THE Q-CONJUGACY CHARACTER TABLE OF DIHEDRAL GROUPS

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Abstract. In a seminal paper published in 1998, Shinsaku Fujita introduced the concept of Q-conjugacy character table of a finite group. He applied this notion to solve some problems in combinatorial chemistry. In this paper, the Q-conjugacy character table of dihedral groups is computed in general. As a consequence, Q-conjugacy character table of molecules with point group symmetries $D_3 \cong C_{3v} \cong Dih_6$, $D_4 \cong D_{2d} \cong C_{4v} \cong Dih_8$, $D_5 \cong C_{5v} \cong Dih_{10}$, $D_{3d} \cong D_{3h} \cong D_6 \cong C_{6v} \cong Dih_{12}$, $D_2 \cong C_{2v} \cong Z_2 \times Z_2 \cong Dih_2$, $D_{4d} \cong Dih_{16}$, $D_{5d} \cong D_{5h} \cong Dih_{20}$, $D_{6d} \cong Dih_{24}$ are computed, where Z_2 denotes a cyclic group of order 2 and Dih_n is the dihedral group of even order n.

Keywords: Q-conjugacy character table, dihedral group, conjugacy class.

1. Introduction

A *representation* of a group G is a homomorphism from G into the group of invertible operators of a vector space V. In this case, we can interpret each element of G as an invertible linear transformation $V \longrightarrow V$. If $n = dimV$ and fix a basis for V then we can construct an isomorphism between the set of all "invertible linear transformation $V \longrightarrow V$" and the set of all "invertible $n \times n$ matrices". For the existence of this isomorphism, we usually use the term *matrix representation* as representation. The irreducible representations

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are precisely the ones that cannot be broken up into smaller pieces. Suppose α is a representation of the group G. Then the function $\hat{\alpha} : G \longrightarrow \mathbb{C}$ given by $\hat{\alpha}(g) = tr(\alpha(g))$, the sum of all diagonal entries, is called the *character* afforded by α. The character of an irreducible representation is called an *irreducible character*.

It is well-known that the number of conjugacy classes is the same as the number of distinct irreducible representations. Also, the group character afforded by a representation is constant on a conjugacy class. Hence, the values of the irreducible characters of a group G can be written as an array in a square matrix $CT(G)$ known as the *character table* of G. In this matrix, the rows are given the irreducible characters and the columns are given the conjugacy classes of the group G, see [11, 12] for details.

Suppose G is a finite group and $\langle h \rangle$ is denoted the cyclic subgroup of G generated by h. The elements g and h are said to be $Q-$*conjugate* to each other if there exist $t \in G$ such that $t^{-1}\langle g \rangle t = \langle h \rangle$. The $Q-$conjugacy is an equivalence relation and generates equivalence classes which is called the $Q-$*conjugacy classes* of G. Suppose that the group G is partitioned into $Q-$conjugacy classes as $G = K_1 \dot{\cup} K_2 \dot{\cup} \cdots \dot{\cup} K_r$. Following Fujita [4], each $Q-$conjugacy class K_i is called a dominant class."

Suppose Γ is an irreducible $Q-$representation of a finite group G and γ is the irreducible character afforded by Γ. It is well-known that the $Q-$character γ is constant on the $Q-$conjugacy classes of G. So, if $g \in K_i$ then we can write $\gamma(K_i)$ as $\gamma(g)$. The $Q-$character table of the group G is an $r \times r$ invertible matrix in which the columns and rows are labeled by $Q-$conjugacy classes and $Q-$irreducible characters of G, respectively. Shinsaku Fujita used this table in a series of papers to solve some problems in combinatorial chemistry. We encourage the interested readers to consult papers $[1-3]$, $[5-10]$ and references therein.

The dihedral group Dih_{2n} is a group of order $2n$ generated by a, b such that $a^n = 1, b^2 = 1, b^{-1}ab = a^{-1}$. The elements of this group has form $a^\alpha b^\beta$, where $1 \leq \alpha \leq n$ and $\beta = 0, 1$. The character table of Dih_{2n} is shown in Tables 1 and 2, when n is odd or even, respectively.

Table 1: The character table of Dih_{2n}, n is Odd.

D_{2n}	$a^\alpha b^\beta$
χ_1	1
χ_2	$(-1)^\beta$
χ_{j+2}	$(1-\beta)(\varepsilon^{j\alpha} + \varepsilon^{-j\alpha})$

Table 2: The character table of Dih_{2n}, n is Even.

D_{2n}	$a^{\alpha}b^{\beta}$
χ_1	1
χ_2	$(-1)^{\beta}$
χ_3	$(-1)^{\alpha}(-1)^{\beta}$
χ_4	$(-1)^{\alpha}$
χ_{j+4}	$(1-\beta)(\varepsilon^{j\alpha}+\varepsilon^{-j\alpha})$

Here, ε denotes the primitive $n-$th root of unity, i.e. $\varepsilon = Cos(\frac{2\pi}{n}) + iSin(\frac{2\pi}{n})$. Our other notations are standard and can be taken from the standard books on group theory.

2. Main results

Suppose G is a finite group and T_1, T_2 are representations of G over a field K of characteristic 0. The representations T_1 and T_2 are called *equivalent* if there exists a non-singular matrix S with entries in K such that $S^{-1}T_1(g)S = T_2(g)$, for each $g \in G$. For simplicity, we use the notion $K-$representation as representation over a field K.

Suppose L and K are fields such that $K \subseteq L$, and U is an L-representation of a group G. We say that U is *realizable* in K if there exists a K-representation T such that U and T are L-equivalent. We now assume that χ is an irreducible character afforded by an irreducible K^*-representation U of G. The field generated by K and all of the values $\chi(g)$ over all elements of G is denoted by $K(\chi)$. The *Schur index* of U with respect to K is defined as $M_K(U) = \min(F : K(\chi))$ in which the minimum is taken over all fields F in which U is realizable. If χ is the character afforded by an irreducible $K-$representation U of G and K has characteristic 0, then $m_K(\chi)$ denotes the Schur index of χ over K. We use the notation $m(\chi)$, when $K = Q$ is the field of rational numbers, see [13] for details.

It is merit to mention here that we can construct a group G such that irreducible characters of G are rational values, but the Q-conjugacy character table are from its character table. The reason is the fact that Schur index of irreducible rational values characters can be different from 1 and so calculations given the papers [14, 17] have to be corrected. A counterexample is as follows:

Example 4. Suppose $SmallGroup(n, i)$ denotes the $i-th$ group of order n in the library of GAP, [15]. Then it is easy to see that $SmallGroup(128, 937)$ is a group isomorphic to $Q_8 \times Q_8$. Consider the characters $\chi_{17}, \chi_{18}, \chi_{19}$ and χ_{20}, see Table 1. Then one can see that these irreducible characters are having Schur index 2. Thus the Q-conjugacy character table of this group can be constructed by the character table in which $2\chi_{17}, 2\chi_{18}, 2\chi_{19}$ and $2\chi_{20}$ are substituted by the corresponding irreducible characters in the parent table.

Table 3: The character table of $G = Q_8 \times Q_8$.

G	$1a$	$2a$	$4a$	$4b$	$4c$	$4d$	$2b$	$2c$	$8a$	$8b$	$4e$	$4f$	$4g$	$4h$	$4i$	$8c$	$4j$	$4k$	$4l$	$4m$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	-1	1	1	1	1	1	1	-1	-1	-1	1	1	1	1	-1	1	1	1	1
χ_3	1	1	-1	1	1	1	1	1	-1	1	1	-1	-1	1	1	-1	-1	-1	-1	1
χ_4	1	-1	-1	1	1	1	1	1	1	-1	-1	-1	-1	1	1	1	-1	-1	-1	1
χ_5	1	1	1	-1	1	1	1	1	1	-1	1	-1	1	-1	1	-1	-1	-1	1	-1
χ_6	1	-1	1	-1	1	1	1	1	-1	1	-1	-1	1	-1	1	1	-1	-1	1	-1
χ_7	1	1	-1	-1	1	1	1	1	-1	-1	1	1	-1	-1	1	1	1	1	-1	-1
χ_8	1	1	-1	-1	1	1	1	1	-1	-1	1	1	-1	-1	1	1	1	1	-1	-1
χ_9	1	-1	-1	-1	1	1	1	1	1	1	-1	1	-1	-1	1	-1	1	1	-1	-1
χ_{10}	2	0	0	2	-2	2	2	2	0	0	0	0	0	-2	-2	0	0	0	0	-2
χ_{11}	2	0	0	2	-2	2	2	2	0	0	0	0	0	-2	-2	0	0	0	0	-2
χ_{12}	2	0	-2	0	2	-2	2	2	0	0	0	0	2	0	-2	0	0	0	2	0
χ_{13}	2	0	0	0	-2	-2	2	2	0	0	0	2	0	0	2	0	-2	-2	0	0
χ_{14}	2	0	0	0	-2	-2	2	2	0	0	0	-2	0	0	2	0	2	2	0	0
χ_{15}	4	2	0	0	0	0	-4	4	0	0	-2	0	0	0	0	0	0	0	0	0
χ_{16}	4	-2	0	0	0	0	-4	4	0	0	2	0	0	0	0	0	0	0	0	0
χ_{17}	4	0	0	0	0	0	0	-4	0	0	0	0	2	-2	0	0	2	-2	-2	2
χ_{18}	4	0	0	0	0	0	0	-4	0	0	0	0	-2	-2	0	0	-2	2	2	2
χ_{19}	4	0	0	0	0	0	0	-4	0	0	0	0	2	2	0	0	-2	2	-2	-2
χ_{20}	4	0	0	0	0	0	0	-4	0	0	0	0	-2	2	0	0	2	-2	2	-2

Theorem 1. [16, Example 1]**.** *The Schur index of any irreducible character of dihedral group Dih_{2n} is equal to 1.*

Let G be a finite group, χ be an irreducible complex character of G and $m_Q(\chi)$ denote the Schur index of χ over Q. If $\Gamma(\chi)$ denotes the Galois group $Q(\chi)$ over Q then it is well-known that $\sum_{\alpha \in \Gamma(\chi)} m(\chi)\chi^{\alpha}$ is an irreducible Q-conjugacy character of G [11]. Thus, by knowing the character table of a group and its Schur indices one can find the $Q-$conjugacy character table of the group.

In [6], Fujita computed the $Q-$conjugacy character table of cyclic groups. For the sake of completeness, we reprove this result by a new method. To do this, we define the Ramanujan sum $c_n(m)$ as $c_n(m) = \sum_{1 \le k \le n, (k,n)=1} e^{\frac{2\pi ikm}{n}}$, where m and n are positive integers.

Theorem 2. *The Q-conjugacy character table of cyclic groups Z_n can be computed by $QCT(Z_n) = [a_{ij}]$, where $a_{i,j} = c_{d_i}(\frac{n}{d_j})$.*

In previous result, it is easy to see that

$$a_{i,j} = \frac{\varphi(d_i)\mu(\frac{d_i}{(n/d_j, d_i)})}{\varphi(\frac{d_i}{(d_i, n/d_j)})},$$

where φ is the Euler function and μ is the Möbius function.

Let $\tau(n)$ denote the number of divisors of n. Then we can sort all divisors in a sequence as $d_1 = 1 < d_2 < \cdots < d_{\tau(n)}$. In the following theorem we obtain the Q-conjugacy character table of dihedral group Dih_{2n}.

Theorem 3. *The Q-conjugacy character table of Dih_{2n} is recorded in Tables 4 and 5, when n is odd or even, respectively.*

Table 4: The Q-conjugacy character table of Dih_{2n}, n is Odd.

D_{2n}	K_1	K_j	$K_{\tau(n)+1}$
γ_1	1	1	1
γ_i	$\phi(d_i)$	$c_{d_i}(j)$	0
$\gamma_{\tau(n)+1}$	1	1	-1

Table 5: The Q-conjugacy character table of Dih_{2n}, n is Even.

D_{2n}	K_1	K_j	$K_{\tau(n)+1}$	$K_{\tau(n)+2}$
1	1	1	1	1
γ_i	$\phi(d_i)$	$c_{d_i}(j)$	0	0
$\gamma_{\tau(n)}$	1	$(-1)^j$	-1	1
$\gamma_{\tau(n)+1}$	1	$(-1)^j$	1	-1
$\gamma_{\tau(n)+2}$	1	1	-1	-1

Proof. Suppose $[x]$ denotes the conjugacy class with x as a representative. Our main proof consider two cases as follows:

Case 1. n is odd. In this case the dominant classes are $K_1 = \{[id_G]\}$, $K_{\tau(n)+1} = \{[b]\}$ and $K_j = \{[a^r], [a^{n-r}] : (r, n) = \frac{n}{d_j}\}$. Let $\gamma_1 = \chi_1$, $\gamma_{\tau(n)+1} = \chi_2$ and $\psi_i = \chi_{2+i}$. Then $\gamma_i = \sum_{(j,n)=\frac{n}{d_i}} \psi_j$ where,

$$\begin{aligned}
\gamma_i(a^j) &= \sum_{\substack{k=1\\(k,n)=\frac{n}{d_i}}}^{k=[n/2]} \psi_k(a^j) \\
&= \sum_{\substack{k=1\\(k,n)=\frac{n}{d_i}}}^{k=[n/2]} \varepsilon^{kj} + \varepsilon^{-kj} \\
&= e \sum_{\substack{k=1\\(k,n)=\frac{n}{d_i}}}^{k=[n/2]} \varepsilon^{kj} + \sum_{\substack{k=1\\(k,n)=\frac{n}{d_i}}}^{k=[n/2]} \varepsilon^{-kj} \\
&= \sum_{\substack{k=1\\(k,n)=\frac{n}{d_i}}}^{k=[n/2]} \varepsilon^{kj} + \sum_{\substack{k=[n/2]+1\\(k,n)=\frac{n}{d_i}}}^{k=n} \varepsilon^{-kj} \\
&= \sum_{\substack{k=1\\(k,n)=\frac{n}{d_i}}}^{k=n} \varepsilon^{kj} \\
&= \sum_{\substack{k=1\\(k,d_i)=1}}^{k=d_i} \varepsilon^{kj} \\
&= c_{d_i}(j).
\end{aligned}$$

Case 2. n is even. In this case, the dominant classes are $K_1 = \{[id_G]\}$, $K_j = \{[a^r], [a^{n-r}] : (r, n) = \frac{n}{d_j}\}$, $K_{\tau(n)+1} = \{[ab]\}$ and $K_{\tau(n)+2} = \{[b]\}$. Let $\gamma_1 = \chi_1$, $\gamma_{\tau(n)+2} = \chi_2$, $\gamma_{\tau(n)+1} = \chi_3$, $\gamma_{\tau(n)} = \chi_4$ and $\psi_i = \chi_{2+i}$. Then $\gamma_i = \sum_{(j,n)=\frac{n}{d_i}} \psi_j$.□

We now consider the following isomorphisms between dihedral groups and the point group symmetries of molecules:

- $D_2 \cong C_{2v} \cong Z_2 \times Z_2 \cong Dih_4$,
- $D_3 \cong C_{3v} \cong Dih_6$,
- $D_4 \cong D_{2d} \cong C_{4v} \cong Dih_8$,
- $D_5 \cong C_{5v} \cong Dih_{10}$,
- $D_{3d} \cong D_{3h} \cong D_6 \cong C_{6v} \cong Dih_{12}$,
- $D_{4d} \cong Dih_{16}$,
- $D_{5d} \cong D_{5h} \cong Dih_{20}$,
- $D_{6d} \cong Dih_{24}$.

Using the previous theorem and above isomorphisms, one can calculate the $\mathbb{Q}-$conjugacy character table of these molecules. In the end of this paper we compute the $\mathbb{Q}-$conjugacy character table of $C_{5v} \cong Dih_{10}$ and $D_{4d} \cong Dih_{16}$. Our calculations are recorded in Tables 6 and 7.

Table 6: The Q-conjugacy character table of $C_{5v} \cong Dih_{10}$.

D_{10}	1	a	b
γ_1	1	1	1
γ_2	4	-1	0
γ_3	1	1	-1

Table 7: The Q-conjugacy character table of $D_{4d} \cong D_{16}$.

D_{16}	1	a	a^2	a^4	ab	b
γ_1	1	1	1	1	1	1
γ_2	2	0	-2	2	0	0
γ_3	4	0	0	-4	0	0
γ_4	1	-1	1	1	-1	1
γ_5	1	-1	1	1	1	-1
γ_6	1	1	1	1	-1	-1

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