

LAWVERE-TIERNEY SHEAVES, FACTORIZATION SYSTEMS, SECTIONS AND j -ESSENTIAL MONOMORPHISMS IN A TOPOS

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Abstract. Let j be a Lawvere-Tierney topology (a topology, for short) on an arbitrary topos \mathcal{E} , B an object of \mathcal{E} , and $j_B = j \times 1_B$ the induced topology on the slice topos \mathcal{E}/B . In this manuscript, we analyze some properties of the pullback functor $B^* : \mathcal{E} \rightarrow \mathcal{E}/B$ which are dealing with topologies. Then for the left cancellable class \mathcal{M} of all j -dense monomorphisms in a topos \mathcal{E} , we achieve some necessary and sufficient conditions for that the pair $(\mathcal{M}, \mathcal{M}^\perp)$ is a factorization system in \mathcal{E} , which is related to the factorization systems in slice topoi \mathcal{E}/B , where B ranges over the class of objects of \mathcal{E} . Among other things, we prove that an arrow $f : X \rightarrow B$ in \mathcal{E} is a j_B -sheaf in \mathcal{E}/B whenever the graph of f , is a section in \mathcal{E}/B as well as the object of sections $S(f)$ of f , is a j -sheaf in \mathcal{E} . Furthermore, we introduce a class of monomorphisms in \mathcal{E} , which we call each member of the class j -essential. Some equivalent forms and some of their properties are presented. Also, we prove that any presheaf in a presheaf topos has a maximal essential extension. Finally, some similarities and differences of the obtained result are discussed if we put a (productive) weak topology j , studied by some authors, instead of a topology.

Keywords: (weak) Lawvere-Tierney topology, sheaf, factorization system, slice topos, essential monomorphism.

1. Introduction and preliminaries

One of the basic tools to construct new topoi from old ones is the notion of Lawvere-Tierney topology. Recently, applications of Lawvere-Tierney topologies in broad topics such as measure theory [7] and quantum physics [14, 15] are observed. In the spacial case, considerable work has been presented that is dedicated to the study of (weak) Lawvere-Tierney topology on a presheaf topos on a small category and especially on a monoid, see [6, 5]. It is clear that Lawvere-Tierney sheaves in a topos are exactly injective objects (of course,

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with respect to dense monomorphisms, not to merely monomorphisms) which are separated too. Injectivity with respect to a class \mathcal{M} of morphisms in a slice category \mathcal{C}/B (which its objects are \mathcal{C} -arrows with codomain B) has been studied in extensive form; for example we refer the reader to [1, 3]. From this perspective, among other things, in this paper we will establish for Lawvere-Tierney sheaves some categorical characterizations of injective objects in slice topoi. The object of sections $S(f)$ of an arbitrary morphism $f : X \rightarrow B$ is a notion related to injective objects in a slice category, see [3]. This object is very useful in synthetic differential geometry (or SDG, for short) (for details, see [11]). For example, considering D as infinitesimals, for any micro-linear object M we have:

- Let τ be the tangent bundle on M , i.e., $\tau : M^D \rightarrow M$, defined by $\tau(t) = t(0)$. Then $S(\tau)$ is all vector fields on M .
- Consider $\eta : M^{D \times D} \rightarrow M$ which assigns to any micro-square Q of $M^{D \times D}$, the element $Q(0, 0)$. Then, $S(\eta)$ is all distributions of dimension 2 on M .

Throughout this paper, \mathcal{E} is a (elementary) topos, two objects $0, 1$ are the initial and terminal objects and the object Ω together with the arrow $1 \xrightarrow{\text{true}} \Omega$ is the subobject classifier of \mathcal{E} . Also, the arrow $\wedge : \Omega \times \Omega \rightarrow \Omega$ is the meet operation on Ω . Now, we express some basic concepts from [12] which will be needed in the sequel.

Definition 1.1. A *Lawvere-Tierney topology* on \mathcal{E} is a map $j : \Omega \rightarrow \Omega$ in \mathcal{E} satisfies the following properties

- (a) $j \circ \text{true} = \text{true}$; (b) $j \circ j = j$; (c) $j \circ \wedge = \wedge \circ (j \times j)$;

$$\begin{array}{ccc}
 \begin{array}{ccc} 1 & \xrightarrow{\text{true}} & \Omega \\ & \searrow \text{true} & \downarrow j \\ & & \Omega \end{array} &
 \begin{array}{ccc} \Omega & \xrightarrow{j} & \Omega \\ & \searrow j & \downarrow j \\ & & \Omega \end{array} &
 \begin{array}{ccc} \Omega \times \Omega & \xrightarrow{\wedge} & \Omega \\ j \times j \downarrow & & \downarrow j \\ \Omega \times \Omega & \xrightarrow{\wedge} & \Omega \end{array}
 \end{array}$$

From now on, we say briefly to a Lawvere-Tierney topology on \mathcal{E} , a *topology* on \mathcal{E} .

Recall [12] that topologies on \mathcal{E} are in one to one correspondence with universal closure operators. For a topology j on \mathcal{E} , considering $\overline{(\cdot)}$ as the universal closure operator corresponding to j , a monomorphism $k : A \rightarrow C$ in \mathcal{E} is called *j -dense* whenever $\overline{A} = C$, as two subobjects of C . Also, we say that k is *j -closed* if we have $\overline{A} = A$, again as subobjects of C .

Definition 1.2. For a topology j on \mathcal{E} , an object F of \mathcal{E} is called a *j -sheaf* whenever for any j -dense monomorphism $m : A \rightarrow E$, one can uniquely extend

any arrow $h : A \rightarrow F$ to a map g on all of E ,

$$(1) \quad \begin{array}{ccc} A & \xrightarrow{h} & F \\ m \downarrow & \nearrow g & \\ E & & \end{array}$$

We say that F is j -separated if the arrow g exists in (1), it is unique.

We will denote the full subcategories of \mathcal{E} consisting of j -sheaves and j -separated objects by $\mathbf{Sh}_j(\mathcal{E})$ and $\mathbf{Sep}_j(\mathcal{E})$, respectively.

We now briefly describe the contents of other sections. We start in Section 2, to study basic properties of the pullback functor $B^* : \mathcal{E} \rightarrow \mathcal{E}/B$, for any object B of \mathcal{E} , along with the unique map $!_B : B \rightarrow 1$. Afterwards, we would like to achieve, for the left cancellable class \mathcal{M} of all j -dense monomorphisms in a topos \mathcal{E} , some necessary and sufficient conditions for that $(\mathcal{M}, \mathcal{M}^\perp)$ to be a factorization system in \mathcal{E} , which is related to the factorization systems in slice topoi \mathcal{E}/B . In section 3, among other things, we prove that an arrow $f : X \rightarrow B$ in \mathcal{E} is a j_B -sheaf whenever the graph of f is a section in \mathcal{E}/B as well as the object of sections $S(f)$ of f is a j -sheaf in \mathcal{E} . In section 4, we introduce a class of monomorphisms in an elementary topos \mathcal{E} , which we call them ‘ j -essential monomorphisms’. Furthermore, we present some equivalent forms to these kind of monomorphisms and get some of their properties. Meanwhile, we prove that any presheaf in a presheaf topos has a maximal essential extension. Also, it is shown that the functor B^* reflects j -essential extensions. It is seen that some of these results hold for a (*productive*) *weak topology* j , studied in [10], instead of a topology as well.

2. Pullback functors, dense monomorphisms and factorization systems

The purpose of this section is to present some basic properties of the pullback functor $B^* : \mathcal{E} \rightarrow \mathcal{E}/B$, for any object B of \mathcal{E} , along with the unique map $!_B : B \rightarrow 1$. Afterwards, for the left cancellable class \mathcal{M} of all j -dense monomorphisms in a topos \mathcal{E} we achieve some necessary and sufficient conditions for that the pair $(\mathcal{M}, \mathcal{M}^\perp)$ to be a factorization system in \mathcal{E} , which is related to the factorization systems in slice topoi \mathcal{E}/B .

To begin with, the following lemma characterizes sheaves in a topos \mathcal{E} .

Lemma 2.1. *Let j be a topology on \mathcal{E} . Then an object E of \mathcal{E} is j -sheaf iff E is j -unique absolute retract; that is, any j -dense monomorphism $u : E \rightarrowtail F$, has a unique retraction $v : F \rightarrow E$.*

Proof. *Necessity.* Since E is a j -sheaf, for any j -dense monomorphism $u : E \rightarrowtail F$, corresponding to the identity map $\text{id}_E : E \rightarrow E$, there exists a unique map

$v : F \rightarrow E$ such that the following diagram commutes.

$$\begin{array}{ccc} E & \xrightarrow{\text{id}_E} & E \\ u \downarrow & \nearrow v & \\ F & & \end{array}$$

Sufficiency. For each j -dense monomorphism $m : U \rightarrow V$ and any map $f : U \rightarrow E$, we construct the following pushout diagram in \mathcal{E} .

$$(2) \quad \begin{array}{ccc} U & \xrightarrow{f} & E \\ m \downarrow & & \downarrow n \\ V & \xrightarrow[\text{p.o.}]{g} & F \end{array}$$

Since in any topos pushouts transfer j -dense monomorphisms (see [9]), n is j -dense in (2), and hence by assumption, there exists a unique retraction $p : F \rightarrow E$ such that $pn = \text{id}_E$. Now, for the arrow $pg : V \rightarrow E$ we have $pgm = pnf = \text{id}_E f = f$. To prove that $pg : V \rightarrow E$ is unique with this property, let $h : V \rightarrow E$ be an arrow in \mathcal{E} in such a way that $hm = f$. Then, in the pushout diagram (2), according to the maps $h : V \rightarrow E$ and $\text{id}_E : E \rightarrow E$, there exists a unique map $k : F \rightarrow E$ such that $kn = \text{id}_E$ and $kg = h$.

$$\begin{array}{ccc} U & \xrightarrow{f} & E \\ m \downarrow & & \downarrow n \\ V & \xrightarrow{g} & F \end{array} \quad \begin{array}{c} \searrow \text{id}_E \\ \nearrow k \\ \searrow h \end{array}$$

Now, k is a retraction of j -dense monomorphism n , so by hypothesis we get $p = k$. Consequently, $pg = kg = h$. \square

For an object B of \mathcal{E} , we consider the pullback functor $B^* : \mathcal{E} \rightarrow \mathcal{E}/B$ along with the unique map $!_B : B \rightarrow 1$, which assigns to any A of \mathcal{E} , the second projection $B^*(A) = \pi_B^A : A \times B \rightarrow B$ and to any $f : A \rightarrow C$, the arrow $f \times \text{id}_B : A \times B \rightarrow C \times B$ in \mathcal{E} such that $\pi_B^C(f \times \text{id}_B) = \pi_B^A$. Recall [12] that the object π_B^Ω together with the arrow

$$\text{true} \times \text{id}_B : \text{id}_B \rightarrow \pi_B^\Omega$$

is the subobject classifier of the slice topos \mathcal{E}/B . Also, in a similar vein, we can observe that the meet operation \wedge_B on π_B^Ω is the arrow $\wedge \times 1_B$ in \mathcal{E} such that $\pi_B^\Omega(\wedge \times 1_B) = \pi_B^{\Omega \times \Omega}$,

$$\begin{array}{ccc} \Omega \times \Omega \times B & \xrightarrow{\wedge \times 1_B} & \Omega \times B \\ \searrow \pi_B^{\Omega \times \Omega} & & \downarrow \pi_B^\Omega \\ & & B. \end{array}$$

Now, by Definition 1.1, we easily get the following lemma.

Lemma 2.2. *Let B be any object in a topos \mathcal{E} . Then any topology $k : \pi_B^\Omega \rightarrow \pi_B^\Omega$ on \mathcal{E}/B is a pair (l, π_B^Ω) , for some arrow $l : \Omega \times B \rightarrow \Omega$ in \mathcal{E} satisfies the following conditions (as arrows in \mathcal{E})*

- (1) $l \circ (l, \pi_B^\Omega) = l$;
- (2) $l \circ (\text{true} \times 1_B) = \text{true} \circ !_B$;
- (3) $l \circ \wedge_B = \wedge \circ (l \circ (\pi_1, \pi_3), l \circ (\pi_2, \pi_3))$, where π_i is the i -th projection on $\Omega \times \Omega \times B$, for $i = 1, 2, 3$.

By Lemma 2.2, for each topology j on \mathcal{E} , considering $l = j \circ \pi_\Omega^B$, it is easily seen that $j \times 1_B = (l, \pi_B^\Omega)$ is a topology on \mathcal{E}/B which we denote it by j_B . In this case j_B is called the *induced topology* on \mathcal{E}/B by j .

One can simply see that if an arrow k is a monomorphism in \mathcal{E}/B , then k as an arrow in \mathcal{E} , is too. Also, for each monomorphism $k : f \rightarrow g$ in \mathcal{E}/B , where $f : X \rightarrow B$ and $g : Y \rightarrow B$ in \mathcal{E} , we can observe

$$(3) \quad \widetilde{f} \xrightarrow{\widetilde{k}} g = (\overline{X} \xrightarrow{g\overline{k}} B) \xrightarrow{\overline{k}} g,$$

where $\overline{(\cdot)}$ and $\widetilde{(\cdot)}$ are the universal closure operators corresponding to the topologies j and j_B on topoi \mathcal{E} and \mathcal{E}/B , respectively, for which the whole and the middle squares of the following diagram are pullbacks in \mathcal{E} ,

$$(4) \quad \begin{array}{ccccc} \overline{X} & \xrightarrow{\quad} & 1 & & \\ \downarrow \overline{k} & & \downarrow \text{true} & & \\ Y & \xrightarrow{\quad} & Y & \xrightarrow{\text{char}(k)} & \Omega & \xrightarrow{j} & \Omega \\ & & \downarrow k & & \downarrow \text{true} & & \\ & & Y & \xrightarrow{\quad} & 1 & & \\ & & \downarrow \text{true} & & \downarrow \text{true} & & \end{array}$$

(for more details, see [12]). One can construct \widetilde{k} in \mathcal{E}/B , similar to the above diagram.

Here, we proceed to improve [2, Vol. III, Proposition 9.2.5] as follows:

Lemma 2.3. *Let j be a topology in a topos \mathcal{E} . For every object B of \mathcal{E} , the pullback functor $B^* : \mathcal{E} \rightarrow \mathcal{E}/B$ preserves and reflects: denseness (closeness) and j -separated objects (j -sheaves).*

Proof. Let j be a topology on \mathcal{E} and B an object of \mathcal{E} . Preserving dense (closed) monomorphisms and sheaves (separated objects) in \mathcal{E} by the pullback functor B^* , is standard and may be found in [2, Vol. III, Proposition 9.2.5]. To prove the rest of Lemma, here we just show that B^* reflects dense (closed) monomorphisms. To verify this claim, let $g : A \rightarrow C$ be an arrow in \mathcal{E} for which $B^*(g)$ is a j_B -dense (j_B -closed) monomorphism. We show that g is j -dense (j -closed) monomorphism. As $B^*(g) = g \times \text{id}_B$ being a monomorphism in \mathcal{E}/B ,

the arrow g is a monomorphism in \mathcal{E} as well. For, let f and h in \mathcal{E} be two arrows such that $gf = gh$, we will have

$$\begin{aligned} gf = gh &\implies (g \times \text{id}_B)(f \times \text{id}_B) = (g \times \text{id}_B)(h \times \text{id}_B) \\ &\implies f \times \text{id}_B = h \times \text{id}_B \quad (g \times \text{id}_B \text{ is a monomorphism}) \\ &\implies f = h. \end{aligned}$$

Considering $\overline{(\cdot)}$ and $\widetilde{(\cdot)}$ as the universal closure operators corresponding to j and j_B , respectively. We get

$$\begin{aligned} \widetilde{B^*(g)} &= \widetilde{g \times \text{id}_B} \\ &= \overline{g \times \text{id}_B} && \text{(by (3))} \\ &= \overline{g} \times \text{id}_B, \end{aligned}$$

where the last equality is true since we have $g \times \text{id}_B = (\pi_C^B)^{-1}(g)$, and because of stability of universal closure operators under pullbacks we get $\overline{(\pi_C^B)^{-1}(g)} = (\pi_C^B)^{-1}(\overline{g})$. The above equalities imply that if $B^*(g)$ is j_B -dense (j_B -closed) in \mathcal{E}/B , then g is j -dense (j -closed) in \mathcal{E} . \square

For any topology j on a topos \mathcal{E} , consider \mathcal{M} as the class of all j -dense monomorphisms in \mathcal{E} . Also, we denote by \mathcal{M}^\perp the class of all arrows $g : C \rightarrow D$ in \mathcal{E} such that for any $f : A \rightarrow E$ in \mathcal{M} and every commutative square

$$(5) \quad \begin{array}{ccc} A & \xrightarrow{u} & C \\ f \downarrow & \nearrow w & \downarrow g \\ E & \xrightarrow{v} & D \end{array}$$

there exists a unique dashed arrow $w : E \rightarrow C$ such that the resulting triangles are commutative. In this case, we say that g is *right orthogonal* to f . Moreover, we say that the pair $(\mathcal{M}, \mathcal{M}^\perp)$ forms a *factorization system* in \mathcal{E} if any arrow f in \mathcal{E} factors as $f = me$, where $m \in \mathcal{M}$ and $e \in \mathcal{M}^\perp$ (for more information, see [1]).

Lemma 2.4. *Let j be a topology on a topos \mathcal{E} . Then for each object B of \mathcal{E} , we have $\mathcal{M}_B^\perp \subseteq \mathcal{M}^\perp$, where \mathcal{M}_B is the class of all j_B -dense monomorphisms in the slice topos in \mathcal{E}/B .*

Proof. By Lemma 2.3 we get $\mathcal{M}_B \subseteq \mathcal{M}$. To reach the conclusion, let $h : f \rightarrow g$ be an arrow in \mathcal{M}_B^\perp , where $f : D \rightarrow B$ and $g : E \rightarrow B$ are arrows in \mathcal{E} . Now, consider the commutative square

$$(6) \quad \begin{array}{ccc} A & \xrightarrow{u} & D \\ m \downarrow & & \downarrow h \\ C & \xrightarrow{v} & E \end{array}$$

where $m : A \rightarrow C$ is in \mathcal{M} . Since by Lemma 2.3 the arrow $m : fu \rightarrow gv$ in \mathcal{E}/B belongs to \mathcal{M}_B and $h \in \mathcal{M}_B^\perp$, there exists a unique arrow $w : gv \rightarrow f$ in \mathcal{E}/B such that the following diagram commutes

$$(7) \quad \begin{array}{ccc} fu & \xrightarrow{u} & f \\ m \downarrow & \nearrow w & \downarrow h \\ gv & \xrightarrow{v} & g \end{array}$$

The arrow $w : C \rightarrow D$ (as an arrow in \mathcal{E}) which commutes the resulting triangles, is unique in the diagram (6). To prove this, let $k : C \rightarrow D$ be an arrow in \mathcal{E} such that $km = u$ and $hk = v$. Now, we have $fk = (gh)k = gv$, so $k : gv \rightarrow f$ is an arrow in \mathcal{E}/B making all triangles in (7) commutative. Thus, $k = w$ and we are done. \square

Definition 2.5. Let j be a topology on a topos \mathcal{E} . We say that \mathcal{E} has enough j -sheaves if for every object A of \mathcal{E} there is a j -dense monomorphism $A \rightarrow F$ where F is a j -sheaf.

Following [1] a class \mathcal{M} of morphisms in \mathcal{E} is a *left cancellable class* if $gf \in \mathcal{M}$ implies $f \in \mathcal{M}$. It is well known that for any object B of \mathcal{E} , the class \mathcal{M}_B of all j_B -dense monomorphisms in \mathcal{E}/B is left cancellable. In the following we will find some equivalent conditions for $(\mathcal{M}, \mathcal{M}^\perp)$ to be a factorization system in \mathcal{E} .

Theorem 2.6. *Let j be a topology on a topos \mathcal{E} . Then the following are equivalent:*

- (i) for any object B of \mathcal{E} , $(\mathcal{M}_B, \mathcal{M}_B^\perp)$ is a factorization system in \mathcal{E}/B ;
- (ii) for any object B of \mathcal{E} , \mathcal{E}/B has enough j_B -sheaves;
- (iii) for any object B of \mathcal{E} , any object of \mathcal{E}/B is j_B -separated;
- (iv) for any object B of \mathcal{E} , any object of \mathcal{E}/B is j_B -sheaf;
- (v) any object of \mathcal{E} is j -sheaf;
- (vi) any object of \mathcal{E} is j -separated;
- (vii) \mathcal{E} has enough j -sheaves;
- (viii) $(\mathcal{M}, \mathcal{M}^\perp)$ is a factorization system in \mathcal{E} .

Proof. That any j -sheaf is j -separated in \mathcal{E} yields that (v) \implies (vi) holds.

(vi) \implies (v). That any object of \mathcal{E} is j -separated it follows that $\mathbf{Sep}_j(\mathcal{E})$ is the topos \mathcal{E} and then, every j -separated object is a j -sheaf as in [8, Theorem 2.1].

(iii) \implies (vi). Setting $B = 1$, then any object of \mathcal{E} is j -separated.

(vi) \implies (iii). The claim follows immediately from the fact that for any object B of \mathcal{E} ,

$$\mathbf{Sep}_{j_B}(\mathcal{E}/B) \cong \mathbf{Sep}_j(\mathcal{E})/B.$$

(see also [9]).

(viii) \implies (vii). By (viii), for any object A of \mathcal{E} , the unique arrow $!_A : A \rightarrow 1$ factors as

$$\begin{array}{ccc} A & \xrightarrow{!_A} & 1 \\ & \searrow m & \nearrow !_C \\ & & C \end{array}$$

where $!_C \in \mathcal{M}_1^\perp = \mathcal{M}^\perp$ and $m \in \mathcal{M}_1 = \mathcal{M}$. It is easy to check that for any object B of \mathcal{E} , j_B -sheaves in \mathcal{E}/B are exactly the class of all objects of \mathcal{E}/B which belong to \mathcal{M}_B^\perp . Since $!_C$ is an object in $\mathcal{E}/1 = \mathcal{E}$ which is in \mathcal{M}_1^\perp , so $!_C$ is a j_1 -sheaf, or equivalently, C is a j -sheaf.

(vii) \implies (viii). Consider an arrow $f : A \rightarrow B$ in \mathcal{E} , using (vii), there exists a j -dense monomorphism $\iota : A \rightarrow F$, where F is a j -sheaf in \mathcal{E} . Now, we factor f as the composite arrow $A \xrightarrow{(\iota, f)} F \times B \xrightarrow{\pi_B^F} B$. Since $\pi_F^B(\iota, f) = \iota \in \mathcal{M}$ and \mathcal{M} is a left cancellable class, so $(\iota, f) \in \mathcal{M}$. Also, F being j -sheaf, by Lemma 2.3 we have π_B^F is a j_B -sheaf in \mathcal{E}/B . By Lemma 2.4 we have $\pi_B^F \in \mathcal{M}_B^\perp \subseteq \mathcal{M}^\perp$, as required.

(vi) \implies (vii). First of all we know that any j -separated object of \mathcal{E} can be embedded into a j -sheaf (see, e.g. [12, Proposition V.3.4]). Let A be an object of \mathcal{E} . Then, by assumption A is j -separated, and there exists an embedding $A \xrightarrow{\iota} F$, where F is a j -sheaf. Now, considering the closure of A in F , since \overline{A} is closed in F , by [12, Lemma V.2.4], it is a j -sheaf. Also since A is j -dense in \overline{A} we get the result.

(vii) \implies (vi). By assumption for any object A of \mathcal{E} , there is a j -dense monomorphism $A \rightarrow F$ in \mathcal{E} , where F is a j -sheaf. Since any subobject of a j -sheaf is j -separated, A is j -separated.

For any object B of \mathcal{E} , setting \mathcal{E}/B instead of \mathcal{E} in (v), (vi), (vii) and (viii), we drive the equivalences (i) \iff (ii) \iff (iii) \iff (iv). \square

In the following, we will introduce two main classes of dense monomorphisms in a topos \mathcal{E} .

Remark 2.7. By diagram (4), one can easily obtain:

(i) Let $j = \text{id}_\Omega$ be the trivial topology on \mathcal{E} . Then j -dense monomorphisms are only the identity maps. Therefore, any object of \mathcal{E} is a j -sheaf. Also, j -closed monomorphisms are exactly all monomorphisms.

(ii) Let j be the topology $\text{true} \circ !_\Omega$ on \mathcal{E} , that is, the characteristic map of id_Ω . Then, j -dense monomorphisms are exactly all monomorphisms. Furthermore, j -closed monomorphisms are just the identity maps.

Recall [1] that $(\text{Mono}, \text{Mono}^\square)$ is a weak factorization system in any topos \mathcal{E} , where Mono is the class of all monomorphisms in \mathcal{E} . By Remark 2.7(ii), the class Mono is the class of all j -dense monomorphisms with respect to the topology $j = \text{true} \circ !_\Omega$ on \mathcal{E} . Since the class Mono is left cancellable, so we can obtain a special case of Theorem 2.6 as follows. (Notice that by Lemma 2.3 for

the topology $j = \text{true} \circ !_{\Omega}$ and any object B of \mathcal{E} , the class Mono_B will be all monomorphisms in \mathcal{E}/B .)

Corollary 2.8. *For the topology $j = \text{true} \circ !_{\Omega}$ on a topos \mathcal{E} , the following are equivalent:*

- (i) *for any object B of \mathcal{E} , $(\text{Mono}_B, \text{Mono}_B^{\perp})$ is a factorization system in \mathcal{E}/B ;*
- (ii) *for any object B of \mathcal{E} , \mathcal{E}/B has enough j_B -sheaves;*
- (iii) *for any object B of \mathcal{E} , any object of \mathcal{E}/B is j_B -sheaf;*
- (iv) *for any object B of \mathcal{E} , any object of \mathcal{E}/B is j_B -separated;*
- (v) *any object of \mathcal{E} is j -sheaf;*
- (vi) *any object of \mathcal{E} is j -separated;*
- (vii) *\mathcal{E} has enough j -sheaves;*
- (viii) *$(\text{Mono}, \text{Mono}^{\perp})$ is a factorization system in \mathcal{E} .*

3. Sheaves and sections of an arrow

In this section, among other things, we investigate a relationship between sheaves and sections of an arrow in a topos \mathcal{E} . We start to recall [12] that for any object B of \mathcal{E} , the pullback functor $B^* : \mathcal{E} \rightarrow \mathcal{E}/B$ has a right adjoint which we denoted it by $S : \mathcal{E}/B \rightarrow \mathcal{E}$ (according to the terminology of [3]) as for any $f : X \rightarrow B$ we have the following pullback

$$(8) \quad \begin{array}{ccc} S(f) & \longrightarrow & 1 \\ \downarrow & & \downarrow i_B \\ X^B & \xrightarrow{f^B} & B^B \end{array}$$

where i_B is the transpose of $\text{id}_B : 1 \times B \cong B \rightarrow B$ and f^B is the transpose of the composition arrow $X^B \times B \xrightarrow{\text{ev}_X} X \xrightarrow{f} B$ by the exponential adjunction $(-) \times B \dashv (-)^B$; that is, $\text{ev}_B(i_B \times \text{id}_B) = \text{id}_B$ and $\text{ev}_B(f^B \times \text{id}_B) = \text{ev}_X$, where the natural transformation $\text{ev} : (-)^B \times B \rightarrow (-)$ is the counit of the exponential adjunction. In fact, in the Mitchell-Bénabou language, we can write

$$S(f) = \{h \mid (\forall c \in B) f \circ (h(c)) = c\}.$$

This means that we can call $S(f)$ *the object of sections of f* .

Since any retract of an object in a topos (or in an arbitrary category) is an equalizer, so the topos $\mathbf{Sh}_j(\mathcal{E})$ is closed under retracts. By Lemma 2.3, the pullback functor B^* preserves dense monomorphisms. Furthermore, as $B^* \dashv S$, the functor S preserves sheaves (for details, see [9, Corollary 4.3.12]). (Roughly, for any object $B \in \mathcal{E}$ and any adjoint $F \dashv G : \mathcal{E} \rightarrow \mathcal{E}/B$ one can easily checked that the functor G preserves sheaves whenever F preserves dense monomorphisms.)

In the following theorem we will find a relationship between sheaves in \mathcal{E}/B and the object of sections of an arrow with codomain B .

Theorem 3.1. *Let j be a topology on a topos \mathcal{E} and $f : X \rightarrow B$ be an object of \mathcal{E}/B . Then, f is a j_B -sheaf in \mathcal{E}/B , whenever the graph of f which stands for the monomorphism $(\text{id}_X, f) : f \rightarrow \pi_B^X$ in \mathcal{E}/B , is a section as well as $S(f)$ is a j -sheaf in \mathcal{E} .*

Proof. We recall that in [3] it was proved if (id_X, f) is a section in \mathcal{E}/B , then f is a retract of $\pi_B^{S(f)}$ in \mathcal{E}/B . As $S(f)$ is a j -sheaf, by Lemma 2.3, $\pi_B^{S(f)}$ is a j_B -sheaf in \mathcal{E}/B . But $\mathbf{Sh}_{j_B}(\mathcal{E}/B)$ being closed under retracts, therefore f is a j_B -sheaf in \mathcal{E}/B . \square

Regarding to the converse of Theorem 3.1, that the section functor S preserves sheaves it yields that if $f : X \rightarrow B$ be a j_B -sheaf in \mathcal{E}/B , then $S(f)$ is a j -sheaf in \mathcal{E} . Also, by Remark 2.7(ii), for $j = \text{true} \circ !_\Omega$, the monomorphism $(\text{id}_X, f) : f \rightarrow \pi_B^X$ is j_B -dense in \mathcal{E}/B and then for a j_B -sheaf $f : X \rightarrow B$, it will be a section in \mathcal{E}/B .

In the rest of this section, for a small category \mathcal{C} we restrict our attention to obtain a version of Theorem 3.1 for injective presheaves in trivial slices of the presheaf topos $\widehat{\mathcal{C}} = \mathbf{Sets}^{\mathcal{C}^{op}}$ which is close to the version of this theorem on j -sheaves for the topology $j = \text{true} \circ !_\Omega$ on $\widehat{\mathcal{C}}$. (See Proposition 3.5 below.) Note that the topology $j = \text{true} \circ !_\Omega$ on $\widehat{\mathcal{C}}$ is associated to the *chaotic or indiscrete Grothendieck topology* on \mathcal{C} .

Recall [12] that in the presheaf topos $\widehat{\mathcal{C}} = \mathbf{Sets}^{\mathcal{C}^{op}}$, the exponential object G^F is defined in each stage C of \mathcal{C} as $G^F(C) = \text{Hom}_{\widehat{\mathcal{C}}}(Y(C) \times F, G)$, where Y is the Yoneda embedding, that is

$$Y : \mathcal{C} \rightarrow \widehat{\mathcal{C}}; \quad Y(C) = \text{Hom}_{\mathcal{C}}(-, C).$$

Now, for an arrow $\alpha : G \rightarrow F$ in $\widehat{\mathcal{C}}$ consider the arrows $i_F : 1 \rightarrow F^F$ and $\alpha^F : G^F \rightarrow F^F$ in $\widehat{\mathcal{C}}$ as the transposes of $\text{id}_F : 1 \times F \cong F \rightarrow F$ and $\alpha \circ \text{ev}_G : G^F \times F \rightarrow F$, respectively, by the exponential adjunction. We can observe

$$(9) \quad \forall C \in \mathcal{C}, \quad (i_F)_C : 1(C) = \{*\} \longrightarrow F^F(C); \quad (i_F)_C(*) = \pi_F^{Y(C)}.$$

Also, for any two objects C, D of \mathcal{C} , any γ in $G^F(C)$ and any (k, y) in $Y(C)(D) \times F(D)$ we have

$$(10) \quad (\alpha_C^F(\gamma))_D(k, y) = \alpha_D(\gamma_D(k, y)).$$

Recall that a presheaf G has a (unique) global section whenever in each stage C of \mathcal{C} there is a (unique) element $\theta_C \in G(C)$ in such a way that for any arrow $k : D \rightarrow C$ in \mathcal{C} we have

$$(11) \quad G(k)(\theta_C) = \theta_D.$$

Here, we give a special case that the exponential object and the object of sections in $\widehat{\mathcal{C}}$ are exactly similar to \mathbf{Sets} . First, we state the following which is required.

Lemma 3.2. *Let j be the topology $\text{true} \circ !_{\Omega}$ on $\widehat{\mathcal{C}}$. Then the following assertions hold:*

(i) *For any j -sheaf G in $\widehat{\mathcal{C}}$, G has a unique global section. More generally, any injective presheaf G of $\widehat{\mathcal{C}}$ has a global section.*

(ii) *For any family $\{G_{\lambda}\}_{\lambda \in \Lambda}$ in $\widehat{\mathcal{C}}$, the presheaf $G = \prod_{\lambda \in \Lambda} G_{\lambda}$ is a j -sheaf (injective) in $\widehat{\mathcal{C}}$ iff for all $\lambda \in \Lambda$, G_{λ} is a j -sheaf (injective) in $\widehat{\mathcal{C}}$.*

Proof. (i) Let G be a j -sheaf in $\widehat{\mathcal{C}}$ and consider the coproduct object $G \sqcup 1$ in $\widehat{\mathcal{C}}$. By Remark 2.7(ii), there exists a unique natural transformation $\eta : G \sqcup 1 \rightarrow G$ in $\widehat{\mathcal{C}}$ such that the following diagram commutes (if G being injective, the arrow η is not necessarily unique)

$$\begin{array}{ccc} G & \xrightarrow{\text{id}_G} & G \\ \downarrow \iota & \nearrow \eta & \\ G \sqcup 1 & & \end{array}$$

in which $\iota : G \rightarrow G \sqcup 1$ is the injection arrow. Now, we will denote $\eta_C(*)$ by an element θ_C in $G(C)$ in each stage C of \mathcal{C} . Since $\eta : G \sqcup 1 \rightarrow G$ is natural, so for any arrow $k : D \rightarrow C$ in \mathcal{C} the following square commutes

$$\begin{array}{ccc} (G \sqcup 1)(D) & \xrightarrow{\eta_D} & G(D) \\ (G \sqcup 1)(k) \uparrow & & \uparrow G(k) \\ (G \sqcup 1)(C) & \xrightarrow{\eta_C} & G(C) \end{array}$$

Then, we have

$$\begin{aligned} G(k)(\theta_C) &= G(k)(\eta_C(*)) \\ &= \eta_D((G \sqcup 1)(k)(*)) \\ &= \eta_D(1(k)(*)) = \theta_D. \end{aligned}$$

This is the required result.

(ii) *Necessity.* Let G be a j -sheaf (injective) in $\widehat{\mathcal{C}}$. For any $\lambda, \mu \in \Lambda$, we define $\alpha^{\lambda\mu} : G_{\lambda} \rightarrow G_{\mu}$ such that in each stage C of \mathcal{C} and for each $x \in G_{\lambda}(C)$, we have $\alpha_C^{\lambda\mu}(x) = \theta_C^{\mu}$, where θ_C^{μ} is the μ -th component of θ_C corresponding to G in (i). Now, we will show that for any $\lambda, \mu \in \Lambda$, $\alpha^{\lambda\mu}$ is a natural transformation in $\widehat{\mathcal{C}}$, that is, for any arrow $k : D \rightarrow C$ in \mathcal{C} the following diagram is commutative

$$\begin{array}{ccc} G_{\lambda}(D) & \xrightarrow{\alpha_D^{\lambda\mu}} & G_{\mu}(D) \\ G_{\lambda}(k) \uparrow & & \uparrow G_{\mu}(k) \\ G_{\lambda}(C) & \xrightarrow{\alpha_C^{\lambda\mu}} & G_{\mu}(C) \end{array}$$

For, consider an element $x \in G_{\lambda}(C)$ we get

$$\begin{aligned} G_{\mu}(k)(\alpha_C^{\lambda\mu}(x)) &= G_{\mu}(k)(\theta_C^{\mu}) \\ &= \theta_D^{\mu} && \text{(by (11))} \\ &= \alpha_D^{\lambda\mu}(G_{\lambda}(k)(x)). \end{aligned}$$

Now, for any $\lambda \in \Lambda$, consider the family $\{\gamma_\mu : G_\lambda \rightarrow G_\mu\}_{\mu \in \Lambda}$ in $\widehat{\mathcal{C}}$ such that $\gamma_\mu = \alpha^{\lambda\mu}$ and $\gamma_\lambda = \text{id}_{G_\lambda}$, for each $\lambda \neq \mu \in \Lambda$. Since G is the product $\prod_{\lambda \in \Lambda} G_\lambda$, so there is a unique natural transformation $\gamma : G_\lambda \rightarrow G$ such that $p_\mu \gamma = \gamma_\mu$ and $p_\lambda \gamma = \text{id}_{G_\lambda}$, for all $\lambda, \mu \in \Lambda$ and the projections p_λ . Thus, for any $\lambda \in \Lambda$, G_λ is a retract of the j -sheaf (injective) G and then, G_λ is a j -sheaf (injective).

Sufficiency. By the universal property of the product presheaf G , the unique arrow in the definition of a sheaf easily follows. \square

We recall [12] that in each stage C of \mathcal{C} the object $\Omega(C)$ of $\widehat{\mathcal{C}}$ is the set of all sieves on C . Also, the arrow $\text{true}_C : 1(C) = \{*\} \rightarrow \Omega(C)$ assigns to $*$, the *maximal sieve* $t(C)$ of $\Omega(C)$, that is the set of all arrows with codomain C of \mathcal{C} .

Remark 3.3. Note that the topology $j = \text{true} \circ !_\Omega$ on $\widehat{\mathcal{C}}$ is the unique topology on $\widehat{\mathcal{C}}$ which satisfies Lemma 3.2. To show this, let j be an arbitrary topology on $\widehat{\mathcal{C}}$. For a j -sheaf G of $\widehat{\mathcal{C}}$, consider the injection $\iota : G \rightarrow G \sqcup 1$ in $\widehat{\mathcal{C}}$. In each stage C of \mathcal{C} we have $\text{char}(\iota)_C(*) = \emptyset$. Now, if ι is j -dense monomorphism, then in each stage C of \mathcal{C} we have $j_C(\emptyset) = t(C)$. For any sieve $S \in \Omega(C)$ by Definition 1.1 we get

$$\begin{aligned} t(C) &= j_C(\emptyset) = j_C(\emptyset \cap S) \\ &= j_C(\emptyset) \cap j_C(S) = t(C) \cap j_C(S) = j_C(S). \end{aligned}$$

Thus, j_C is the constant function on $t(C)$, as required.

Let F be the constant presheaf on a set A . One can easily checked that the exponential adjunction $(-) \times F \dashv (-)^F$ is determined by, for any presheaf G in $\widehat{\mathcal{C}}$, the exponential presheaf G^F assigns to any object C of \mathcal{C} , the hom-set $\text{Hom}_{\mathbf{Sets}}(A, G(C))$ and to any arrow $f : C \rightarrow D$ of \mathcal{C} , the function

$$G^F(f) : \text{Hom}_{\mathbf{Sets}}(A, G(D)) \longrightarrow \text{Hom}_{\mathbf{Sets}}(A, G(C))$$

given by $G^F(f)(g) = G(f) \circ g$. As any function $f : A \rightarrow G(C)$ can be considered as a sequence $(x_a)_{a \in A} \in \prod_A G(C)$, it yields that one has

$$(12) \quad G^F(C) \cong \prod_A G(C) \quad \text{for all objects } C \text{ in } \mathcal{C}.$$

By (12), (9) and (10), it is convenient to see that for each arrow $\alpha : G \rightarrow F$ in $\widehat{\mathcal{C}}$, in which F stands for the constant presheaf on a set A , we get

$$(13) \quad \forall C \in \mathcal{C}, \quad S(\alpha)(C) \cong \prod_{a \in A} \alpha_C^{-1}(a).$$

Now, we will extract a special case of Theorem 3.1 in $\widehat{\mathcal{C}}$. First, let $\alpha : G \rightarrow F$ be an arrow in $\widehat{\mathcal{C}}$ in which F is the constant presheaf on a set A . For each element a of A , consider the subpresheaf H_a of G such that $H_a(C) = \alpha_C^{-1}(a)$, for any object C of \mathcal{C} . Since limits in $\widehat{\mathcal{C}}$ are constructed pointwise, so (13) shows that $S(\alpha) \cong \prod_{a \in A} H_a$.

Proposition 3.4. *Let j be the topology $\text{true} \circ !_{\Omega}$ on $\widehat{\mathcal{C}}$ and $\alpha : G \rightarrow F$ an arrow in $\widehat{\mathcal{C}}$, where F is the constant presheaf on a set A . Then, α is a j_F -sheaf in $\widehat{\mathcal{C}}/F$ iff the monomorphism $(\text{id}_G, \alpha) : \alpha \mapsto \pi_F^G$ is a section in $\widehat{\mathcal{C}}/F$ as well as for any $a \in A$, the subpresheaf H_a of G is a j -sheaf in $\widehat{\mathcal{C}}$.*

Proof. We deduce the result by Theorem 3.1, Lemma 3.2(ii) and (13). \square

Since in topoi regular monomorphisms are exactly monomorphisms, so by [3, Theorem 1.2], Lemma 3.2(ii) and (13), we get the following.

Proposition 3.5. *Let $\alpha : G \rightarrow F$ be an arrow in $\widehat{\mathcal{C}}$, where F is the constant presheaf on a set A . Then, α is injective in $\widehat{\mathcal{C}}/F$ iff the monomorphism $(\text{id}_G, \alpha) : \alpha \mapsto \pi_F^G$ is a section in $\widehat{\mathcal{C}}/F$ as well as for any $a \in A$, the subpresheaf H_a of G is injective.*

When \mathcal{C} is a monoid, we obtain the following example.

Example 3.6. Let M be a monoid and $M\text{-Sets}$ the topos of all (right) representations of a fixed monoid M . Since M is a small category with just one object, for two M -sets X, B we have $X^B = \text{Hom}_{M\text{-Sets}}(M \times B, X)$, where $M \times B$ has the componentwise action. Hence, by (9) and (10), for any equivariant map $f : X \rightarrow B$, in the diagram (8) we observe

$$(14) \quad i_B(*) = \pi_B^M : M \times B \rightarrow B,$$

and

$$(15) \quad \forall h \in X^B, \forall (m, b) \in M \times B, (f^B(h))(m, b) = fh(m, b).$$

Note that one writes any equivariant map $h : M \times B \rightarrow X$ in X^B as a sequence $((x_{m,b})_{b \in B})_{m \in M}$, consisting of elements $x_{m,b} = h(m, b)$ of X , for any $(m, b) \in M \times B$. Also, h being equivariant map means that

$$\forall n, m \in M, \forall b \in B, \quad x_{mn, bn} = x_{m, b}n.$$

Hence, we obtain that X^B is equal to

$$(16) \quad \{((x_{m,b})_b)_m \in \prod_{m \in M} \prod_{b \in B} X \mid \forall n, m \in M, \forall b \in B, x_{mn, bn} = x_{m, b}n\}.$$

Now, by (8), (14) and (15) we have

$$\begin{aligned} S(f) &= \{((x_{m,b})_b)_m \in X^B \mid f^B(((x_{m,b})_b)_m) = \pi_B^M = ((b)_b)_m\} \\ &= \{((x_{m,b})_b)_m \in X^B \mid ((f(x_{m,b}))_b)_m = ((b)_b)_m\} \\ &= \{((x_{m,b})_b)_m \in X^B \mid \forall m \in M, \forall b \in B, x_{m,b} \in f^{-1}(b)\}, \end{aligned}$$

Hence, by (16) we interpret a simple form of underlying set of the M -set $S(f)$ in the topos $M\text{-Sets}$ as follows

$$\{((x_{m,b})_b)_m \in \prod_{m \in M} \prod_{b \in B} f^{-1}(b) \mid \forall n, m \in M, \forall b \in B, x_{mn, bn} = x_{m, b}n\}.$$

If B has the trivial action “.”, that is, $\cdot = \pi_1 : B \times M \rightarrow B$ the first projection, then by (12) and (13) we can obtain $X^B \cong \prod_B X$ and $S(f) \cong \prod_{b \in B} f^{-1}(b)$.

Furthermore, recall [12] that for a group G and two G -sets X, B , we have

$$(17) \quad X^B = \{h : B \rightarrow X \mid h \text{ is a function}\} \cong \prod_B X$$

as two sets. According to the action on X^B , under the isomorphism (17), the action on $\prod_B X$ is given by $(x_b)_{b \in B} \cdot g = (x_{bg^{-1}} \cdot g)_{b \in B}$, for any $g \in G$ and $(x_b)_{b \in B} \in \prod_B X$. Also, by (17) for any equivariant map $f : X \rightarrow B$ in $G\text{-Sets}$, in a similar way to (13), we have $S(f) \cong \prod_{b \in B} f^{-1}(b)$.

4. j -essential extensions in a topos

This section is devoted to introduce a class of monomorphisms in an elementary topos, which we call its members ‘ j -essential monomorphisms’. We present some equivalent forms of these monomorphisms and state some of their properties. Meanwhile, we prove that any presheaf in a presheaf topos has a maximal essential extension.

Recall that a monomorphism $\iota : A \rightarrow B$ is called *essential* whenever for each arrow $g : B \rightarrow C$ such that $g\iota$ is a monomorphism, then g is a monomorphism also. Now, we define a j -essential monomorphism in a topos \mathcal{E} as follows.

Definition 4.1. For a topology j on \mathcal{E} , a monomorphism $\iota : A \rightarrow B$ is called *j -essential* whenever it is j -dense as well as essential. In this case, we say that B is a *j -essential extension* of A and we write $A \subseteq_j B$.

We call an arrow $f : A \rightarrow B$ in \mathcal{E} to be *j -dense* whenever the subobject $f(A)$, which is the image of f , is j -dense in B . In this way, any epimorphism in \mathcal{E} becomes j -dense. (For the definition of image of an arrow in a topos, see [12].)

In the following we give some equivalent definitions of j -essential monomorphisms in a topos \mathcal{E} .

Lemma 4.2. *Let j be a topology on \mathcal{E} and $\iota : A \rightarrow B$ a j -dense monomorphism. Then, the following are equivalent:*

- (i) *for any $g : B \rightarrow C$, g is a monomorphism whenever $g\iota$ is a monomorphism;*
- (ii) *for any $g : B \rightarrow C$, g is a j -dense monomorphism whenever $g\iota$ is a j -dense monomorphism;*
- (iii) *for any $g : B \rightarrow C$, g is a monomorphism whenever $g\iota$ is a j -dense monomorphism.*

Proof. (i) \implies (ii) and (iii) \implies (ii) are proved by [9, A.4.5.11(iii)].

(ii) \implies (iii) is clear.

(ii) \implies (i). Consider an arrow $g : B \rightarrow C$ for which $g\iota$ is a monomorphism.

We show that g is a monomorphism too. Assume that $B \xrightarrow{k} g(B) \xrightarrow{m} C$ is the

image factorization of the arrow g . Since $g\iota = m(k\iota)$ and $g\iota$ is monomorphism, it follows that the arrow $k\iota$ is a monomorphism. Meanwhile, we get

$$\begin{aligned} g(B) &= k(B) && \text{(as } k \text{ is epic)} \\ &= \overline{k(A)} && \text{(as } \iota \text{ is dense)} \\ &\subseteq \overline{k(A)} \\ &\subseteq g(B). \end{aligned}$$

Therefore, $g(B) = \overline{k(A)} = \overline{k\iota(A)}$. It follows that the compound monomorphism $k\iota : A \rightarrow g(B)$ is a dense monomorphism and so is k , by (ii). That k is monomorphism and so isomorphism, yields that g is a monomorphism. \square

We point out that the proof of (ii) \implies (i) of Lemma 4.2 shows that any composite $k\iota$, for an epic k and a dense monomorphism ι , is dense.

The following shows that j -essential monomorphisms in \mathcal{E} are closed under composition.

Proposition 4.3. *Let j be a topology on \mathcal{E} . Then for two subobjects $A \xrightarrow{\iota} A' \xrightarrow{\iota'}$ B in \mathcal{E} , we have $A \subseteq_j B$ iff $A \subseteq_j A'$ and $A' \subseteq_j B$.*

Proof. By [9, A.4.5.11(iii)], one has $\iota'\iota$ is j -dense iff ι' and ι are j -dense.

Necessity. First, by Lemma 4.2(i), we show that $A \subseteq_j A'$. To do so, consider an arrow $f' : A' \rightarrow C$ for which $f'\iota$ is a monomorphism. Now, by [12, Corollary IV. 10. 3], the object C can be embedded into an injective object D as in $C \xrightarrow{\nu} D$ and hence there is an arrow $\tilde{f}' : B \rightarrow D$ such that $\tilde{f}'\iota' = \nu f'$. Since $A \subseteq_j B$ and $\tilde{f}'\iota'\iota = \nu f'\iota$ is a monomorphism, we deduce that \tilde{f}' is a monomorphism. As $\tilde{f}'\iota' = \nu f'$ it follows that f' is a monomorphism.

To prove $A' \subseteq_j B$, choose an arrow $f : B \rightarrow C$ for which $f\iota$ is a monomorphism. Then, $f\iota'$ is also a monomorphism. Now $A \subseteq_j B$ implies that f is a monomorphism, as required.

Sufficiency. Let $f : B \rightarrow C$ be an arrow in \mathcal{E} such that $f\iota'$ is a monomorphism. Since $A \subseteq_j A'$ and $(f\iota')\iota = f\iota'\iota$ is a monomorphism, it concludes that $f\iota$ is a monomorphism. Using $A' \subseteq_j B$, we achieve that f is a monomorphism and hence $A \subseteq_j B$. \square

In the following, we obtain another property of j -essential monomorphisms in \mathcal{E} .

Lemma 4.4. *Let j be a topology on \mathcal{E} . If $A \subseteq_j B$ and A is embedded in a j -sheaf F , then B is also embedded in F .*

Proof. Let $\iota : A \rightarrow B$ be a j -essential monomorphism and $m : A \rightarrow F$ an arbitrary embedding. Since F is a j -sheaf, there exists a unique morphism $f : B \rightarrow F$ making the diagram below commutative;

$$\begin{array}{ccc} A & \xrightarrow{m} & F \\ \iota \downarrow & \nearrow f & \\ B & & \end{array}$$

As $A \subseteq_j B$ being j -essential, f is a monomorphism, as required. \square

By Remark 2.7(ii), essential monomorphisms in a topos \mathcal{E} are exactly j -essential monomorphisms in \mathcal{E} with respect to the topology $j = \text{true} \circ !\Omega$ on \mathcal{E} .

Now, we would like to prove that any presheaf in $\widehat{\mathcal{C}}$ has a maximal essential extension.

Theorem 4.5. *Any presheaf in $\widehat{\mathcal{C}}$ has a maximal essential extension.*

Proof. Let F be a presheaf in $\widehat{\mathcal{C}}$ and G an injective presheaf into which F can be embedded. By Lemma 4.4, we can assume that both F and all its essential extensions are subpresheaves of G . Consider Σ as the set of all essential extensions of F which is a poset under subpresheaf inclusion \subseteq . Since the arrow id_F is an essential extension of F , it follows that Σ is non-empty. If

$$\dots \subseteq F_i \subseteq \dots,$$

$i \in I$, is a chain in Σ , then the subpresheaf H of G given by $H(C) = \bigcup_{i \in I} F_i(C)$ for any object C in \mathcal{C} is an upper bound of this chain. Now we show that H lies in Σ , i.e., H is an essential extension of F . To achieve this, let $\alpha : H \rightarrow K$ be an arrow in $\widehat{\mathcal{C}}$ such that the restriction arrow $\alpha|_F$ is a monomorphism. We prove that α is a monomorphism. To verify this claim, we show that for any $C \in \widehat{\mathcal{C}}$, the function $\alpha_C : \bigcup_{i \in I} F_i(C) \rightarrow K(C)$ is one to one. Take $a, b \in \bigcup_{i \in I} F_i(C)$, $a \neq b$. Then there is a $j \in I$ such that $a, b \in F_j(C)$. Denote $\alpha|_{F_j}$ by α_j . Since F_j is an essential extension of F and $\alpha_j|_F = \alpha|_F$, it implies that α_j is a monomorphism. Now

$$\alpha_C(a) = (\alpha_j)_C(a) \neq (\alpha_j)_C(b) = \alpha_C(b).$$

Therefore, α is a monomorphism. Thus, $H \in \Sigma$. Now it follows from Zorn's Lemma that there is a maximal element M in Σ . Then, M is a maximal essential extension of F . \square

It is straightforward to see that any essential extension of B can be embedded in any injective extension of B .

For a topology j on a topos \mathcal{E} , by a j -injective object we mean an injective object with respect to the class of all j -dense monomorphisms in \mathcal{E} .

The following shows that the j -injective presheaves (j -sheaves) in $\widehat{\mathcal{C}}$ have no proper j -essential extension.

Proposition 4.6. *Let j be a topology on $\widehat{\mathcal{C}}$ and F a j -injective presheaf (j -sheaf) in $\widehat{\mathcal{C}}$. Then, F has no proper j -essential extension.*

Proof. Suppose that G is a proper j -essential extension of F , so F is a j -dense subpresheaf of G and $F \neq G$. Thus there is an object C of \mathcal{C} such that $G(C) \not\subseteq F(C)$ and then, an $a \in G(C)$ such that $a \notin F(C)$. Since F is j -injective (j -sheaf) implies that there is an arrow $\alpha : G \rightarrow F$ for which $\alpha|_F = \text{id}_F$. That $a \notin F(C)$ and $\alpha_C(a) \in F(C)$ follows that $a \neq \alpha_C(a)$. But $\alpha_C(\alpha_C(a)) = \alpha_C(a)$. Then, α_C and so α is not a monomorphism although $\alpha|_F = \text{id}_F$ is. This shows that G is not a proper j -essential extension of F and it is a contradiction. \square

The following shows that the pullback functor B^* reflects j -essential extensions.

Proposition 4.7. *Let j be a topology in a topos \mathcal{E} . For every object $B \in \mathcal{E}$, the pullback functor $B^* : \mathcal{E} \rightarrow \mathcal{E}/B$ reflects j -essential monomorphisms.*

Proof. Let $f : A \rightarrow C$ be an arrow in \mathcal{E} such that $B^*(f)$ is a j_B -essential monomorphism. We show that f is a j -essential monomorphism. By Lemma 2.3, f is a j -dense monomorphism in \mathcal{E} . Let $g : C \rightarrow D$ be an arrow in \mathcal{E} such that gf is a monomorphism. We show that g is too. Since gf is a monomorphism, the arrow $(g \times \text{id}_B)B^*(f) = (gf) \times \text{id}_B$ is a monomorphism also. As $B^*(f)$ is j_B -essential, $g \times \text{id}_B$ is a monomorphism. Then g is a monomorphism. This is the required result. \square

Recall [10] that a *weak topology* on a topos \mathcal{E} is a morphism $j : \Omega \rightarrow \Omega$ such that:

(i) $j \circ \text{true} = \text{true}$;

(ii) $j \circ \wedge \leq \wedge \circ (j \times j)$, in which \leq stands for the internal order on Ω .

Meanwhile, a weak topology j on \mathcal{E} is said to be *productive* if $j \circ \wedge = \wedge \circ (j \times j)$.

In what follows, we review the whole paper for a weak topology j on a topos \mathcal{E} instead of a topology.

Remark 4.8. Similar to [9, A.4.5.11(ii)], one can easily check that for a weak topology j on \mathcal{E} pushouts also preserve dense monomorphisms. Hence, we can obtain a version of Lemma 2.1 for a weak topology j on \mathcal{E} as well. One can observe that completely analogous assertions to Lemmas 4.2, 4.4 and 2.3, Proposition 4.6 and Theorem 3.1, hold for a weak topology j on \mathcal{E} . But, note that by [10], in the proof of Theorem 2.6, the part (vi) \implies (vii) is true for a productive weak topology j on \mathcal{E} . The rest parts of this proof is correct for weak topologies.

Recall [10] that, for a weak topology j on \mathcal{E} , it is convenient to see that if the composite subobject mn is dense then so does for m and n . In contrast with topologies [9, A.4.5.11(iii)], the converse is not necessarily true. Hence, the sufficiency part of Proposition 4.3 does not necessarily hold for a weak topology j on a topos \mathcal{E} . The necessity part of this proposition is true for a weak topology j as well.

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