α-NILPOTENT GROUPS DERIVED FROM HYPERGROUPS WITH ξ^*-RELATION

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Abstract. This paper deals with hypergroups, as a generalization of classical groups. An important tool in the theory of hyperstructures is the fundamental relation, which brings us into the classical algebra. In this paper for an automorphism α we introduce and study the construction of α-nilpitent fundamental relation in hypergroups. We will characterize α-nilpitent groups via strongly regular relations and several results on the topic are presented.
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1. Introduction and preliminaries

Nilpotent groups in terms of the certain normal series of subgroups are defined in [2]. This approach demonstrates that there is a connection between nilpotent groups and commutators. In [2] Barzegar and Erfanian introduced the α-nilpitent group and α-commutator and its preliminary properties. Also the relative nilpotent groups with respect to a certain automorphism are discussed.

Let G be a group and $\alpha \in Aut(G)$, for two elements $x, y \in G$, x and y commutes under the automorphism α whenever $yx = xy^{\alpha}$. Moreover, $x^{-1}y^{-1}xy^{\alpha}$ is called α- commutator of x, y and denoted by $[x, y]_{\alpha}$.

It is clear that if α is the identity automorphism, then we have ordinary commutator. One can define a α-commutator of weight n as follows:

$$[x_1, x_2, \dots, x_n]_{\alpha} = [x_1, [x_2, \dots, x_n]_{\alpha}]_{\alpha}.$$

Let X_1 and X_2 be two non-empty subsets of a group G. The *α-commutator subgroup* of X_1 and X_2 is defined as follows:

$$[X_1, X_2]_{\alpha} = \langle [x_1, x_2]_{\alpha} : x_1 \in X_1, x_2 \in X_2 \rangle.$$

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It is obvious that $[X_1, X_2]_\alpha$ is not equal to $[X_2, X_1]_\alpha$ in general. Barzegar and Erfanian introduced α-center of the group G as follows:

$$Z^\alpha(G) = \{y \in G : [x, y]_\alpha = 1 \quad for \quad all \quad x \in G\}.$$

Let N be a normal subgroup of G and $N^\alpha = N$. Then $(xN)^{\overline{\alpha}} = x^\alpha N$, where $\overline{\alpha} : G/N \Rightarrow G/N$. It is trivial $[\overline{x_1}, \overline{x_2}, \ldots, \overline{x_n}]_{\overline{\alpha}} = [x_1, x_2, \ldots, x_n]_\alpha N$.

Definition 1.1. A group G is called *α-nilpotent* if it has a *α-central series*, that is, a normal series $\{1\} = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G$, such that G_{i+1}/G_i is contained in the $\overline{\alpha}$-center of G/G_i for all i. The length of a shortest α-central series of G is called relative nilpotency class of G. If G is α-nilpotent of class c_α, then it is nilpotent of class at most c_α.

For the group G and $\alpha \in Aut(G)$, a *lower central α-series* is the following normal series

$$G = \Gamma_1^\alpha(G) \trianglerighteq \Gamma_2^\alpha(G) \trianglerighteq \ldots$$

where, $\Gamma_2^\alpha(G) = \langle [x, y]_\alpha | x, y \in G \rangle$ and for $n \geq 1, \Gamma_{n+1}^\alpha(G) = [G, \Gamma_n^\alpha(G)]_\alpha$. Moreover, $\Gamma_n^\alpha(G/N) = \Gamma_n^\alpha(G)N/N$, where $N^\alpha = N$.

Also an *upper α-central series* is the following normal series

$$1 \subseteq Z_0^\alpha(G) \trianglelefteq Z_1^\alpha(G) \trianglelefteq \ldots,$$

where $Z_1^\alpha(G) = Z^\alpha(G)$ and for $i \geq 1, \frac{Z_i^\alpha}{Z_{i-1}^\alpha} = Z^\alpha(\frac{G}{Z_{i-1}^\alpha(G)})$. Clearly, $Z_i^\alpha(G) \trianglelefteq G$ and in general, these two series will not stop, but if so, we will prove that G is α-nilpotent and its converse is valid. Thus we find equivalent definitions for a α-nilpotent group G.

Theorem 1.2 ([2]). *Suppose G is a group and $\alpha \in Aut(G)$. Then*
1) $Z_n^\alpha(G) = Z_n^{\alpha^{-1}}(G)$;
2) $x \in Z_n^\alpha(G)$ *if and only if* $[g_1, \ldots, g_n, x]_\alpha = 1$ *for all* $g_1, g_2, \ldots, g_n \in G$;
3) $Z_n^\alpha(G) \subseteq Z_n(G)$.

Theorem 1.3 ([2]). *For a group G the following is equivalent.*
1) *G is α- nilpotent;*
2) *There is an integer s such that $Z_s^\alpha(G) = G$.*

Hyperstructure theory was first initiated by Marty [11] in 1934 when he defined hypergroups and started to analyze their properties. Since there are extensive application in many branches of mathematics and applied sciences, the theory of algebraic hyperstructures has nowadays become a well-established branch in algebraic theory. Some investigations of the theory hyperstructures are accessible in the book of Corsini [3], Davvaz and Leoreanu-Fotea [5], Corsini and Leoreanu [4] and Vougiouklis [12]. A *hyperstructure (or hypergroupoid)* is a nonempty set H with a *hyperoperation* $\circ$ defined on H, that is, a mapping of $H \times H$ into the family of non-empty subsets of H. If $(x, y) \in H \times H$, its image under $\circ$ is denoted by $x \circ y$. If A, B are non-empty subsets of H then $A \circ B$ is given by $A \circ B = \bigcup\{x \circ y | x \in A, y \in B\}$. $x \circ A$ is used for $\{x\} \circ A$ and $A \circ x$ for $A \circ \{x\}$. Generally, the singleton a is identified with its member a. The structure $(H, \circ)$ is called a *semihypergroup* if $a \circ (b \circ c) = (a \circ b) \circ c$ for all $a, b, c \in H$, and a semihypergroup $(H, \circ)$ is called a *hypergroup* in the sense of Marty if

$$x \circ H = H \circ x = H, \forall x \in H,$$

which is called the *reproduction axiom*. This axiom means that for any $x, y \in H$ there exist $u, v \in H$ such that $y \in x \circ u$ and $y \in v \circ x$.

Definition 1.4. A hypergroup $(P, \circ)$ is said to be a *polygroup* if the following are satisfied:

(i) P has an identity element, that is there exists an element $e \in P$, such that $x \in e \circ x \cap x \circ e$ for all $x \in P$;

(ii) every element $x \in P$ has inverse, that is there exists $x^{-1} \in P$, such that $e \in x \circ x^{-1} \cap x^{-1} \circ x$;

(iii) for all $x, y, z \in P$:

$$x \in y \circ z \Leftrightarrow y \in x \circ z^{-1} \Leftrightarrow z \in y^{-1} \circ x.$$

A function $f : (H, \circ) \longrightarrow (H, *)$ is called a *homomorphism* if $f(a \circ b) \subseteq f(a) \star f(b)$ for all $a, b \in H$. We say that f is a *good homomorphism* if for all a and b in $H, f(a \circ b) = f(a) \star f(b)$. Let $(H, \circ)$ be a hypergroup and $\rho \subseteq H \times H$ be an equivalence relation. For non-empty subsets A and B of H, we define

$$A\overline{\overline{\rho}}B \Longleftrightarrow a\rho b, \forall a \in A, \forall b \in B.$$

The relation ρ is called *strongly regular on the left (on the right)* if $x\overline{\overline{\rho}}y \Longrightarrow a \circ x\overline{\overline{\rho}}a \circ y (x \circ a\overline{\overline{\rho}}y \circ a$, respectively), for all $x, y, a \in H$.

Moreover, ρ is called *strongly regular* if it is strongly regular on the right and on the left.

Theorem 1.5 ([3]). *If $(H, \cdot)$ is a hypergroup and ρ is a strongly regular relation on H, then H/ρ is a group under the operation:*

$$\rho(x) \otimes \rho(y) = \rho(z), \forall z \in x \cdot y.$$

For all $n \geq 1$, we define the relation β_n on a semihypergroup H, as follows: $a\beta_n b \Longleftrightarrow \exists(x_1, ..., x_n) \in H^n : \{a, b\} \subseteq \prod_{i=1}^n x_i$, and $\beta = \cup_{n \geq 1}\beta_n$, where $\beta_1 = \{(x, x); x \in H\}$, is the diagonal relation on H. This relation was introduced by Koskas [9]. Suppose that β^* is the *transitive closure* of β, the relation β^* is a *strongly regular relation* [3]. Also, we have:

Theorem 1.6 ([6]). *If H is hypergroup then $\beta = \beta^*$.*

Freni in [7] introduced a new fundamental relation $\gamma = \cup_{n\geq 1}\gamma_n$, where γ_1 is the diagonal relation and for every integer $n > 1$, γ_n is the relation defined as follows:

$$x\gamma_n y \Longleftrightarrow \exists(z_1, ..., z_n) \in H^n, \exists\tau \in \mathbb{S}_n : x \in \prod_{i=1}^{n} z_i, y \in \prod_{i=1}^{n} z_{\tau(i)},$$

where $\mathbb{S}_n$ is the *symmetric group* of order n. Suppose that γ^* is the transitive closure of γ. The relation γ^* is a strongly regular relation [7].

The relation β^* is the least equivalence relation on a hypergroup H, such that the quotient H/β^* is a group, while γ^* is the least equivalence relation on a hypergroup H, such that the quotient H/γ^* is an Abelian group.

As the fundamental relations play an important role in the study of theory of algebraic hyperstructures, it has been studied by many authors (for more details see [8], [3], [13] and [7]). As it is well known that abelian groups are contained in α- *nilpotent* groups. The smallest equivalence relation γ^* on a hypergroup H such that the quotient H/γ^*, the set of all equivalence classes, is an Abelian group was introduced in [7]. Now in this paper we introduce and analyze a new strongly regular relation ξ^* on a hypergroup H such that the quotient group H/ξ^* is an α-nilpotent group. Also, we study the relationship between α-nilpotent hypergroups and Abelian hypergroups. In particular, we will characterize α-nilpotent groups via strongly regular relations and we obtain several results on the topic.

2. Construct of α-nilpotent groups by a new strongly regular relation ξ_n^*

Now in this paper we introduce and analyze a new strongly regular relation ξ^* on a hypergroup H such that the quotient group H/ξ^* is an α-nilpotent group. Let $\alpha \in Aut(P)$ for two elements $x, y \in P$, we define α-commutator of x, y as follows:

$$[x, y]_\alpha = \{t|t \in xyx^{-1}y^{-\alpha}\}.$$

Definition 2.1. Let H be a hypergroup and $\alpha \in Aut(H)$. We define:
(1) $L_1^\alpha(H) = H$;
(2) $L_{k+1}^\alpha(H) = \{h \in H|yx \cap hx^\alpha y \neq \emptyset, x \in L_k^\alpha(H), y \in H\}, \forall k \geq 1$.

Theorem 2.2. *Let P be a polygroup. Then for all x, y, h in P*

$$\{h|yx \cap hx^\alpha y \neq \emptyset\} = \{h|h \in yxy^{-1}x^{-\alpha}\} = [y, x]_\alpha.$$

Proof. Let $h \in \{h|yx \cap hx^\alpha y \neq \emptyset\}$. Then there is $z \in yx \cap hx^\alpha y$, which implies that $h \in z(x^\alpha y)^{-1} \subseteq yxy^{-1}x^{-\alpha}$. Thus $h \in \{h|h \in [y, x]_\alpha\}$. Conversly if $h \in \{h|h \in yxy^{-1}x^{-\alpha}\}$, then $h \in (yx)(x^\alpha y)^{-1}$. Therefore $h \in sr$ for some $s \in (yx)$, $r \in (x^\alpha y)^{-1}$. Consequently $h \in s(x^\alpha y)^{-1}$ implies that $s \in h(x^\alpha y)$. Also $s \in yx$. Hence $yx \cap hx^\alpha y \neq \emptyset$. □

By the above theorem it is obvios that for a polygroup P we have $L^{\alpha}_{k+1}(P) = \{h \in P | h \in [y,x]_{\alpha}, y \in P, x \in L^{\alpha}_{k}\}$

Let $n \in \mathbb{N}$ and $\xi_n = \bigcup_{m \geq 1} \xi_{m,n}$, where $\xi_{1,n}$ is the diagonal relation and for every integer $m \geq 1$, and $\xi_{m,n}$ is the relation defined as follows:

$$x\xi_{m,n}y \Longleftrightarrow \exists(z_1,...,z_m) \in H^m; \exists\delta \in \mathbb{S}_m :$$
$$\delta(i) = i \ \text{ if } \ z_i \notin L^{\alpha}_n(H) \ \text{ such that } x \in \prod_{i=1}^{m} z_i, y \in \prod_{i=1}^{m} z_{\delta(i)}.$$

Obviously, for every $n \geq 1$, the relation ξ_n is reflexive and symmetric. Now let ξ^*_n be the transitive closure of ξ_n.

Corollary 2.3. *For every $n \in \mathbb{N}$, we have $\beta^* \subseteq \xi^*_n \subseteq \gamma^*$.*

Theorem 2.4. *For every $n \in \mathbb{N}$, the relation ξ^*_n is a strongly regular relation.*

Proof. Let $n \in \mathbb{N}$. It is obvious that $\overline{\overline{\xi^*_n}}$ is an equivalence relation. Now we show that if $x\xi_n y$, then $x.z\overline{\overline{\xi^*_n}}y.z$ and $z.x\overline{\overline{\xi^*_n}}z.y$. For every $z \in H$, $x\xi_n y$ implies that $m \in \mathbb{N}$ such that $x\xi_{m,n}y$. So there exist $(z_1, z_2, \ldots, z_m) \in H^m, \sigma \in \mathbb{S}_m$ with $\sigma(i) = i$ if $z_i \notin L^{\alpha}_n(H)$, such that $x \in \prod_{i=1}^{m} z_i, y \in \prod_{i=1}^{m} z_{\sigma(i)}$. Therefore, for $z \in P$, $x.z \in \prod_{i=1}^{m} z_i.z$, $\quad y.z \in \prod_{i=1}^{m} z_{\sigma(i)}.z$ and $\sigma(i) = i$ if $z_i \notin L^{\alpha}_n(H)$. Now suppose that $z_{m+1} = z$. We define the permutation $\sigma' \in \mathbb{S}_{m+1}$, with $\sigma'(i) = \sigma(i)$, for all $1 \leq i \leq m$ and $\sigma'(m+1) = m+1$. This implies that $x.z \subseteq \prod_{i=1}^{m+1} z_i.z, y.z \subseteq \prod_{i=1}^{m+1} z_{\sigma'(i)}.z$ such that $\sigma'(i) = i$ if $z_i \notin L^{\alpha}_n(P)$. Therefore, $x.z\overline{\overline{\xi^*_n}}y.z$. Similarly we have $z.x\xi^*_n z.y$. Now, if $x\xi^*_n y$, then there exists $k \in \mathbb{N}$ and $(x = u_0, u_1, \ldots, u_k = y) \in P^{k+1}$ such that $x = u_0\xi_n u_1\xi_n u_2 \ldots \xi_n u_{k-1}\xi_n u_k = y$. Hence by the above result we have $x.z = u_0.z\overline{\overline{\xi^*_n}}u_1.z\overline{\overline{\xi^*_n}}u_2.z \ldots \overline{\overline{\xi^*_n}}u_{k-1}.z\overline{\overline{\xi^*_n}}u_k.z = y.z.$. and so $x.z\overline{\overline{\xi^*_n}}y.z$. Similarly we can proof that $z.x\overline{\overline{\xi^*_n}}z.y$ and the proof completes. □

Proposition 2.5. *For every $n \in \mathbb{N}$, we have $\xi^*_{n+1} \subseteq \xi^*_n$.*

Proof. Let $x\xi_{n+1}y$ so $\exists(z_1,...,z_m) \in H^m; \exists\delta \in \mathbb{S}_m : \delta(i) = i$ if $z_i \notin L^{\alpha}_{n+1}(H)$, such that $x \in \prod_{i=1}^{m} z_i$, $y \in \prod_{i=1}^{m} z_{\delta(i)}$. Now let $\delta_1 = \delta$, since $L^{\alpha}_{n+1}(H) \subseteq L^{\alpha}_n(H)$ so $x\xi_n y$. □

The next result immediately follows from previous theorem.

Corollary 2.6. *If P is a commutative hypergroup, then $\beta^* = \xi^*_n = \gamma^*$.*

Theorem 2.7. *If H is a hypergroup and φ is a strongly regular relation on H, then $L^{\overline{\alpha}}_{k+1}(H/\varphi)) = \{[\overline{s}, \overline{t}]; t \in L^{\alpha}_k((H), s \in H\}$.*

Proof. The proof is based on induction. Put $G = H/\varphi$ and $\overline{x} = \varphi(x)$, for all $x \in H$ and $\overline{\alpha} \in Aut(G)$. If $k = 1$, then $L^{\overline{\alpha}}_2(G) = \{[\overline{s}, \overline{t}]_{\overline{\alpha}} \mid t \in L^{\alpha}_1(H), s \in H\}$. Now, put $\overline{a} = [\overline{s}, \overline{t}]_{\overline{\alpha}}$ where $t \in L^{\alpha}_{k+1}(H), s \in H$, so there exist $x \in L^{\alpha}_k(H)$ and

$y \in H$ such that $yx \cap tx^{\alpha}yt \neq \emptyset$. Then $\overline{t} = [\overline{y}, \overline{x}]_{\overline{\alpha}}$. By induction hypotheses we have $\overline{t} \in L_{k+1}^{\overline{\alpha}}(G)$. Hence $\overline{a} = [\overline{s}, \overline{t}]_{\overline{\alpha}} \in L_{k+2}^{\overline{\alpha}}(G)$. Conversely, let $\overline{a} \in L_{k+2}^{\overline{\alpha}}(G)$. Then $\overline{a} = [\overline{y}, \overline{x}]_{\overline{\alpha}}$, where $\overline{x} \in L_{k+1}^{\overline{\alpha}}(G)$ and $\overline{y} \in G$. So induction hypotheses implies that $\overline{x} = [\overline{v}, \overline{u}]_{\overline{\alpha}}$, where $u \in L_k^{\alpha}(H)$ and $v \in H$. Since H is a hypergroup there exists $t \in H; vu \cap tuv \neq \emptyset$ such that $\overline{t} = [\overline{v}, \overline{u}]_{\overline{\alpha}} = \overline{x}$ and $t \in L_{k+1}^{\alpha}(H), y \in H$. Hence, $\overline{a} = [\overline{y}, \overline{x}]_{\overline{\alpha}} = [\overline{y}, \overline{t}]_{\overline{\alpha}} \in \{[\overline{s}, \overline{t}]; t \in L_k^{\alpha}((H), s \in H\}$. □

Theorem 2.8. *H/ξ_n^* is a α-nilpotent group.*

Proof. Let $G = H/\xi_n^*$ and $\overline{x} = \xi_n^*(x)$, for all $x \in H$. We show that $L_{n+1-i}^{\overline{\alpha}}(G) \subseteq Z_i^{\overline{\alpha}}(G)$ for all s. Let $i = 0$ then $L_{n+1}^{\overline{\alpha}}(G) \subseteq Z_0^{\overline{\alpha}}(G) = \{\overline{e}\}$.Then $L_{n+1}^{\overline{\alpha}}(G) = \{\overline{e}\}$.

Now let $\overline{a} \in L_{n+1-i-1}^{\overline{\alpha}}(G)$. we show that $a \in Z_{i+1}^{\overline{\alpha}}(G)$. Since $\overline{a} \in L_{n+1-i}^{\overline{\alpha}}(G)$, so $[\overline{s}, \overline{a}]_{\overline{\alpha}} \in L_{n+1-i}^{\overline{\alpha}}(G)$. By hypotheses of induction $L_{n+1-i}^{\overline{\alpha}}(G) \subseteq Z_i^{\overline{\alpha}}(G)$. Thus by theorem 1.2 $[\overline{s_i}, \ldots, \overline{s_1}, [\overline{s}, \overline{a}]_{\overline{\alpha}}]_{\overline{\alpha}} = \overline{e}$. Thus $\overline{a} \in Z_{i+1}^{\overline{\alpha}}(G)$. Now let $i = n$ so $L_1^{\alpha}(G) \subseteq Z_n^{\overline{\alpha}}(G)$. Therefore $G = Z_n^{\overline{\alpha}}(G)$. Hence G is α- nilpotent as desired. □

Example 2.9. Let $H = \{e, a, b, c, d, f, g\}$. Consider the non-commutative hypergroup $(H, \cdot)$, where $\cdot$ is defined on H as follows:

.	e	a	b	c	d	f	g
e	e	a	b	c	d	f,g	f,g
a	a	e	d	f,g	b	c	c
b	b	f,g	e	d	c	a	a
c	c	d	f,g	e	a	b	b
d	d	c	a	b	f,g	e	e
f	f,g	b	c	a	e	d	d
g	f,g	b	c	a	e	d	d

Then $H/\beta^* \cong S_3$ (for more details see [3]). Since S_3 is not α-nilpotent and $H/\xi_n^* \subseteq H/\beta^* \cong (\mathbb{S}_3)$. It concluded that $\xi_n^* \neq \beta^*$.

3. On α-nilpotent groups derived from finite polygroups

In this section we try to construct α-nilpotent group from a given hypergroup H and the smallest strongly regular relation on H. Let H be a finite hypergroup. Then we define the relation ξ^* on H by

$$\xi^* = \bigcap_{n \geq 1} \xi_n^*.$$

Theorem 3.1. *The relation ξ^* is a strongly regular relation on a finite hypergroup H such that H/ξ^* is a α-nilpotent group.*

Proof. Since ξ^* is a strongly regular relation on H we have $\xi^* = \bigcap_{n \geq 1} \xi_n^*$. Now, by Proposition 2.5, there exists $k \in \mathbb{N}$ such that $\xi_{k+1}^* = \xi_k^*$. Hence $\xi^* = \xi_k^*$ for some $k \in \mathbb{N}$. This complete the proof. □

Theorem 3.2. *The relation ξ^* is the smallest strongly regular relation on a finite hypergroup H such that H/ξ^* is a α-nilpotent group.*

Proof. Suppose ρ is a strongly regular relation on H such that $K = H/\rho$ is a α-nilpotent group. If $x\xi y$, then there exists $n \in \mathbb{N}$ such that $x\xi_n y$. Therefore, for some $m \in \mathbb{N}, x\xi_{mn} y$ if and only if there exists $(z_1, ..z_m) \in H^m$ and $\delta \in \mathbb{S}_m$ such that $\delta(i) = i$ if $z_i \notin L_n^\alpha(H)$ where $x \in \prod_{i=1}^m z_i$, $y \in \prod_{i=1}^m z_{\delta(i)}$. Therefore, $L_{c+1}^\alpha(H/\rho) = \{[\rho(t), \rho(s)] \mid t \in L_c^\alpha(H), s \in H\} = \{\rho(e)\}$. and so for every $z_i \in L_c^\alpha(H)$ and $v \in L_c^\alpha H$, $\rho(z_i)\rho(v) = \rho(v)\rho(z_i)$. This implies that $\rho(x) = \rho(y)$ so $x\rho y$. □

4. Some important properties of ξ^*

Definition 4.1. Let X be a non-empty subset of H. Then we say that X is a ξ-part of H if for every $k \in \mathbb{N}$ and $(z_1, ..., z_k) \in H^k$ and for every $\sigma \in \mathbb{S}_k$ such that $\sigma(i) = i$ if $z_i \notin \cup_{n\geq 1} L_n^\alpha(H)$, then

$$\prod_{i=1}^k z_i \cap X \neq \emptyset \Longrightarrow \prod_{i=1}^k z_{\sigma(i)} \subseteq X.$$

Theorem 4.2. *Let X be a non-empty subset of a hypergroup H. Then the following conditions are equivalent:*

1) *X is a ξ-part of H,*
2) *$x \in X$, $x\xi y \Longrightarrow y \in X$,*
3) *$x \in X$, $x\xi^* y \Longrightarrow y \in X$.*

Proof. $(1 \Longrightarrow 2)$: if $(x, y) \in H^2$ is a pair such that $x \in X, x\xi y$, then there exist $(z_1, ..., z_k) \in H^k$; $x \in \prod_{i=1}^k z_i \cap X$, $y \in \prod_{i=1}^k z_{\sigma(i)}$ and $\sigma(i) = i$ if $z_i \notin \cup_{n\geq 1} L_n^\alpha(H)$. Since X is a ξ-part of H, we have $\prod_{i=1}^k z_{\sigma(i)} \subseteq X$ and so $y \in X$.

$(2 \Longrightarrow 3)$: Suppose that $(x, y) \in H^2$ is a part such that $x \in X$ and $x\xi^* y$. Then there is $(z_1, ..., z_k) \in H^k$ such that $x = z_0\xi z_1\xi ...\xi z_k = y$. Now by using (2) k times we obtain $y \in X$.

$(3 \Longrightarrow 1)$: Suppose that $x \in \prod_{i=1}^k z_i \cap X$ and $\sigma \in \mathbb{S}_k$ such that $\sigma(i) = i$ if $z_i \notin \cup_{n\geq 1} L_n^\alpha(H)$ .Let $y \in \prod_{i=1}^k z_{\sigma(i)}$. Since $x\xi y$, by (3) we have $y \in X$. Consequently, $\prod_{i=1}^k z_{\sigma(i)} \subseteq X$ and so X is a ξ-part. □

Theorem 4.3. *The following conditions are equivalent:*

(1) *for every $a \in H$, $\xi(a)$ is a ξ-part of H,*
(2) *ξ is transitive.*

Proof. $(1) \Longrightarrow (2)$: Suppose that $x\xi^* y$. Then there is $(z_1, ..., z_k) \in H^k$ such that $x = z_0\xi z_1\xi ...\xi z_k = y$ since $\xi(z_i)$ for all $0 \leq i \leq k$, is a ξ-part, we have $z_i \in \xi(z_{i-1})$, for all $1 \leq i \leq k$. Thus $y \in \xi(x)$, which means that $x\xi y$.

$(2) \Longrightarrow (1)$: Suppose that $x \in H$, $z \in \xi(x)$ and $z\xi y$. By transitivity of ξ, we have $y \in \xi(x)$. Now according to the last theorem, $\xi(x)$ is a ξ-part of H. □

Definition 4.4. The intersection of all ξ-parts which contain A is called ξ-closure of A in H and it will be denoted by $K(A)$.

In what follows, we determine the set $W(A)$, where A is a non-empty subset of H. We set

1) $W_1(A) = A$ and

2) $W_{n+1}(A) = \{x \in H | \exists (z_1, ..., z_k) \in H^k, x \in \prod_{i=1}^{k} z_{(i)}, \exists \sigma \in \mathbb{S}_k$ such that $\sigma(i) = i$, if $z_i \notin \cup_{t \geq 1} L_t^{\alpha}(H)$ and $\prod_{i=1}^{k} z_{\sigma(i)} \bigcap W_n(A) \neq \emptyset\}$.

We denote $W(A) = \bigcup_{n \geq 1} W_n(A)$.

Theorem 4.5. *For any non-empty subset of H, the following statements hold:*

1) $W(A) = K(A)$,

2) $K(A) = \cup_{a \in A} K(a)$.

Proof. 1) It is enough to prove:

(a) $W(A)$ i a ξ-part,

(b) if $A \subseteq B$ and B is a ξ-part, then $W(A) \subseteq B$.

In order to prove (a), suppose that $\prod_{i=1}^{k} z_i \bigcap W(A) \neq \emptyset$ and $\sigma \in \mathbb{S}_k$ such that $\sigma(i) = i$,if $z_i \notin \cup_{n \geq 1} L_n^{\alpha}(H)$. Therefore, there exist $n \in \mathbb{N}$ such that $\prod_{i=1}^{k} z_i \cap W_n(A) \neq \emptyset$ where it follows that $\prod_{i=1}^{k} z_{\sigma(i)} \subseteq W_{n+1}(A) \subseteq W(A)$. Now, we prove (b) by induction on n. We have $W_1(A) = A \subseteq B$. Suppose that $W_n(A) \subseteq B$. We prove that $W_{n+1}(A) \subseteq B$. If $z \in W_{n+1}(A)$, then $z \in \prod_{i=1}^{k} z_i$ and there exists $\sigma \in \mathbb{S}_k$ such that $\sigma(i) = i$,if $z_i \notin \cup_{t \geq 1} L_t^{\alpha}(H)$ and also $\prod_{i=1}^{k} z_{\sigma_i} \cap W_n(A) \neq \emptyset$. Therefore, $\prod_{i=1}^{k} z_{\sigma_i} \cap B \neq \emptyset$ and hence $z \in \prod_{i=1}^{k} z_i \subseteq B$.

2) It is clear that for all $a \in A$, $K(a) \subseteq K(A)$. By part (1), we have $K(A) = \cup_{n \geq 1} W_n(A)$ and $W_1(A) = A = \cup_{a \in A} \{a\}$. It is enough to prove that $W_n(A) = \cup_{a \in A} W_n(a)$, for all $n \in \mathbb{N}$. We follow by induction on n. Suppose it is true for n. We prove that $W_{n+1}(A) = \cup_{a \in A} W_{n+1}(a)$. If $z \in W_{n+1}(A)$, then $z \in \prod_{i=1}^{k} z_i$ and there exists $\sigma \in \mathbb{S}_k$ such that $\sigma(i) = i$,if $z_i \notin \cup_{t \geq 1} L_t^{\alpha}(H)$ and also $\prod_{i=1}^{k} z_{\sigma(i)} \cap W_n(A) \neq \emptyset$. By the hypotheses of induction $\prod_{i=1}^{k} z_{\sigma(i)} \cap W_n(a^{'}) \neq \emptyset$, for some $a^{'} \in A$. Therefore, $z \in W_{n+1}(a^{'})$, and so $W_{n+1}(A) \subseteq \cup_{a \in A} W_{n+1}(a)$. Hence $K(A) = \cup_{a \in A} K(a)$. □

Theorem 4.6. *The following relation is equivalence relation on H.*

$$xWy \Longleftrightarrow x \in W(y),$$

for every $(x, y) \in H^2$, where $W(y) = W(\{y\})$).

Proof. It is easy to see that W is reflexive and transitive. We prove that W is symmetric. To this end, we check that:

1) for all $n \geq 2$ and $x \in H$, $W_n(W_2(x)) = W_{n+1}(x)$,

2) $x \in W_n(y)$ if and only if $y \in W_n(x)$.

We prove 1) by induction on n. Suppose that $z \in W_2(W_2(x))$. Then $z \in \prod_{i=1}^{k} z_i$ and there is $\sigma \in \mathbb{S}_k$ such that $\sigma(i) = i$,if $z_i \notin \cup_{t \geq 1} L_t^{\alpha}(H)$ and also $\prod_{i=1} z_{\sigma(i)} \cap W_2(x) \neq \emptyset$. Thus $z \in W_3(x)$. If $z \in W_{n+1}(W_2(x))$, then $z \in \prod_{i=1}^{k} z_i$

and there exist $\sigma \in \mathbb{S}_k$ such that $\sigma(i) = i$,if $z_i \notin \cup_{t\geq 1} L_t^{\alpha}(H)$ and also $\prod_{i=1}^{k} z_{\sigma(i)} \cap W_n(W_2(x)) \neq \emptyset$. By hypotheses of induction, we have $\prod_{i=1}^{k} z_{\sigma(i)} \cap W_{n+1}(x) \neq \emptyset$ and so $z \in W_{n+2}(x)$. Now we prove (2) by induction on n, too. It is clear that $x \in W_2(y)$ if and only if $y \in W_2(x)$. Then $x \in \prod_{i=1}^{k} z_i$ and there exists $\sigma \in \mathbb{S}_k$ such that $\sigma(i) = i$,if $z_i \notin \cup_{t\geq 1} L_t^{\alpha}(H)$ and also $\prod_{i=1}^{k} z_{\sigma(i)} \cap W_n(y) \neq \emptyset$. Suppose that $b \in \prod_{i=1}^{k} z_{\sigma(i)} \cap W_n(y)$. Then we have $y \in W_n(b)$. From $x \in \prod_{i=1}^{k} z_i \cap W_1(x)$ and $b \in \prod_{i=1}^{k} z_{\sigma(i)}$ we conclude that $b \in W_2(x)$. Therefore, $y \in W_n(W_2(x)) = W_{n+1}(x)$. □

Definition 4.7. Let H be a hypergroup and $p : H \longrightarrow H/\xi$ be the canonical projection. We denote by 1 the identity of the group H/ξ, the set $\rho^{-1}(1)$ is called the ξ-part of H and it is denoted by α_ξ.

Theorem 4.8. *α_ξ is the smallest subhypergroup of H, which is also a ξ-part of H.*

Proof. First we show that α_ξ is a subhypergroup of H. If $x, y \in \alpha_\xi$ and $z \in x\cdot y$, then $\xi(z) = \xi(x)\xi(y) = 1$, so $z \in \alpha_\xi$. On the other hand, there exists $u \in H$ such that $x \in u.y$ and so $1 = \xi(x) = \xi(u)\xi(y) = \xi(u)$. Therefore, $u \in \alpha_\xi$. This means that $\alpha_\xi y = \alpha_\xi$, for all $y \in H$. Similarly, $y\alpha_\xi = \alpha_\xi$ which follows that α_ξ is a subhypergroup of H. Now we prove that $K(x) = \rho^{-1}(\rho(x)) = \alpha_\xi x = x\alpha_\xi$, for every $x \in H$. Indeed we have $z \in \rho^{-1}(\rho(x))$ if and only if $\rho(z) = \rho(x)$ which means $z\xi x$. Therefore, $z \in V(x) = K(x)$. Hence, $K(x) = \rho^{-1}(\rho(x))$. It is easy to see that $\rho^{-1}(\rho(x)) = \alpha_\xi x = x\alpha_\xi$. From the above we obtain if $x \in \alpha_\xi$, then $K(x) = \alpha_\xi$ which means that α_ξ is a ξ-part of H. Moreover, if L is a subhypergroup of H which is also a ξ-part, then $L = K(L) = \bigcup_{a\in L} K(a) = \bigcup_{a\in L} \alpha_\xi a = \alpha_\xi L$. Thus $\alpha_\xi \subseteq L$.

□

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