(z)$$

have at most finitely many common zeros.

Proof of Theorem 1.1. Suppose that $f(z)$ is a finite order transcendental meromorphic solution of equation (1.3). We divide this proof into the following two cases.

Case 1. Suppose that $c = 0$.

Firstly, we suppose that $n = 0$. By (1.3), we get

$$H_0(z, f) := f(z + 1)(f(z) + P_3(z)) - P_1(z)f(z) - P_2(z) = 0. \tag{2.1}$$

Thus,

$$H_0(z, 0) = -P_2(z) \not\equiv 0$$

since $P_2(z)$ is a nonzero polynomial. Thus, by Lemma 2.1, we have that

$$m\left(r, \frac{1}{f(z)}\right) = S(r, f(z)) \tag{2.2}$$

holds for all r outside of a possible exceptional set with finite logarithmic measure. So, by (2.2), we obtain

$$N\left(r, \frac{1}{f(z)}\right) = T(r, f(z)) + S(r, f(z)) \tag{2.3}$$

holds for all r outside of a possible exceptional set with finite logarithmic measure. Hence, by Lemma 2.6 and (2.3), we see that $\lambda(f(z)) = \sigma(f(z))$.

Now suppose that $n = 1$. Set $g(z) = f(z + 1)$. Thus, by (1.3), we get

$$g(z + 1) = \frac{P_1(z + 1)g(z) + P_2(z + 1)}{g(z) + P_3(z + 1)}. \tag{2.4}$$

For (2.4), applying the conclusion for $n = 0$ above and Lemma 2.5, we obtain

$$\lambda(f(z + 1)) = \lambda(g(z)) = \sigma(g(z)) = \sigma(f(z)).$$

Continuing to use the similar method as above, we can obtain

$$\lambda(f(z + n)) = \sigma(f(z)) \quad n = 0, 1, 2, \ldots.$$

Case 2. Suppose that $c \in \mathbb{C} \setminus \{0\}$. Firstly, we suppose that $n = 0$. By (1.3), we get (2.1). Thus, by (2.1), we get

$$H_0(z,c) = cP_3(z) - cP_1(z) - P_2(z) + c^2.$$

By $c \neq 0$ and (1.2), we see that $H_0(z,c) \not\equiv 0$. Thus, by Lemma 2.1, we have that

$$m\left(r, \frac{1}{f(z)-c}\right) = S(r, f(z))$$

holds for all r outside of a possible exceptional set with finite logarithmic measure. So, we have that

$$N\left(r, \frac{1}{f(z)-c}\right) = T(r, f(z)) + S(r, f(z)) \tag{2.5}$$

holds for all r outside of a possible exceptional set with finite logarithmic measure. So, by Lemma 2.6 and (2.5), we obtain $\lambda(f(z) - c) = \sigma(f(z))$.

Now suppose that $n = 1$, By (1.3), we obtain

$$f(z+1) - c = \frac{(P_1(z) - c)f(z) + P_2(z) - cP_3(z)}{f(z) + P_3(z)}. \tag{2.6}$$

If $P_1(z) = c$, then by (2.6) we have that

$$f(z+1) - c = \frac{P_2(z) - cP_3(z)}{f(z) + P_3(z)}. \tag{2.7}$$

By (1.2) and $P_1(z) = c$, we get $\deg P_2(z) \neq \deg P_3(z)$. Thus $P_2(z) - cP_3(z) \not\equiv 0$. From (2.7), we see that if $f(z) = \infty$, then $f(z+1) - c = 0$. Thus, we get $\lambda(f(z+1) - c) = \lambda(\frac{1}{f})$.

By (1.3), we get

$$(f(z) + P_3(z))f(z+1) = P_1(z)f(z) + P_2(z). \tag{2.8}$$

By Lemma 2.2 and (2.8), we have that

$$m(r, f(z+1)) = O(r^{\sigma(f)-1+\varepsilon}) + S(r, f) \tag{2.9}$$

holds for all r outside of a possible exceptional set with finite logarithmic measure. By Lemma 2.3, we get

$$T(r, f(z+1)) = T(r, f(z)) + S(r, f). \tag{2.10}$$

From Lemma 2.4, we obtain

$$N(r, f(z+1)) \leq N(r+1, f(z)) = N(r, f(z)) + S(r, f) \tag{2.11}$$

holds for all r outside of a possible exceptional set with finite logarithmic measure. Thus, by (2.9)-(2.11), we get

$$T(r,f(z)) \leq N(r,f(z)) + O(r^{\sigma(f)-1+\varepsilon}) + S(r,f) \tag{2.12}$$

holds for all r outside of a possible exceptional set with finite logarithmic measure. Hence, by Lemma 2.6 and (2.12), we see that $\lambda\left(\frac{1}{f}\right) = \sigma(f)$. Furthermore, we get $\lambda(f(z+1)-c) = \sigma(f)$.

If $P_1(z) - c \not\equiv 0$ and $cP_3(z) \not\equiv P_2(z)$, then by (2.6) we have that

$$f(z+1) - c = \frac{(P_1(z)-c)\left(f(z) - \frac{cP_3(z)-P_2(z)}{P_1(z)-c}\right)}{f(z)+P_3(z)}. \tag{2.13}$$

By Lemma 2.7, we see that $\left(f(z) - \frac{cP_3(z)-P_2(z)}{P_1(z)-c}\right)$ and $f(z)+P_3(z)$ have at most finitely many common zeros. So by (2.13), we only need to prove that

$$\lambda\left(f(z) - \frac{cP_3(z)-P_2(z)}{P_1(z)-c}\right) = \sigma(f(z)). \tag{2.14}$$

Suppose that $\lambda(f(z) - \frac{cP_3(z)-P_2(z)}{P_1(z)-c}) < \sigma(f(z))$. By $\sigma(f(z) - \frac{cP_3(z)-P_2(z)}{P_1(z)-c}) = \sigma(f(z))$ and Hadamard factorization theorem, $f(z) - \frac{cP_3(z)-P_2(z)}{P_1(z)-c}$ can be rewritten as a form

$$f(z) - \frac{cP_3(z)-P_2(z)}{P_1(z)-c} = z^m \frac{b_0(z)}{H_0(z)} e^{h(z)}, \tag{2.15}$$

where $h(z)$ is a polynomial with $\deg h(z) \leq \sigma(f(z))$, $b_0(z)$ and $H_0(z)$ are canonical products ($b_0(z)$ may be a polynomial) formed by nonzero zeros and poles of $f(z) - \frac{cP_3(z)-P_2(z)}{P_1(z)-c}$ respectively, m is an integer, if $m \geq 0$ then $b(z) = z^m b_0(z)$, $H(z) = H_0(z)e^{-h(z)}$; if $m < 0$ then $b(z) = b_0(z)$, $H(z) = z^{-m}H_0(z)e^{-h(z)}$. By property of canonical product, we see that

$$\begin{cases} \lambda(b(z)) = \sigma(b(z)) = \lambda\left(f(z) - \frac{cP_3(z)-P_2(z)}{P_1(z)-c}\right) < \sigma(f(z)), \\ \lambda(H(z)) = \sigma(H(z)) = \sigma(f(z)). \end{cases} \tag{2.16}$$

By (2.15), we obtain

$$\begin{cases} f(z) = \frac{cP_3(z)-P_2(z)}{P_1(z)-c} + b(z)y(z), \\ f(z+1) = \frac{cP_3(z+1)-P_2(z+1)}{P_1(z+1)-c} + b(z+1)y(z+1), \end{cases} \tag{2.17}$$

where $y(z) = \frac{1}{H(z)}$. Thus, by (2.16) and Lemma 2.5, we have that

$$\sigma(y(z)) = \sigma(H(z)) = \sigma(f(z)),\ \sigma(b(z+1)) = \sigma(b(z)) < \sigma(y(z)).$$

Substituting (2.17) into (1.3), we obtain

$$\begin{aligned} K_1(z,y) := &\left\{\frac{cP_3(z+1)-P_2(z+1)}{P_1(z+1)-c}+b(z+1)y(z+1)\right\} \\ &\cdot\left\{\frac{cP_3(z)-P_2(z)}{P_1(z)-c}+b(z)y(z)+P_3(z)\right\} \\ &-P_1(z)\left\{\frac{cP_3(z)-P_2(z)}{P_1(z)-c}+b(z)y(z)\right\}-P_2(z)=0. \end{aligned} \tag{2.18}$$

By (2.18), we see that

$$\begin{aligned} K_1(z,0) &= \frac{cP_3(z+1)-P_2(z+1)}{P_1(z+1)-c}\cdot\left\{\frac{cP_3(z)-P_2(z)}{P_1(z)-c}+P_3(z)\right\} \\ &\quad -P_1(z)\frac{cP_3(z)-P_2(z)}{P_1(z)-c}-P_2(z) \\ &= \frac{P_3(z)P_1(z)-P_2(z)}{P_1(z)-c}\left(\frac{cP_3(z+1)-P_2(z+1)}{P_1(z+1)-c}-c\right). \end{aligned}$$

That is

$$K_1(z,0)=\frac{P_3(z)P_1(z)-P_2(z)}{P_1(z)-c}\cdot\frac{cP_3(z+1)-P_2(z+1)-cP_1(z+1)+c^2}{P_1(z+1)-c}.$$

Since $P_1(z)f(z)+P_2(z)$ and $f(z)+P_3(z)$ are relatively prime polynomials in f, we see that $P_3(z)P_1(z)-P_2(z)\not\equiv 0$. And by (1.2), we get $cP_3(z+1)-P_2(z+1)-cP_1(z+1)+c^2\not\equiv 0$. Thus, we obtain

$$K_1(z,0)\not\equiv 0. \tag{2.19}$$

Thus, by (2.16), (2.19), Lemma 2.1 and its Remark 2.1, we obtain for any given $\varepsilon(0<\varepsilon<\sigma(f(z))-\sigma(b(z)))$

$$N\left(r,\frac{1}{y(z)}\right)=T(r,y(z))+S(r,y(z))+O(r^{\sigma(b(z))+\varepsilon}) \tag{2.20}$$

holds for all r outside of a possible exceptional set with finite logarithmic measure.

On the other hand, by $y(z)=\frac{1}{H(z)}$ and the fact that $H(z)$ is an entire function, we see that

$$N\left(r,\frac{1}{y(z)}\right)=N(r,H(z))=0.$$

Thus, (2.20) is a contradiction. Hence, $\lambda\left(f(z)-\frac{cP_3(z)-P_2(z)}{P_1(z)-c}\right)=\sigma(f(z))$.

If $P_1(z)-c\not\equiv 0$ and $cP_3(z)-P_2(z)\equiv 0$, then by (2.6) we have that

$$f(z+1)-c=\frac{(P_1(z)-c)f(z)}{f(z)+P_3(z)}. \tag{2.21}$$

By Lemma 2.7, we see that $(P_1(z)-c)f(z)$ and $f(z)+P_3(z)$ have at most finitely many common zeros. So that we have that $\lambda(f(z+1)-c)=\lambda(f(z))$. By Case 1, we see that $\lambda(f(z))=\sigma(f(z))$. Thus, we obtain $\lambda(f(z+1)-c)=\sigma(f(z))$.

Now suppose that $n=2$, By (2.6), we have that

$$f(z+2)-c=\frac{(P_1(z+1)-c)f(z+1)+P_2(z+1)-cP_3(z+1)}{f(z+1)+P_3(z+1)}. \tag{2.22}$$

Set $g(z)=f(z+1)$. Thus, (2.22) is transformed as

$$g(z+1)-c=\frac{(P_1(z+1)-c)g(z)+P_2(z+1)-cP_3(z+1)}{g(z)+P_3(z+1)}. \tag{2.23}$$

Since $\deg P_j(z+1)=\deg P_j(z)$, thus, $P_j(z+1)(j=1,2,3)$ satisfy (1.2). For (2.23), applying the conclusion for $n=1$ above, we obtain $\lambda(f(z+2)-c)=\lambda(g(z+1)-c)=\sigma(g(z))=\sigma(f(z))$. Continuing to use the same method as above, we can obtain $\lambda(f(z+n)-c)=\sigma(f(z)) \quad n=0,1,2,\ldots$.

Theorem 1.1 is proved.

3. Proof of Theorem 1.2

Suppose that $f(z)$ is a finite order transcendental meromorphic solution of equation (1.3).

Firstly, we suppose that $n=0$. Set $f(z)-z=g(z)$. So, $g(z)$ is transcendental, $T(r,g(z))=T(r,f(z))+O(\log r)$ and $S(r,g)=S(r,f)$. Substituting $f(z)=z+g(z)$ into (1.3), we get $H_0(z,g):=(g(z+1)+z+1)(g(z)+z+P_3(z))-P_1(z)(g(z)+z)-P_2(z)=0$. Thus,

$$H_0(z,0)=(z+1)P_3(z)-zP_1(z)-P_2(z)+z(z+1). \tag{3.1}$$

By (1.4) or (1.5) and (3.1), we see that $H_0(z,0)\not\equiv 0$. Thus, by Lemma 2.1 and $H_0(z,0)\not\equiv 0$, we have that

$$N\left(r,\frac{1}{g(z)}\right)=T(r,g(z))+S(r,g(z))$$

holds for all r outside of a possible exceptional set with finite logarithmic measure. So, we obtain

$$N\left(r,\frac{1}{f(z)-z}\right)=T(r,f(z))+S(r,f(z)) \tag{3.2}$$

holds for all r outside of a possible exceptional set with finite logarithmic measure. Hence, by Lemma 2.6 and (3.2), we see that $\tau(f(z))=\sigma(f(z))$.

Now suppose that $n=1$, By (1.3), we obtain

$$f(z+1)-z=\frac{(P_1(z)-z)f(z)+P_2(z)-zP_3(z)}{f(z)+P_3(z)}. \tag{3.3}$$

If $P_1(z) = z$, then by (3.3) we have that

$$f(z+1) - z = \frac{P_2(z) - zP_3(z)}{f(z) + P_3(z)}. \tag{3.4}$$

By $P_1(z) = z$, we see that $P_1(z)$, $P_2(z)$, $P_3(z)$ satisfy (1.5). Thus $P_2(z) - zP_3(z) \not\equiv 0$. From (3.4), we see that if $f(z) = \infty$, then $f(z+1) - z = 0$. Thus, we get $\tau(f(z+1)) = \lambda\left(\frac{1}{f}\right)$. From the Proof of Theorem 1.1, we see that $\lambda\left(\frac{1}{f}\right) = \sigma(f)$. Furthermore, we get $\tau(f(z+1)) = \sigma(f)$.

If $P_1(z) - z \not\equiv 0$ and $zP_3(z) \not\equiv P_2(z)$, then by (3.3), we have that

$$f(z+1) - z = \frac{(P_1(z) - z)\left(f(z) - \frac{zP_3(z)-P_2(z)}{P_1(z)-z}\right)}{f(z) + P_3(z)}. \tag{3.5}$$

By Lemma 2.8, we see that $\left(f(z) - \frac{zP_3(z)-P_2(z)}{P_1(z)-z}\right)$ and $f(z)+P_3(z)$ have at most finitely many common zeros. So by (3.5), we only need to prove that

$$\lambda\left(f(z) - \frac{zP_3(z) - P_2(z)}{P_1(z) - z}\right) = \sigma(f(z)). \tag{3.6}$$

Suppose that $\lambda(f(z) - \frac{zP_3(z)-P_2(z)}{P_1(z)-z}) < \sigma(f(z))$. By $\sigma(f(z) - \frac{zP_3(z)-P_2(z)}{P_1(z)-z}) = \sigma(f(z))$ and Hadamard factorization theorem, $f(z) - \frac{zP_3(z)-P_2(z)}{P_1(z)-z}$ can be rewritten as a form

$$f(z) - \frac{zP_3(z) - P_2(z)}{P_1(z) - z} = z^m \frac{b_0(z)}{H_0(z)} e^{h(z)}, \tag{3.7}$$

where $h(z)$ is a polynomial with $\deg h(z) \le \sigma(f(z))$, $b_0(z)$ and $H_0(z)$ are canonical products ($b_0(z)$ may be a polynomial) formed by nonzero zeros and poles of $f(z) - \frac{zP_3(z)-P_2(z)}{P_1(z)-z}$ respectively, m is an integer, if $m \ge 0$ then $b(z) = z^m b_0(z)$, $H(z) = H_0(z)e^{-h(z)}$; if $m < 0$ then $b(z) = b_0(z)$, $H(z) = z^{-m}H_0(z)e^{-h(z)}$. By property of canonical product, we see that

$$\begin{cases} \lambda(b(z)) = \sigma(b(z)) = \lambda\left(f(z) - \frac{zP_3(z)-P_2(z)}{P_1(z)-z}\right) < \sigma(f(z)), \\ \lambda(H(z)) = \sigma(H(z)) = \sigma(f(z)). \end{cases} \tag{3.8}$$

By (3.7), we obtain

$$\begin{cases} f(z) = \frac{zP_3(z)-P_2(z)}{P_1(z)-z} + b(z)y(z), \\ f(z+1) = \frac{(z+1)P_3(z+1)-P_2(z+1)}{P_1(z+1)-(z+1)} + b(z+1)y(z+1), \end{cases} \tag{3.9}$$

where $y(z) = \frac{1}{H(z)}$. Thus, by (3.8) and Lemma 2.5, we have that

$$\sigma(y(z)) = \sigma(H(z)) = \sigma(f(z)),\ \sigma(b(z+1)) = \sigma(b(z)) < \sigma(y(z)).$$

Substituting (3.9) into (1.3), we obtain

$$
\begin{aligned}
K_1(z,y) := &\left\{\frac{(z+1)P_3(z+1)-P_2(z+1)}{P_1(z+1)-(z+1)} + b(z+1)y(z+1)\right\} \\
&\cdot \left\{\frac{zP_3(z)-P_2(z)}{P_1(z)-z} + b(z)y(z) + P_3(z)\right\} \\
&- P_1(z)\left\{\frac{zP_3(z)-P_2(z)}{P_1(z)-z} + b(z)y(z)\right\} - P_2(z) = 0.
\end{aligned} \tag{3.10}
$$

By (3.10), we see that

$$
\begin{aligned}
K_1(z,0) &= \frac{(z+1)P_3(z+1)-P_2(z+1)}{P_1(z+1)-(z+1)} \cdot \left\{\frac{zP_3(z)-P_2(z)}{P_1(z)-z} + P_3(z)\right\} \\
&\quad -P_1(z)\frac{zP_3(z)-P_2(z)}{P_1(z)-z} - P_2(z) \\
&= \frac{P_3(z)P_1(z)-P_2(z)}{P_1(z)-z}\left(\frac{(z+1)P_3(z+1)-P_2(z+1)}{P_1(z+1)-(z+1)} - z\right).
\end{aligned}
$$

That is

$$
\begin{aligned}
K_1(z,0) = &\frac{P_3(z)P_1(z)-P_2(z)}{P_1(z)-z} \\
&\cdot \frac{(z+1)P_3(z+1)-zP_1(z+1)-P_2(z+1)+z(z+1)}{P_1(z+1)-(z+1)}.
\end{aligned}
$$

Since $P_1(z)f(z)+P_2(z)$ and $f(z)+P_3(z)$ are relatively prime polynomials in f, we see that $P_3(z)P_1(z)-P_2(z) \not\equiv 0$. And by (1.4) or (1.5), we get $(z+1)P_3(z+1)-zP_1(z+1)-P_2(z+1)+z(z+1) \not\equiv 0$. Thus, we obtain

$$
K_1(z,0) \not\equiv 0. \tag{3.11}
$$

Thus, by (3.8), (3.11), Lemma 2.1 and its Remark 2.1, we obtain for any given $\varepsilon(0<\varepsilon<\sigma(f(z))-\sigma(b(z)))$

$$
N\left(r, \frac{1}{y(z)}\right) = T(r,y(z)) + S(r,y(z)) + O(r^{\sigma(b(z))+\varepsilon}) \tag{3.12}
$$

holds for all r outside of a possible exceptional set with finite logarithmic measure.

On the other hand, by $y(z) = \frac{1}{H(z)}$ and the fact that $H(z)$ is an entire function, we see that

$$
N\left(r, \frac{1}{y(z)}\right) = N(r, H(z)) = 0.
$$

Thus, (3.12) is a contradiction. Hence, $\lambda(f(z) - \frac{zP_3(z)-P_2(z)}{P_1(z)-z}) = \sigma(f(z))$. Furthermore, $\tau(f(z+1)) = \lambda(f(z+1)-z) = \lambda(f(z) - \frac{zP_3(z)-P_2(z)}{P_1(z)-z}) = \sigma(f(z))$.

If $P_1(z)-z\not\equiv 0$ and $zP_3(z)-P_2(z)\equiv 0$, then by (3.3) we have that $f(z+1)-z=\frac{(P_1(z)-z)f(z)}{f(z)+P_3(z)}$. By Lemma 2.8, we see that $(P_1(z)-z)f(z)$ and $f(z)+P_3(z)$ have at most finitely many common zeros. So that we only need to prove that $\lambda(f(z+1)-z)=\lambda(f(z))$. By Case 1 of Theorem 1.1, we see that $\lambda(f(z))=\sigma(f(z))$. Thus, we obtain $\lambda(f(z+1)-z)=\sigma(f(z))$, that is $\tau(f(z+1))=\sigma(f(z))$.

Now suppose that $n=2$, By (1.3), we have that

$$g(z+1)=\frac{P_1(z+1)g(z)+P_2(z+1)}{g(z)+P_3(z+1)}, \tag{3.13}$$

where $g(z)=f(z+1)$. By Lemma 2.5, we have that $\sigma(g(z))=\sigma(f(z))$. By (1.4) or (1.5), we get $\deg P_1(z+1)>\max\{\deg P_2(z+1),\deg P_3(z+1)\}$ and $\deg P_1(z+1)\geq 2$ or $\deg P_3(z+1)>\max\{\deg P_2(z+1),\deg P_1(z+1)\}$ and $\deg P_3(z+1)\geq 2$. Thus, for (3.13), applying the conclusion for $n=1$ above, we obtain $\tau(f(z+2))=\tau(g(z+1))=\sigma(g(z))=\sigma(f(z))$. Continuing to use the same method as above, we can obtain $\tau(f(z+n))=\sigma(f(z))$, $n=0,1,2,\ldots$.

Theorem 1.2 is proved.

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