ON SOME PROPERTIES OF ROUGH APPROXIMATIONS OF SUBRINGS VIA COSETS

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Abstract. In 1982, Zdzislaw Pawlak introduced the theory of Rough sets to deal with the problems involving imperfect knowledge. This present research article studies some interesting properties of Rough approximations of subrings via an equivalence relation involving cosets of an ideal. In this present work, a ring structure is assigned to the universe set and a few results on the Rough approximations of subrings of the universe set are established.

Keywords: ring, subring, ideal, maximal ideal, prime ideal, rough set, upper approximation, lower approximation, information system, equivalence relation.

Introduction. The problem of imperfect knowledge has been tackled for a long time by philosophers, logicians and mathematicians. Recently it became also a crucial issue for computer scientists, particularly in the area of Artificial Intelligence. There are many approaches to the problem of how to understand and manipulate imperfect knowledge. The most successful approaches to tackle this problem are the Fuzzy set theory and the Rough set theory. Theories of Fuzzy sets and Rough sets are powerful mathematical tools for modeling various types of uncertainties. Fuzzy set theory was introduced by L. A. Zadeh in his classical paper [7] of 1965.

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A polish applied mathematician and computer scientist Zdzislaw Pawlak introduced Rough set theory in his classical paper [4] of 1982. Rough set theory is a new mathematical approach to imperfect knowledge. This theory presents still another attempt to deal with uncertainty or vagueness.

The Rough set theory has attracted the attention of many researchers and practitioners who contributed essentially to its development and application. Rough sets have been proposed for a very wide variety of applications.

In particular, the Rough set approach seems to be important for Artificial Intelligence and cognitive sciences, especially for machine learning, knowledge discovery, data mining, pattern recognition and approximate reasoning.

In this present work, we construct Rough sets by considering cosets of an Ideal in a ring. We investigate a few results on lower and upper approximations of subrings.

1. Rings and ideals

In this section, some basic definitions of Rings and Ideals that are necessary for further study of this work are presented.

Definition 1.1. A non-empty set of elements G is said to form a *semi-group* if in G there is defined a binary operation, called the product and denoted by o,such that

a. $a, b \in G \Rightarrow aob \in G$ (Closure);

b. $ao(boc) = (aob)oc$, for all a, b, c in G (Associative law).

Definition 1.2. A non-empty set of elements G is said to form a *group* if in G there is defined a binary operation, called the product and denoted by o , such that

a. $a, b \in G \Rightarrow aob \in G$ (Closure);

b. $ao(boc) = (aob)oc$, for all a, b, c in G (Associative law);

c. there exists an element $e \in G$ such that $aoe = eoa = a$, for all $a \in G$. The element e is called the identity element in G. (Existence of identity);

d. for every $a \in G$ there exists an element$a^{-1} \in G$ such that $aoa^{-1} = a^{-1}oa = e$. The element a^{-1} is called the inverse element of a in G. (Existence of inverse).

We denote a non-empty set G with respect to a binary operation o by the symbol (G, o).

Definition 1.3. A group G with respect to a binary operation o is said to be an *abelian group* if $aob = boa$, for all $a, b \in G$.

Definition 1.4. A non-empty set U is said to be a *ring* if in there are defined two operations, denoted by $+$ and $\cdot$ respectively, such that

a. $(U, +)$ is an abelian group;

b. $(U, \cdot)$ is a semi-group;

c. $x \cdot (y + z) = x \cdot y + x \cdot z$, for all $x, y, z \in U$;

d. $(x + y) \cdot z = x \cdot y + x \cdot z$, for all $x, y, z \in U$.

We denote a ring U with respect to the binary operations $+$ and $\cdot$ by the symbol $(U, +, \cdot)$. We denote the identity element in $(U, +)$ by the symbol 0 and we call this element, the zero element of U . If $a \in U$ then the inverse element of a in $(U, +)$ by the symbol $-a$.

Definition 1.5. A non-empty subset S of a ring $(U, +, \cdot)$ is said to be a *subring* of $(U, +, \cdot)$ if and only if $a - b \in S$ for every a and b in S .

Definition 1.6. A non-empty subset I of a ring $(U, +, \cdot)$ is said to be an *ideal* of U if

a. $x \in I$ and $y \in I \Rightarrow x - y \in I$;

b. $u \in U$ and $x \in I \Rightarrow u \cdot x \in I$;

c. $u \in U$ and $x \in I \Rightarrow x \cdot u \in I$.

Definition 1.7. An ideal P of a ring $(U, +, \cdot)$ is said to be a *prime ideal* if $x \cdot y \in P$ then either $x \in P$ or $y \in P$.

Definition 1.8. An ideal M of a ring $(U, +, \cdot)$ is said to be a *maximal ideal* if for any ideal I of U, $M \subseteq I \subseteq U \Rightarrow M = I$ or $I = U$.

2. Rough sets

This section is devoted to the basic concepts of Rough set theory. In what follows ϕ and U stand for the empty set and the universe set respectively.

Definition 2.1. A relation R on a non-empty set S is said to be an *equivalence relation* on S if

a. xRX, for all $x \in S$ (reflexivity);

b. $xRy \Rightarrow yRx$ (symmetry);

c. xRy and $yRz \Rightarrow xRz$ (transtivity).

We denote the equivalence class of an element $x \in S$ with respect to the equivalence relation R by the symbol $R[x]$ and $R[x] = \{y \in S : yRx\}$.

Definition 2.2. Let $X \subseteq U$. Let R be an equivalence relation on U. Then we define the following

a. The *lower approximation* of X with respect to R is the set of all objects, which can be for certain classified as X using R. That is the set $R_*(X) = \{x : R[x] \subseteq X\}$.

b. The *upper approximation* of X with respect to R is the set of all objects, which can be possibly classified as X using R. That is the set $R^*(X) = \{x : R[x] \cap X \neq \phi\}$.

c. The *boundary region* of X with respect to R is the set of all objects, which can be classified neither as X nor as not$-X$ using R. That is the set $B_R(X) = R^*(X) - R_*(X)$.

It is clear that $R_*(X) \subseteq X \subseteq R^*(X)$.

Definition 2.3. A set $X \subseteq U$ is said to be a *Roughset* with respect to an equivalence relation R on U, if the boundary region $B_R(X) = R^*(X) - R_*(X)$ is non-empty.

3. Construction of rough sets

In the Literature of Rough set theory, information systems are considered. An *information system* is a pair(U, A) where A is a set of attributes. Each attribute $a \in A$ is a mapping $a : U \to V_a$ where V_a is the range set of the attribute $a \in A$. Corresponding to each attribute $a \in A$, an equivalence relation $R_a$is defined on U such that $xR_ay \Leftrightarrow a(x) = a(y)$. Rough sets are constructed through this relation as usual.

In this section, we slightly deviate from the above traditional setting to construct Rough sets. We consider an equivalence relation on a ring to construct Rough sets and present a few results in this context.

In what follows U stands for a ring and we take our universe set to be the ring $(U, +, \cdot)$.

Definition 3.1. Let I be an ideal of a ring U such that $I \neq \{0\}$ and $I \neq U$.

We define a relation R on U as follows.

For $x, y \in U$, $xRy \Leftrightarrow y - x \in I$.

Proposition 3.1. *The relation R on U is an equivalence relation.*

Remark 3.1. If $x \in U$ then we denote the equivalence class of x under the equivalence relation R by the symbol $R[x]$ and we have $R[x] = \{y \in U : yRx\} = X + I$ thus the equivalence class of $x \in U$ is the left coset $X + I$ of I in U.

Proposition 3.2. *For any subset A of U, the following conditions are equivalent to one another.*

(a) $R_*(A) = A$;

(b) $R^*(A) = A$;

(c) $R_*(A) = R^*(A)$.

Proof. For any subset A of U, We always have $R_*(A) \subseteq A \subseteq R^*(A)$.
Suppose $R_*(A) = A$ and let $x \in R^*(A)$. Then $R[x] \cap A \neq \phi$
$\Rightarrow R_*(A) \cap R^*(A) \neq \phi$
$\Rightarrow$ there exists a point y in U such that $y \in R[x] \cap R_*(A)$
$\Rightarrow yRx$ and $y \in R_*(A)$
$\Rightarrow R[x] = R[y]$ and $R[y] \subseteq A$
$\Rightarrow R[x] \subseteq A$
$\Rightarrow x \in A$
This shows that $R^*(A) \subseteq A$. Hence $R^*(A) = A$.
This completes the proof of (a)$\Rightarrow$(b).
Now consider $R^*(A) = A$. Let $x \in A$. Then $x \in R^*(A)$.
$\Rightarrow R[x] \cap A \neq \phi$
Let $z \in R[x]$. Then $xRz \Rightarrow R[z] = R[x]$
$\Rightarrow R[z] \cap A \neq \phi$
$\Rightarrow z \in R^*(A) \Rightarrow z \in A$.
This proves that $R[x] \subseteq A$ and hence $x \in R_*(A)$.
Hence $R_*(A) = A$. This completes the proof of (b)$\Rightarrow$(a).
Obviously (c) is equivalent to both (a) and (b).

4. Rough approximations of subrings

In this section, we present a few properties of lower and upper rough approximations of subrings of U.

Proposition 4.1. $\{0\}$*is a Rough set.*

Proof. Since $R_*(\{0\}) \subseteq \{0\} \subseteq R^*(\{0\})$, either $R_*(\{0\}) = \phi$ or $R_*(\{0\}) = \{0\}$
Assume that $R_*(\{0\}) = \{0\}$.
Then $0 \in R_*(\{0\}) \Rightarrow I \subseteq \{0\} \Rightarrow I = \{0\}$.
This is a contradiction. Hence $R_*(\{0\}) = \phi$.
Let $x \in R^*(\{0\})$. Then $x + I \cap \{0\} \neq \phi$.
$\Rightarrow 0 \in x + I$.
$\Rightarrow -x \in I$.
$\Rightarrow x \in I$.
This shows that

$$R^*(\{0\}) \subseteq I \tag{4.1}$$

Let $x \in I$. Then $x + I = I$
$\Rightarrow x + I \cap \{0\} = I \cap \{0\} = \{0\} \neq \phi$.
$\Rightarrow x \in R^*(\{0\})$.

This shows that

$$I \subseteq R^*(\{0\}) \tag{4.2}$$

From (4) and (4), $R^*(\{0\}) = I$.
Then $B_R(\{0\}) = I$.
Hence $\{0\}$ is a Rough set.

Remark 4.1. By the above Proposition-4.1, it follows that the lower approximation of a subring of $U$is not necessarily a subring of U. Even though $\{0\}$ is a subring of U, its lower approximation is empty and hence $R_*(\{0\})$ is not a subring of U.

Proposition 4.2. *I is not a Rough set.*

Proof. Clearly $R_*(I) \subseteq I \subseteq R^*(I)$. Let $x \in I$.
Then $x + I = I \Rightarrow x \in R_*(I)$.
Then $I = R_*(I)$.
By Proposition-3.2, $R_*(I) = I = R^*(I)$.
Hence I is not a Rough set.

Proposition 4.3. *Let K be a subringof U.Then the following are equivalent.*

(a) $I \subseteq K$;

(b) $R_*(K) = K = R^*(K)$;

(c) $R_*(K)$ *is a subring of* U.

Proof. Let K be a subring of U. Suppose that $I \subset K$
Clearly $R_*(K) \subseteq K \subseteq R^*(K)$.
Let $x \in K$. Since $I \subseteq K$, $x + I \subseteq x + K = K$
$\Rightarrow x \in R_*(K)$.
This shows that $K \subseteq R_*(K) \Rightarrow K = R_*(K)$.
Hence $R_*(K) = K = R^*(K)$.
This proves (a)$\Rightarrow$(b).
Suppose that $R_*(K) = K = R^*(K)$.
Then $R_*(K)$ is a subring of U.
This proves (b)$\Rightarrow$(c).
Suppose that $R_*(K)$ is a subring of U.
$\Rightarrow 0 \in R_*(K)$
$\Rightarrow I = 0 + I \subseteq K$
This proves (c)$\Rightarrow$(a).

Proposition 4.4. *If K is a subring of U then $R^*(K)$ is a subring of U.*

Proof. Let K be a subring of U.
Let $x \in R^*(K)$.
$\Rightarrow x + I \cap K \neq \phi$

$\Rightarrow$ there exists a point z_1 such that $z_1 \in x + I \cap K$
$\Rightarrow z_1 \in x + I$ and $z_1 \in K$
$\Rightarrow z_1 - x \in I$ and $z_1 \in K$
Let $y \in R^*(K)$.
$\Rightarrow y + I \cap K \neq \phi$
$\Rightarrow$ there exists a point z_2 such that $z_2 \in y + I \cap K$
$\Rightarrow z_2 \in y + I$ and $z_2 \in K$
$\Rightarrow z_2 - y \in I$ and $z_2 \in K$
Since I is an ideal of U, $z_1 - x \in I$ and $z_2 - y \in I$ we have $z_1 - z_2 - x + y \in I$
$\Rightarrow z_1 - z_2 \in x - y + I$
Since K is a subring of U, $z_1 \in K$ and $z_2 \in K$ we have $z_1 - z_2 \in K$
Hence $z_1 - z_2 \in x - y + I \cap K$.
This shows that $x - y + I \cap K$ is non-empty.
$\Rightarrow x - y \in R^*(K)$
Since $z_2 \in U$ and $z_1 - x \in I$, it follows that $z_1z_2 - xz_2 \in I$
Since $x \in U$ and $z_2 - y \in I$, it follows that $xz_2 - xy \in I$
Then $z_1z_2 - xz_2 + xz_2 - xy \in I$
$\Rightarrow z_1z_2 - xy \in I$
$\Rightarrow z_1z_2 \in xy + I$
Since K is a subring of U, $z_1 \in K$ and $z_2 \in K$ we have $z_1z_2 \in K$
$\Rightarrow xy + I \cap K$ is non-empty.
$\Rightarrow xy \in R^*(K)$
Hence $R^*(K)$ is a subring of U.

Proposition 4.5. *If K is an ideal of U then $R^*(K)$is an ideal of U.*

Proof. Since K is an ideal of U, it is a subring of U.
Then $R^*(K)$ is a subring of U.
Let $u \in U$ and $x \in R^*(K)$. Then $(x + I) \cap K \neq \phi$.
$\Rightarrow$ there exists a point $y \in K$ such that $y - x \in I$.
$\Rightarrow uy \in K$ and $uy - ux \in I$
$\Rightarrow uy \in K$ and $uy \in ux + I$
$\Rightarrow uy \in ux + I \cap K$
$\Rightarrow ux + I \cap K \neq \phi$
$\Rightarrow ux \in R^*(K)$
Similarly $xu \in R^*(K)$. Hence $R^*(K)$is an ideal of U.

Remark 4.2. The proof of the proposition 4.6 follows easily from the definitions and hence can be omitted.

Proposition 4.6. *If K is an maximal ideal of U then $R^*(K)$is an maximal ideal of U.*

References

[1] B. Davvaz, *Roughness in rings*, InfSci, 144-163, 2004.

[2] I.N. Herstein, *Topics in algebra*, 2nd edition, Wiley and Sons, New York, 1975.

[3] J.L. Kelley, *Genaral topology*, Springer-Verlag, New York, 1955.

[4] Z. Pawlak, *Rough sets*, International Journal of Computer and Information Sciences, Vol-11, 341-356, 1982.

[5] Z. Pawlak, *Rough sets theoretical aspects of reasoning about data*, Kluwer Academic Publishers, Dordrecht, 1991.

[6] L. Polkowski, *Rough sets*, Mathematical Foundations, SpringerVerlag, Berlin, 2002.

[7] L.A. Zadeh, *Fuzzy sets*, Inform. Control., vol-8, 338-353, 1965.

[8] J. Zhan, B. Davvaz, *Notes on roughness in rings*, Information Sciences, 488-490, 2016.

Accepted: 1.11.2016