ON GENERALIZED ZERO-DIVISOR GRAPH ASSOCIATED WITH A COMMUTATIVE RING

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Abstract. Let R be a commutative ring with identity, and let $Z(R)$ be the set of zero-divisors of R. The generalized zero-divisor graph of R is defined as the graph $\Gamma_g(R)$ with the vertex set $Z(R)^* = Z(R) \setminus \{0\}$, and two distinct vertices x and y are adjacent if and only if $ann_R(x)+ann_R(y)$ is an essential ideal of R. It follows that each edge (path) of the zero-divisor graph $\Gamma(R)$ is an edge (path) of $\Gamma_g(R)$. It is proved that $\Gamma_g(R)$ is connected with diameter at most three and with girth at most four, if $\Gamma_g(R)$ contains a cycle. Furthermore, all rings with the same generalized zero-divisor and zero-divisor graphs are characterized. Among other results, we show that the generalized zero-divisor graph associated with an Artinian ring is weakly perfect, i.e., its vertex chromatic number equals its clique number.
Keywords: generalized zero-divisor graph, zero-divisor graph, complete graph, chromatic number, clique number.

1. Introduction

Assigning a graph to a ring gives us the ability to translate algebraic properties of rings into graph theory language and vice versa. It leads to arising interesting algebraic and combinatorics problems. Therefore, the study of graphs associated

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with rings has attracted many researches. There are a lot of papers which apply combinatorial methods to obtain algebraic results in ring theory (see for instance [1], [5], [9] and [10]). Moreover, for the most recent study in this field see [6], [12] and [14].

Throughout this paper, all rings are assumed to be commutative with identity and they are not fields. We denote by $\mathrm{Max}(R)$, $\mathrm{Min}(R)$, $\mathrm{Nil}(R)$ and $\mathrm{U}(R)$, the set of all maximal ideals of R, the set of all minimal prime ideals of R, the set of all nilpotent elements of R and the set of all invertible elements of R, respectively. For a subset A of a ring R, we let $A^* = A \setminus \{0\}$. For every ideal I of R, we denote the *annihilator* of I by $ann_R(I)$. A non-zero ideal I of R is said to be *minimal* if there is no non-trivial ideal of R contained in I. A non-zero ideal I of R is called *essential*, denoted by $I \leq_e R$, if I has a non-zero intersection with every non-zero ideal of R. The *krull dimension* of R, denoted by $\dim(R)$, is the supremum of the lengths of all chains of prime ideals. We say that $\mathrm{depth}(R) = 0$, if every non-unit element of R is a zero-divisor. The ring R is said to be *reduced* if it has no non-zero nilpotent element. A non-zero ideal K of R is called *weakly prime*, if $(0) \neq IJ \subseteq K$ for some ideals I, J of R, then $I \subseteq K$ or $J \subseteq K$ (see [2], for more details). For any undefined notation or terminology in ring theory, we refer the reader to [4, 7].

Let $G = (V, E)$ be a graph, where $V = V(G)$ is the set of vertices and $E = E(G)$ is the set of edges. By $\overline{G}$, $\mathrm{diam}(G)$ and $\mathrm{gr}(G)$, we mean the complement, the diameter and the girth of G, respectively. We write $u - v$, to denote an edge with ends u, v. A graph $H = (V_0, E_0)$ is called a *subgraph of* G if $V_0 \subseteq V$ and $E_0 \subseteq E$. Moreover, H is called an *induced subgraph by* V_0, denoted by $G[V_0]$, if $V_0 \subseteq V$ and $E_0 = \{\{u, v\} \in E \mid u, v \in V_0\}$. Let G_1 and G_2 be two disjoint graphs. The *join* of G_1 and G_2, denoted by $G_1 \vee G_2$, is a graph with the vertex set $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1), v \in V(G_2)\}$. Also G is called a *null graph* if it has no edge. A complete bipartite graph with part sizes m, n is denoted by $K_{m,n}$. If $m = 1$, then the complete bipartite graph is called *star graph*. Also, a complete graph of n vertices is denoted by K_n. By $\overline{K}_\infty$, we mean a null graph with infinitely many vertices. A *clique* of G is a maximal complete subgraph of G and the number of vertices in the largest clique of G, denoted by $\omega(G)$, is called the *clique number* of G. For a graph G, let $\chi(G)$ denote the *vertex chromatic number* of G, i.e., the minimal number of colors which can be assigned to the vertices of G in such a way that every two adjacent vertices have different colors. Clearly, for every graph G, $\omega(G) \leq \chi(G)$. A graph G is said to be *weakly perfect* if $\omega(G) = \chi(G)$. For any undefined notation or terminology in graph theory, we refer the reader to [13].

The *zero- divisor graph* of a ring R, denoted by $\Gamma(R)$, is a graph with the vertex set $Z(R)^*$ and two distinct vertices x and y are adjacent if and only if $xy = 0$. This is the most important graph associated with a ring and it was first introduced and studied in [3]. Obviously, the properties of zero-divisor graphs are depend on the zero-divisors. Let $x, y \in Z(R)^*$ and $xy \neq 0$ whereas the behavior of the ideal $ann_R(x) + ann_R(y)$ is same to the case that $xy = 0$. These elements motivate us to define a new graph containing the zero-divisor graph. The *generalized zero-divisor graph* of R is defined as the graph $\Gamma_g(R)$ with the vertex set $Z(R)^* = Z(R) \setminus \{0\}$, and two distinct vertices x and y are adjacent if and only if $ann_R(x) + ann_R(y)$ is an essential ideal of R. In this paper, we study some connections between the graph-theoretic properties of $\Gamma_g(R)$ and some algebraic properties of rings. Moreover, we investigate the affinity between generalized zero-divisor graph and zero-divisor graph associated with a ring. Especially, we focus on the conditions under which theses two graphs are identical. Finally, the coloring of generalized zero-divisor graph is studied.

2. Basic properties of generalized zero-divisor graph

In this section, we study some of fundamental properties of $\Gamma_g(R)$. For instance, it is shown that $\Gamma_g(R)$ is always connected and $\mathrm{diam}(\Gamma_g(R)) \leq 3$. Moreover, we prove that if $\Gamma_g(R)$ contains a cycle, then $\mathrm{girth}(\Gamma_g(R)) \leq 4$. Moreover, all rings with complete or star generalized zero-divisor graph are characterized.

We begin with the following lemma.

Lemma 2.1. *Let R be a ring. Then the following statements hold.*

(1) *If $x \in R$, then $Rx^n + ann_R(x)$ is an essential ideal of R, for every positive integer n.*

(2) *$ann_R(x)$ is an essential ideal of R, for every $x \in \mathrm{Nil}(R)$.*

(3) *x is adjacent to all other vertices in $\Gamma_g(R)$, for every $x \in \mathrm{Nil}(R)^*$.*

(4) *If $x - y$ is an edge of $\Gamma(R)$, then $x - y$ is an edge of $\Gamma_g(R)$.*

Proof. (1) Let $x \in R$ and n be a positive integer. We show that $(Rx^n + ann_R(x)) \cap Ra \neq (0)$, for every $a \in R$. Suppose that $ann_R(x) \cap Ra = (0)$, for some $a \in R$. We have to prove that $Rx^n \cap Ra \neq (0)$. Since $ann_R(x) \cap Ra = (0)$, $ax \neq 0$. If $axx = ax^2 = 0$, then $ann_R(x) \cap Ra \neq (0)$, a contradiction. This implies that $ax^2 \neq 0$. By continuing this procedure, $ax^n \neq 0$. Hence $ax^n \in Rx^n \cap Ra$ and so $Rx^n \cap Ra \neq (0)$, as desired.

(2) It follows directly from part (1).

(3) Let $x \in \mathrm{Nil}(R)^*$. By part (2), $ann_R(x)$ is an essential ideal of R and thus $ann_R(x) + ann_R(y)$ is essential, for every $y \in Z(R)^*$. Hence x is adjacent to all other vertices.

(4) Let $x - y$ be an edge of $\Gamma(R)$. Then $y \in ann_R(x)$. By part (1), $Ry + ann_R(y)$ is an essential ideal. This, together with $Ry \subseteq ann_R(x)$, imply that $ann_R(x) + ann_R(y)$ is essential. Therefore, $x - y$ is an edge of $\Gamma_g(R)$. □

Let R be a ring. By [3, Theorem 2.3], the zero-divisor graph $\Gamma(R)$ is a connected graph and $\mathrm{diam}(\Gamma(R)) \leq 3$. Moreover, if $\Gamma(R)$ contains a cycle, then girth($\Gamma(R)$) ≤ 4 (see [11]). By using theses facts and Lemma 2.1, we have the following result.

Theorem 2.1. *Let R be a ring. Then $\Gamma_g(R)$ is connected and* $\mathrm{diam}(\Gamma_g(R)) \leq 3$. *Moreover, if $\Gamma_g(R)$ contains a cycle, then* $\mathrm{girth}(\Gamma_g(R)) \leq 4$.

Proof. By part (4) of Lemma 2.1, $\Gamma(R)$ is a subgraph of $\Gamma_g(R)$ and so $\Gamma_g(R)$ is connected, $\mathrm{diam}(\Gamma_g(R)) \leq 3$ and $\mathrm{girth}(\Gamma_g(R)) \leq 4$, if $\Gamma_g(R)$ contains a cycle. □

To classify all rings with complete or star generalized zero-divisor graph, the following lemma is needed.

Lemma 2.2. *Let R be a ring. Then the following statements hold.*

(1) *If R is non-reduced and $\Gamma_g(R)$ is complete, then R is an indecomposable ring.*

(2) *If R is reduced and $x \in Z(R)^*$, then $ann_R(x) = ann_R(x^2)$.*

Proof. (1) Suppose that $R \cong R_1 \times R_2$, where R_1 and R_2 are two rings. With no loss of generality, one may assume that $|\mathrm{Nil}(R_1)| \geq 2$. Thus $2 \leq |\mathrm{Nil}(R_1)| \leq |U(R_1)|$. Let $a = (1, 0)$ and $b = (u, 0)$, where $1 \neq u \in U(R_1)$. Then $ann_R(a) + ann_R(b) = (0) \times R_2$ and so it is not an essential ideal of R, a contradiction. Thus R is indecomposable.

(2) It is clear by [5, Lemma 3.2]. □

Theorem 2.2. *Let R be a ring. Then the following statements hold.*

(1) *If R is a non-reduced ring, then the following statements are equivalent:*

  (*i*) *$\Gamma_g(R)$ is complete.*

  (*ii*) *$ann_R(x^2)$ is an essential ideal of R, for every $x \in Z(R)^*$.*

(2) *If R is a reduced ring, then the following statements are equivalent:*

  (*i*) *$\Gamma_g(R)$ is complete.*

  (*ii*) *$R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.*

  (*iii*) *$\Gamma_g(R) = K_{1,1}$.*

Proof. (1) (*i*) $\Longrightarrow$ (*ii*) Let $x \in Z(R)^*$. If $x \in \mathrm{Nil}(R)$, then by part (2) of Lemma 2.1, $ann_R(x)$ is an essential ideal of R and thus $ann_R(x^2)$ is an essential ideal of R. So let $x \in Z(R) \setminus \mathrm{Nil}(R)$. Since $\Gamma_g(R)$ is complete, by Lemma 2.2, R is an indecomposable ring and thus $x \neq x^2$. Hence x is adjacent to x^2 and so $ann_R(x) + ann_R(x^2) = ann_R(x^2)$ is essential.

(*ii*) $\Longrightarrow$ (*i*) Let $x \in Z(R)^*$. We need only to show that $ann_R(x)$ is an essential ideal of R. Suppose to the contrary, $ann_R(x) \cap Ra = (0)$, for some $a \in R$. Since $ann_R(x^2) \cap Ra \neq (0)$, there exists an element $r \in R$ such that

$rax^2 = 0$ and $ra \neq 0$. Since $ann_R(x) \cap Ra = (0)$, $rax \neq 0$. This means that $ann_R(x) \cap Ra \neq (0)$, a contradiction. Thus $ann_R(x)$ is an essential ideal of R.

(2) $(i) \Longrightarrow (ii)$ Let $x \in Z(R)^*$. As R is a reduced ring, $Rx \cap ann_R(x) = (0)$ and thus $ann_R(x)$ is not an essential ideal of R. By part (2) of Lemma 2.2, $ann_R(x) = ann_R(x^2)$ and so $ann_R(x)+ann_R(x^2) = ann_R(x^2)$ is not an essential ideal of R. Since $\Gamma_g(R)$ is complete, we deduce that $x = x^2$. Hence $R \cong R_1 \times R_2$, where R_1 and R_2 are two rings. If $a \in R_1 \setminus \{0, 1\}$, then it is easily seen that $(1, 0)$ is not adjacent to $(a, 0)$ (since $ann_R(a)$ is not essential in R_1). Thus $|R_1| = 2$. Similarly, $|R_2| = 2$. Hence $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

$(ii) \Longrightarrow (iii)$ and $(iii) \Longrightarrow (i)$ are clear. □

The following result shows that, if $\Gamma(R) = K_n \vee \overline{K}_m$, for some n, m, then either $\Gamma(R) = \Gamma_g(R)$ or $\Gamma_g(R)$ is a complete graph.

Theorem 2.3. *Let R be a non-reduced ring and $\Gamma(R) = K_n \vee \overline{K}_m$ for some n, m. Then the following statements are equivalent:*

(1) $\Gamma(R) \neq \Gamma_g(R)$.

(2) $\Gamma_g(R)$ *is a complete graph and* $Z(R)^2 \neq (0)$.

Proof. (1) ⇒ (2) Since $\Gamma(R) = K_n \vee \overline{K}_m$, every vertex of $V(K_n)$ is adjacent to all other vertices and thus by [3, Theorem 2.5], $ann_R(a) = Z(R)$, for every vertex $a \in V(K_n)$. This implies that $ann_R(a)$ is an essential ideal of R. Moreover, $\Gamma(R) = K_n \vee \overline{K}_m$ implies that $ann_R(x) = \{0\} \cup V(K_n)$, for every $x \in Z(R) \setminus \{0\} \cup V(K_n)$. The inequality $\Gamma(R) \neq \Gamma_g(R)$ shows that there exist $x, y \in V(\overline{K}_m)$ such that x is adjacent to y (note that $m \geq 2$). Thus $ann_R(x) + ann_R(y)$ is an essential ideal of R. Since $ann_R(x) = ann_R(y) = \{0\} \cup V(K_n)$, we conclude that $\{0\} \cup V(K_n)$ is essential. Hence $ann_R(x)$ is an essential ideal of R, for every $x \in Z(R)^*$ and thus $\Gamma_g(R)$ is a complete graph. Finally, if $Z(R)^2 = (0)$, then $\Gamma(R)$ is complete and since $\Gamma(R)$ is a connected subgraph of $\Gamma_g(R)$, we infer that $\Gamma(R) = \Gamma_g(R) = K_{n+m}$, a contradiction.

(2)⇒(1) Since $Z(R)^2 \neq (0)$, $\Gamma(R)$ is not complete and hence $\Gamma(R) \neq \Gamma_g(R)$. □

The following example explains Theorem 2.3.

Example 2.1. (1) Let $R = \mathbb{Z}_8$. Then $\Gamma(R) = K_{1,2}$, $\Gamma_g(R) = K_3$ and so $n = 1$ and $m = 2$, in Theorem 2.3.

(2) Let $R = \mathbb{Z}_2[X, Y]/(XY, X^2)$, $x = X+(XY+X^2)$ and $y = Y+(XY+X^2)$. Then $Z(R) = (x, y)R$, $\mathrm{Nil}(R) = \{0, x\}$ and $\Gamma(R)$ is an infinite star. It is easily checked that $y \neq y^2$ and y is not adjacent to y^2 in $\Gamma_g(R)$. This means that $\Gamma_g(R)$ is not a complete graph and since $Z(R)^2 \neq (0)$, we deduce that $\Gamma(R) = \Gamma_g(R)$.

The last result of this section is devoted to identifying non-reduced rings with star generalized zero-divisor graph.

Theorem 2.4. *Let R be a non-reduced ring with $|Z(R)^*| \geq 2$. Then the following statements are equivalent:*

(1) $\Gamma_g(R)$ *is a star graph.*

(2) $\mathrm{girth}(\Gamma_g(R)) = \infty$.

(3) $ann_R(Z(R))$ *is a prime ideal of* R *and* $|ann_R(Z(R))| = 2$ *or* $|ann_R(Z(R))| = |Z(R)| = 3$.

(4) *Either* $\Gamma_g(R) = K_{1,1}$ *or* $\Gamma_g(R) = K_{1,\infty}$.

Proof. (1) $\Rightarrow$ (2) and (4) $\Rightarrow$ (1) are clear.

(2) $\Rightarrow$ (3) Since $\mathrm{girth}(\Gamma_g(R)) = \infty$, by part (3) of Lemma 2.1, $|\mathrm{Nil}(R)| \in \{2, 3\}$. If $|\mathrm{Nil}(R)| = 3$, then part (3) of Lemma 2.1 implies that $|ann_R(Z(R))| = |Z(R)| = 3$ and so $ann_R(Z(R))$ is a prime ideal. If $|\mathrm{Nil}(R)| = 2$, then one may let $\mathrm{Nil}(R) = \{0, a\}$ and thus a is adjacent to all other vertices in $\Gamma_g(R)$. Since $\mathrm{girth}(\Gamma_g(R)) = \infty$, $\Gamma_g(R)$ is a star graph. By part (4) of Lemma 2.1, $\Gamma(R)$ is a subgraph of $\Gamma_g(R)$. This implies that $\mathrm{Nil}(R) = ann_R(Z(R))$ and thus $|ann_R(Z(R))| = 2$. We show that $\mathrm{Nil}(R)$ is a prime ideal of R. For this, let $xy \in \mathrm{Nil}(R) = \{0, a\}$ such that $x \notin \mathrm{Nil}(R)$ and $y \notin \mathrm{Nil}(R)$. If $xy = 0$, then $x - a - y - x$ is a cycle of length three in $\Gamma_g(R)$, a contradiction. If $xy = a$, then $x^2y^2 = 0$ again a contradiction. Thus $ann_R(Z(R))$ is a prime ideal of R.

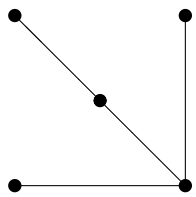
(3) $\Rightarrow$ (4) It is clear that, if $|ann_R(Z(R))| = |Z(R)| = 3$, then $\Gamma_g(R) = K_{1,1}$. Hence, suppose that $ann_R(Z(R)) = \{0, a\}$, for some $a \in R$ and $ann_R(Z(R))$ is a prime ideal. Clearly, $ann_R(Z(R)) \neq Z(R)$. If $Z(R)$ is a finite set, then since $Z(R)$ is an ideal of R, we deduce that R is an Artinian local ring and this means $Z(R)$ is a prime ideal. Thus we find the chain $ann_R(Z(R)) \subseteq Z(R)$ of primes, a contradiction (As $\dim(R) = 0$). Hence $Z(R)$ is an infinite set. We claim that $ann_R(Z(R))$ is not an essential ideal of R. To see this, let $x \in Z(R) \setminus ann_R(Z(R))$. If $Rx \cap ann_R(Z(R)) \neq (0)$, then $rx = a$, for some $r \in R$ and thus $rx^2 = 0$, a contradiction. So the claim is proved. Since $ann_R(Z(R))$ is a prime ideal of R, we deduce that $ann_R(x) = ann_R(Z(R))$, for every $x \in Z(R) \setminus ann_R(Z(R))$. To complete the proof, assume that $x - y$ is an edge of $\Gamma_g(R)$, $x \neq a$ and $y \neq a$. Obviously, $ann_R(x) + ann_R(y) = ann_R(x) = ann_R(Z(R))$, a contradiction (As $ann_R(Z(R))$ is not essential) and thus $\Gamma_g(R) = K_{1,\infty}$. $\square$

3. When generalized zero-divisor graphs and zero-divisor graphs are identical?

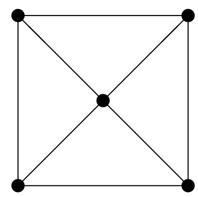
As we observed in the past section, generalized zero-divisor graphs and zero-divisor graphs are close to each other. The following simple example describes two different situations $\Gamma(R) \neq \Gamma_g(R)$ and $\Gamma(R) = \Gamma_g(R)$. Thus it is interesting to study rings whose generalized zero-divisor graphs and zero-divisor graphs are identical. This section is devoted to study of such rings.

Example 3.1.

(1) Let $R = \mathbb{Z}_2 \times \mathbb{Z}_4$. Then $\Gamma(R) \neq \Gamma_g(R)$.

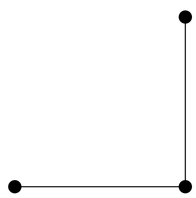


$\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_4)$

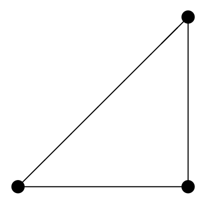


$\Gamma_g(\mathbb{Z}_2 \times \mathbb{Z}_4)$

(2) Let $R = \mathbb{Z}_8$. Then $\Gamma(R) \neq \Gamma_g(R)$.

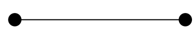


$\Gamma(\mathbb{Z}_8)$

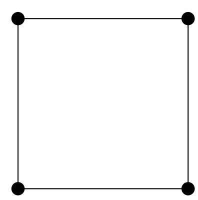


$\Gamma_g(\mathbb{Z}_8)$

(3) Let $R = \mathbb{Z}_9$ and $S = \mathbb{Z}_3 \times \mathbb{Z}_3$. Then $\Gamma(R) = \Gamma_g(R)$ and $\Gamma(S) = \Gamma_g(S)$.



$\Gamma(R) = \Gamma_g(R)$



$\Gamma(S) = \Gamma_g(S)$

To characterize all non-reduced rings with the same generalized zero-divisor graphs and zero-divisor graphs, we need the following lemma.

Lemma 3.1. *Let R be a ring and $x, y \in Z(R)$. Then $ann_R(x) + ann_R(y)$ is an essential ideal of R if and only if $ann_R(x)+ann_R(y^2)$ is an essential ideal of R.*

Proof. Since $ann_R(x) + ann_R(y) \subseteq ann_R(x) + ann_R(y^2)$, one side is clear. To prove the other side, suppose that $ann_R(x)+ann_R(y^2)$ is an essential ideal of R. Let $(ann_R(x)+ann_R(y)) \cap Ra = (0)$, for some $a \in R$. This implies that $rax \neq 0$ and $ray \neq 0$, for every $0 \neq ra \in Ra$. Since $(ann_R(x) + ann_R(y^2)) \cap Ra \neq (0)$, there exist $t \in ann_R(x)$ and $s \in ann_R(y^2)$ such that $t + s = ra \neq 0$, for some $r \in R$. Hence $ty^2 = ray^2$ and thus $0 \neq ty^2 \in (ann_R(x) \cap Ra)$, a contradiction. □

Theorem 3.1. *Let R be a non-reduced ring. Then $\Gamma(R) = \Gamma_g(R)$ if and only if all of the following statements hold.*

(1) *$ann_R(a) = Z(R)$, for every $a \in \mathrm{Nil}(R)^*$.*

(2) *If $ann_R(x)$ is an essential ideal of R, for some $x \in Z(R)^*$, then $x \in \mathrm{Nil}(R)^*$.*

(3) *$\mathrm{Nil}(R)$ is a weakly prime ideal of R.*

Proof. First suppose that $\Gamma(R) = \Gamma_g(R)$.

If $a \in \mathrm{Nil}(R)^*$, then by part (3) of Lemma 2.1, a is adjacent to all other vertices of $\Gamma(R)$. Since R is non-reduced, [3, Theorem 2.5] follows that $ann_R(a) = Z(R)$

Let $ann_R(x)$ is an essential ideal of R, for some $x \in Z(R)^*$. Hence $ann_R(x)+ ann_R(y)$ is an essential ideal, for every $y \in Z(R)$, and thus x is adjacent to all other vertices in $\Gamma_g(R)$. Since $\Gamma(R) = \Gamma_g(R)$, x is adjacent to all other vertices in $\Gamma(R)$. By [3, Theorem 2.5], $Z(R) = ann_R(x)$ and so $x \in \mathrm{Nil}(R)$.

Now, we show that $\mathrm{Nil}(R)$ is a weakly prime ideal of R. Let $0 \neq xy \in \mathrm{Nil}(R)^*$ such that $x \notin \mathrm{Nil}(R)^*$ and $y \notin \mathrm{Nil}(R)^*$. Therefore, $x, y \in Z(R)$ and $x \neq y$. By part (1), $yxy = xy^2 = 0$ and thus $ann_R(x) + ann_R(y^2)$ is an essential ideal, as $\Gamma(R) = \Gamma_g(R)$. By Lemma 3.1, $ann_R(x)+ann_R(y)$ is essential, a contradiction. Hence $\mathrm{Nil}(R)$ is a weakly prime ideal of R.

Conversely, suppose that all of the statements (1), (2) and (3) are hold. We prove that $\Gamma(R) = \Gamma_g(R)$. Let $x - y$ be an edge of $\Gamma_g(R)$. It is enough to show that $xy = 0$. Assume to the contrary, $xy \neq 0$. Since $ann_R(x) + ann_R(y)$ is an essential ideal of R and $ann_R(y) \subseteq ann_R(xy)$, we conclude that $ann_R(x) + ann_R(xy) = ann_R(xy)$ is essential. By part (2), $xy \in \mathrm{Nil}(R)^*$ and thus by part (3) follows that $x \in \mathrm{Nil}(R)$ or $y \in \mathrm{Nil}(R)$. This contradicts part (1). Therefore, $xy = 0$, as desired. □

Theorem 3.2. *Let R be a non-reduced ring and $\mathrm{Nil}(R)$ be a prime ideal of R that is not essential. Then the following statements are equivalent:*

(1) *$\Gamma(R) = \Gamma_g(R)$.*

(2) $\Gamma(R) = K_{|\mathrm{Nil}(R)^*|} \vee \overline{K}_\infty$.

(3) $\Gamma_g(R) = K_{|\mathrm{Nil}(R)^*|} \vee \overline{K}_\infty$ *and* $ann_R(x) \in \{\mathrm{Nil}(R), Z(R)\}$, *for every* $x \in Z(R)^*$.

Proof. (1) $\Rightarrow$ (2) Since $\Gamma(R) = \Gamma_g(R)$, we need only to show that $\Gamma_g(R) = K_{|\mathrm{Nil}(R)^*|} \vee \overline{K}_\infty$. By part (3) of Lemma 2.1, every vertex of $\mathrm{Nil}(R)^*$ is adjacent to all other vertices in $\Gamma_g(R)$. Let $x, y \in Z(R) \setminus \mathrm{Nil}(R)$ and $x \neq y$. Since $\mathrm{Nil}(R)$ is prime that is not essential, $ann_R(x) + ann_R(y)$ is not essential and thus x is not adjacent to y. Next we show that $Z(R)\setminus\mathrm{Nil}(R)$ is infinite. If $Z(R) = \mathrm{Nil}(R)$, then $Z(R)$ is an ideal of R which is not an essential ideal of R, a contradiction. Choose $x \in Z(R) \setminus \mathrm{Nil}(R)$. If $x^m = x^n$, for some positive integers $n > m \geq 1$, then $x^m(1 - x^{n-m}) = 0$ and thus $1 - x^{n-m} \in ann_R(x^m)$. This implies that $Rx^{n-m} + ann_R(x^m) = R$. This contradicts the fact $ann_R(x^m) \subseteq \mathrm{Nil}(R)$. Hence $Z(R) \setminus \mathrm{Nil}(R)$ is infinite and so $\Gamma_g(R) = K_{|\mathrm{Nil}(R)^*|} \vee \overline{K}_\infty$.

(2) $\Rightarrow$ (3) Since $\mathrm{Nil}(R)$ is not an essential ideal of R, $Z(R) \neq \mathrm{Nil}(R)$. Let $x \in Z(R) \setminus \mathrm{Nil}(R)$. The equality $\Gamma(R) = K_{|\mathrm{Nil}(R)^*|} \vee \overline{K}_\infty$ implies that $Z(R)$ is an ideal of R and thus $x \neq x^2$. Since $\mathrm{Nil}(R)$ is a prime ideal of R that is not essential, we can easily get x is not adjacent to x^2. This implies that $\Gamma_g(R)$ is not a complete graph. Thus by Theorem 2.3, $\Gamma(R) = \Gamma_g(R)$. Hence $\Gamma_g(R) = K_{|\mathrm{Nil}(R)^*|} \vee \overline{K}_\infty$. The equality $\Gamma(R) = K_{|\mathrm{Nil}(R)^*|} \vee \overline{K}_\infty$, for every $x \in Z(R)^*$, now shows that $ann_R(x) \in \{\mathrm{Nil}(R), Z(R)\}$.

(3) $\Rightarrow$ (1) We show that $ann_R(x) = Z(R)$, for every $x \in \mathrm{Nil}(R)^*$. For if not, one may suppose that $ann_R(x) = \mathrm{Nil}(R)$, for some $x \in \mathrm{Nil}(R)^*$. Since $\mathrm{Nil}(R)$ is not an essential ideal of R, x is not adjacent to y, for every $y \in Z(R) \setminus \mathrm{Nil}(R)$ (As $\mathrm{Nil}(R)$ is prime), a contradiction. Thus $ann_R(x) = Z(R)$ and so $\Gamma(R) = K_{|\mathrm{Nil}(R)^*|} \vee \overline{K}_\infty$. Hence $\Gamma(R) = \Gamma_g(R)$. □

All conditions in Theorems 3.1 and 3.2 are necessary. In the following example, we examine condition (1) of Theorem 3.1 and the condition of $\mathrm{Nil}(R)$ not to be essential in Theorem 3.2.

Example 3.2. Let $D = \mathbb{Z}_2[X, Y, W]$, $I = (X^2, Y^2, XY, XW)D$, and $R = D/I$. Consider three elements $x = X + I$, $y = Y + I$, and $w = W + I$ of R. Then $\mathrm{Nil}(R) = (x, y)R$ and $Z(R) = (x, y, w)R$ is an ideal of R. It is not hard to check that $\mathrm{Nil}(R)$ is a prime ideal of R and $\Gamma(R) \neq \Gamma_g(R)$ (As $y - w$ is an edge of $\Gamma_g(R)$ which is not an edge of $\Gamma(R)$). Since $ann_R(a) \neq Z(R)$, for some $a \in \mathrm{Nil}(R)^*$, we have $\Gamma(R) \neq \Gamma_g(R)$.

Also, $\Gamma_g(R) = K_{|\mathrm{Nil}(R)^*|} \vee \overline{K}_\infty$ and $ann_R(x) \in \{\mathrm{Nil}(R), Z(R)\}$, for every $x \in Z(R)^*$. Since $\mathrm{Nil}(R)$ is an essential ideal of R, $\Gamma(R) \neq K_{|\mathrm{Nil}(R)^*|} \vee \overline{K}_\infty$.

We close this section with the following result.

Theorem 3.3. *Let R be a reduced ring and $x, y \in Z(R)$. Then the following statements are equivalent:*

(1) $\Gamma(R) = \Gamma_g(R)$.

(2) *If* $xy \neq 0$, *then* $ann_R(ann_R(x) + ann_R(y)) \neq (0)$.

Proof. (1) $\Rightarrow$ (2) Let $x, y \in Z(R)$ and $xy \neq 0$. If $x = y$, then $ann_R(x) = ann_R(y)$ and thus $ann_R(ann_R(x)+ann_R(y)) = ann_R(ann_R(x)) \neq (0)$. If $x \neq y$, then the equality $\Gamma(R) = \Gamma_g(R)$ shows that $ann_R(x) + ann_R(y)$ is not essential. Hence $ann_R(ann_R(x) + ann_R(y)) \neq (0)$ (note that if $ann_R(I) = (0)$, then I is an essential ideal of R).

(2) $\Rightarrow$ (1) Let $x - y$ be an edge of $\Gamma_g(R)$. It is enough to show that $xy = 0$. Assume to the contrary, $xy \neq 0$. Let $ann_R(x) + ann_R(y) = I$. Then $ann_R(I) \neq (0)$. Since R is reduced, $I \cap ann_R(I) = (0)$. This implies that I is not essential, a contradiction. Thus $xy = 0$, as desired. □

4. Coloring of generalized zero-divisor graphs

The main goal of this section is to study the coloring of the generalized zero-divisor graphs associated with Artinian rings. It is shown that the graph $\Gamma_g(R)$ is weakly perfect, for every Artinian ring R. Moreover, the exact value of the $\chi(\Gamma_g(R))$ is given.

It is known that if R is a ring such that $\mathrm{depth}(R) \neq 0$, then R is infinite. Also, Ganesan ([8]) proved that if R is infinite and $Z(R) \neq (0)$, then $Z(R)$ must be infinite (in fact $|R| \leq |Z(R)|^2$ when $2 \leq |Z(R)| < \infty$). In view of these facts, we state the following theorem.

Theorem 4.1. *Let R be a non-reduced indecomposable ring and $\omega(\Gamma_g(R)) < \infty$. Then the following statements are equivalent.*

(1) *$ann_R(x)$ is an essential ideal of R, for every $x \in Z(R)^*$.*

(2) *$\Gamma_g(R)$ is a complete graph.*

(3) *$\omega(\Gamma_g(R)) = \chi(\Gamma_g(R)) = |\mathrm{Nil}(R)| - 1$.*

(4) *R is an Artinian local ring.*

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (3) Since $\Gamma_g(R)$ is a complete graph, $\omega(\Gamma_g(R)) = \chi(\Gamma_g(R))$. We show that $Z(R) = \mathrm{Nil}(R)$. Let $x \in Z(R) \setminus \mathrm{Nil}(R)$. Since R is an indecomposable ring, $x^m \neq x^n$, for every pair of distinct positive integers n, m. Since $\omega(\Gamma_g(R)) < \infty$, a contradiction and thus $\omega(\Gamma_g(R)) = \chi(\Gamma_g(R)) = |\mathrm{Nil}(R)| - 1$.

(3) $\Rightarrow$ (4). We claim that $Z(R) = \mathrm{Nil}(R)$. Let $x \in Z(R) \setminus \mathrm{Nil}(R)$. By part (3) of Lemma 2.1, every vertex of $\mathrm{Nil}(R)^*$ is adjacent to x in $\Gamma_g(R)$ and thus $\omega(\Gamma_g(R)) \geq |\mathrm{Nil}(R)|$, a contradiction and so the claim is proved. Thus $\Gamma_g(R)$ is a complete graph. Since $\omega(\Gamma_g(R)) < \infty$, we conclude that $|Z(R)| < \infty$, and hence $|R| < \infty$. This means that R is an Artinian ring. As R is indecomposable, we deduce that R is local.

(4) $\Rightarrow$ (1) is clear, by part (2) of Lemma 2.1. □

Theorem 4.2. *Let R be a non-reduced ring and* $\dim(R) = 0$. *If* $|\mathrm{Max}(R)| < \infty$ *and $\omega(\Gamma_g(R)) < \infty$, then the following statements hold.*

(1) *$|\mathrm{Max}(R)| \leq \omega(\Gamma_g(R))$.*

(2) *If R is nonlocal, then*

$$\omega(\Gamma_g(R)) = \chi(\Gamma_g(R)) = |J(R)| + |\mathrm{Max}(R)| - 1.$$

Proof. If R is local ($|\mathrm{Max}(R)| = 1$), then there is nothing to prove. Let R be nonlocal. Since R is non-reduced, $|J(R)| > 1$ and hence it is enough to prove (2).

(2) By part (3) of Lemma 2.1, $\Gamma_g(R)[\mathrm{Nil}(R)^*]$ is a complete subgraph of $\Gamma_g(R)$ and thus $|\mathrm{Nil}(R)| < \infty$. This means that $\mathrm{Nil}(R)$ is a nilpotent ideal of R. Since $\dim(R) = 0$, $\mathrm{Min}(R) = \mathrm{Max}(R)$ and so $\mathrm{Nil}(R) = J(R)$. Hence $J(R)$ is a nilpotent ideal of R. Since $\mathrm{Max}(R) < \infty$, $\mathfrak{m}_1^m \cdots \mathfrak{m}_n^m = (0)$, for some positive integers n, m. By the Chinese Remainder Theorem, $R \cong R_1 \times \cdots \times R_n$, where $\dim(R_i) = 0$, for every $1 \leq i \leq n$. This implies that $R_i = \mathrm{Nil}(R_i) \cup U(R_i)$. Put:

$A := \{(x_1, \ldots, x_n) \in V(\Gamma_g(R)) \mid x_i \in \mathrm{Nil}(R_i),\ \text{for all } 1 \leq i \leq n\}$ and

$B := \{(x_1, \ldots, x_n) \in V(\Gamma_g(R)) \mid x_i \notin \mathrm{Nil}(R_i),\ \text{for some } 1 \leq i \leq n\}$.

It is not hard to check that $V(\Gamma_g(R)) = A \cup B$, $A \cap B = \varnothing$ and so $\{A, B\}$ is a partition of $V(\Gamma_g(R))$. We show that $\Gamma_g(R) = \Gamma_g(R)[A] \vee \Gamma_g(R)[B]$, where $\Gamma_g(R)[A]$ is a complete subgraph of $\Gamma_g(R)$ and $\Gamma_g(R)[B]$ is an n-partite subgraph of $\Gamma_g(R)$ which is not an $(n-1)$-partite subgraph of $\Gamma_g(R)$. To see this, by Part (3) of Lemma 2.1, $\Gamma_g(R))[A]$ is a complete subgraph of $\Gamma_g(R)$ and every vertex $x \in A$ is adjacent to all other vertices.

Now, let $B_i = \{(x_1, \ldots, x_n) \in B \mid x_i \in U(R)\}$, for every $1 \leq i \leq n$. It is easy to see that there is no adjacency between two vertices of B_i, for every $1 \leq i \leq n$. This together with this fact that the set $\{(1, 0, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, 0, \ldots, 0, 1)\}$ is a clique of $EG(R)[B]$ imply that $\Gamma_g(R)[B]$ is an n-partite subgraph of $\Gamma_g(R)$ which is not an $(n-1)$-partite subgraph of $\Gamma_g(R)$. Therefore, $\Gamma_g(R) = \Gamma_g(R)[A] \vee \Gamma_g(R)[B]$ and so $\omega(\Gamma_g(R)) = \chi(\Gamma_g(R)) = \omega(\Gamma_g(R)[A]) + \omega(\Gamma_g(R)[B]) = |\mathrm{Nil}(R)| - 1 + |\mathrm{Max}(R)|$. □

Corollary 4.1. *Let R be a non-reduced ring and* $\dim(R) = 0$. *Then the following statements hold.*

(1) *If $\omega(\Gamma_g(R)) < \infty$, then R is an Artinian ring.*

(2) *If R is an Artinian ring, then $\omega(\Gamma_g(R)) = \chi(\Gamma_g(R)) = |J(R)| + |\mathrm{Max}(R)| - 1$.*

We close this paper with the following example.

Example 4.1. Let $R = \mathbb{Z}_2[X, Y]/(XY, X^2)$, $x = X + (XY + X^2)$ and $y = Y + (XY + X^2)$. Then $Z(R) = (x, y)R$, $\mathrm{Nil}(R) = \{0, x\}$ and $\Gamma_g(R) = K_1 \vee \overline{K}_\infty$. So $\omega(\Gamma_g(R)) = 2 < \infty$. Since $\dim(R) \neq 0$, R is not an Artinian ring.

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