A WEAKER QUANTITATIVE CHARACTERIZATION OF THE SPORADIC SIMPLE GROUPS

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Abstract. It is proved in this paper that all the sporadic simple groups can be characterized by their orders and one special conjugacy class sizes, such as largest conjugacy class sizes, and smallest conjugacy class sizes greater than 1.
Keywords: Sporadic groups, conjugacy class sizes, characterization.

1. Introduction

All groups considered in this paper are finite.

In recent years, it is an interesting topic to characterize finite simple groups by their quantitative properties such as element orders and conjugacy class sizes. For example, in a private communication to W.J. Shi, J.G. Thompson proposed the following conjecture (see [3]).

Thompson's conjecture. *Let G be a group with $Z(G) = 1$ and N is a non-abelian simple group satisfying that $cs(G) = cs(N)$. Then $G \simeq N$.*

In the above conjecture, $cs(G) = \{|G : C_G(g)| : g \in G\}$.

In 1994, G. Y. Chen proved in his Ph. D. dissertation [1] that if G is a group with $Z(G) = 1$, and N a non-abelian simple group with non-connected prime graph such that $cs(G) = cs(N)$, then $G \simeq N$ (also ref. to [2, 3, 4]).

What we can see in Thompson's conjecture is that all the conjugacy class sizes of the sporadic groups are involved. Thus, naturally, one can ask whether this condition can be weakened? For instance, if we just consider some special conjugacy class sizes of a group G, what information about G can we obtain? The purpose of this paper is devoted to this direction.

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In order to state our results, we need the some notation. Let G be a group. Let

$$lcs(G) = max\{|G : C_G(g)| : g \in G\}$$

and

$$scs(G) = min\{|G : C_G(g)| : g \in G \setminus Z(G)\}.$$

denote the largest conjuagcy class size of G and the smallest conjugacy class size of G greater than 1, respectively. Furthermore, set

$$sscs(G) = min\ cs(G)\backslash\{1, scs(G)\}.$$

Our results are as follows.

Theorem 1.1. *Let N be one of the following sporadic groups:*

$$M_{11},\ M_{12},\ M_{22},\ M_{23},\ M_{24},\ J_1,\ J_3,\ J_4,$$

$$Co_1,\ Co_3,\ McL,\ He,\ Ru,\ HN,\ Ly,\ Th,\ M.$$

Let G be a group such that $|G| = |N|$ and $scs(G) = scs(N)$. Then $G \simeq N$.

For other sporadic simple groups, we have that

Theorem 1.2. *Let N be one of J_2, HS, Suz, $O'N$, Co_2, Fi_{22}, Fi_{23}, Fi'_{24}. If G is a group with $|G| = |N|$ and $lcs(G) = lcs(M)$, then $G \simeq N$.*

Proposition 1.3. *Let G be a group such that $|G| = |B|$, $scs(G) = scs(B)$ and $sscs(G) = sscs(B)$. Then $G \simeq B$.*

By [1, 2, 3, 4], we know that if G is a group with $Z(G) = 1$ and N is a non-abelian simple group such that $cs(G) = cs(N)$ and the prime graph of N is non-connected, then $|G| = |N|$. Since, the prime graphs of all the sporadic simple groups are not connected, as a corollary, we have

Corollary 1.4. *Let G be a group with $Z(G)$ trivial and N be an arbitrary sporadic group. If $cs(G) = cs(N)$, then $G \simeq N$.*

2. Preliminaries

In this section, we collect some elementary facts which are useful in our proof.

For a group G, define its prime graph $\Gamma(G)$ as follows: the vertices are the primes dividing the order of G, two vertices p and q are joined by an edge if and only if G contains an element of order pq (see [8]). Denote the connected components of the prime graph by $T(G) = \{\pi_i(G) | 1 \leqslant i \leqslant t(G)\}$, where $t(G)$ is the number of the prime graph components of G. If the order of G is even, assume that the prime 2 is always contained in $\pi_1(G)$.

For $x \in G$, x^G denotes the conjugacy class in G containing x and $C_G(x)$ denotes the centralizer of x in G. Then $|x^G| = |G :\ C_G(x)|$.

A simple group whose order has exactly n distinct primes is called a simple K_n-group. For example, it is well known that there are 8 simple K_3-groups. In addition, for a group G, we call G a 2-Frobenius group if G has a normal series $1 \subseteq H \subseteq K \subseteq G$ such that K and G/H are Frobenius groups with kernels H and K/H respectively.

For other notation and terminologies not mentioned in this paper, the reader is referred to ATLAS [5] if necessary.

Lemma 2.1. *Let G be a group with more than one prime graph component. Then G is one of the following:*

(i) *a Frobenius or 2-Frobenius group;*

(ii) *G has a normal series $1 \subseteq H \subseteq K \subseteq G$, where H is a nilpotent π_1-group, K/H is a non-abelian simple group and G/K is a π_1-group such that $|G/K|$ divides the order of the outer automorphism group of K/H. Besides, for $i \geq 2$, $\pi_i(G)$ is also a component of $\Gamma(K/H)$.*

Proof. It follows straight forward from Lemmas 1-3 in [8], Lemma 1.5 in [2] and Lemma 7 in [4]. □

Lemma 2.2. *Suppose that G is a Frobenius group of even order and H, K are the Frobenius kernel and the Frobenius complement of G, respectively. Then $t(G) = 2$, $T(G) = \{\pi(H), \pi(K)\}$ and G has one of the following structures:*

(i) *$2 \in \pi(H)$ and all Sylow subgroups of K are cyclic;*

(ii) *$2 \in \pi(K)$, H is an abelian group, K is a solvable group, the Sylow subgroups of K of odd order are cyclic groups and the Sylow 2-subgroups of K are cyclic or generalized quaternion groups;*

(iii) *$2 \in \pi(K)$, H is abelian, and there exists a subgroup K_0 of K such that $|K : K_0| \leq 2, K_0 = Z \times SL(2,5), (|Z|, 2 \times 3 \times 5) = 1$, and the Sylow subgroups of Z are cyclic.*

Proof. This is Lemma 1.6 in [3]. □

Lemma 2.3. *Let G be a 2-Frobenius group of even order. Then $t(G) = 2$ and G has a normal series $1 \subseteq H \subseteq K \subseteq G$ such that $\pi(K/H) = \pi_2$, $\pi(H) \cup \pi(G/K) = \pi_1$, the order of G/K divides the order of the automorphism group of K/H, and both G/K and K/H are cyclic. Especially, $|G/K| < |K/H|$ and G is soluble.*

Proof. This is Lemma 1.7 in [3]. □

The following lemma is well-known (see [7, Theorem 3.3.20]).

Lemma 2.4. *Let $R = R_1 \times \cdots \times R_k$, where R_i is a direct product of n_i isomorphic copies of a non-abelian simple group H_i and H_i and H_j are not isomorphic if $i \neq j$. Then*

$$Aut(R) \simeq Aut(R_1) \times \cdots \times Aut(R_k) \text{ and } Aut(R_i) \simeq (Aut(H_i)) \wr S_{n_i}.$$

Moreover,

$$Out(R) \simeq Out(R_1) \times \cdots \times Out(R_k) \text{ and } Out(R_i) \simeq (Out(R_i)) \wr S_{n_i}.$$

3. Proof of Theorem 1.1

It has been proved in [6] that the theorem holds for M_{12} and M_{23}. Therefore we need to treat the remaining cases.

From now on, we suppose that G is group satisfying the condition of Theorem 1.1.

Lemma 3.1. *Every minimal normal subgroup of $\overline{G} = G/Z(G)$ is non-soluble. Therefore, if M is the product of all minimal normal subgroups of $\overline{G}$, then $M = S_1 \times S_2 \times \cdots \times S_k$ and $M \leq \overline{G} \leq Aut(M)$, where every S_i is a non-abelian simple group.*

Proof. By the hypothesis and [5], it is clear that $Z(G)$ is a proper subgroup of G. Let S be any minimal normal subgroup of $\overline{G}$. Suppose that S is soluble. Then S is an elementary abelian group, from which we get the preimage T of S in G is a nilpotent group. If $|S| = r^t$, then the Sylow r-subgroup R of T is normal in G. Moreover, R can not be contained in $Z(G)$. Thus there exists an element y of R which is not contained in $Z(G)$ such that $1 < |y^G| \leqslant |R|$. However, by [5], we have that $scs(G) = scs(N) > |N_p|$, where N_p is any Sylow p-subgroup of M with $p \in \pi(M)$. This contradiction shows that every minimal normal subgroup of $\overline{G}$ is non-soluble, as desired. It follows that the second assertion holds. □

Lemma 3.2. *If $N \simeq M_{11}$, then $G \simeq M_{11}$.*

Proof. By the hypothesis, $|G| = |M| = 2^4 \cdot 3^2 \cdot 5 \cdot 11$ and $scs(G) = scs(M) = 3 \cdot 5 \cdot 11$. Then G has an element x such that $|C_G(x)| = 2^4 \cdot 3$ and $|x^G| = 3 \cdot 5 \cdot 11$. Thus, $5, 11 \notin \pi(Z(G))$. Without loss of generality, assume that $11 \in \pi(S_1)$. Then, by [5], we see that S_1 is isomorphic to M_{11} or $L_2(11)$. If $S_1 \simeq L_2(11)$, then $M = L_2(11)$ and $|Z(G)| \geq 6$. Let $y \in G$ with order 11. Then $y \notin Z(G)$ and $|C_G(y)| > |C_G(x)|$, a contradiction. Hence $S_1 \simeq M_{11}$ and so is G. □

Lemma 3.3. *If $N \simeq M_{22}$, then $G \simeq M_{22}$.*

Proof. In this case, $|G| = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$ and $scs(G) = 3 \cdot 5 \cdot 7 \cdot 11$. Therefore for some $x \in G$, we have $|C_G(x)| = 2^7 \cdot 3$ and $|x^G| = 3 \cdot 5 \cdot 7 \cdot 11$. This implies that $5, 7, 11 \notin \pi(Z(G))$. Assume that $11 \in \pi(S_1)$. Then by [5], we know that S_1 is isomorphic to one of the following groups:

$$L_2(11),\ M_{11},\ M_{22}.$$

If $S_1 \simeq L_2(11)$, then $M = S_1 \times S_2$, where S_2 is a simple K_3-group with $7 \in \pi(S_2)$. Then we can pick an element $y \in G$ of order 7 such that $1 < |y^G| < |x^G|$, a contradiction. If $S_1 \simeq M_{11}$, then one can conclude that $M = M_{11}$ and so $7 \in \pi(Z(G))$, a contradiction. Hence we have that S_1 must be isomorphic to M_{22}, as wanted. □

Lemma 3.4. *If $N \simeq M_{24}$, then $G \simeq M_{24}$.*

Proof. By the hypothesis, we have that $|G| = 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$ and $scs(G) = scs(M) = 3^2 \cdot 5 \cdot 11 \cdot 23$. Then there is an element $x \in G$ such that $|C_G(x)| = 2^{10} \cdot 3 \cdot 7$ and $|x^G| = scs(G)$. It is easy to see that $11, 23 \notin \pi(Z(G))$. As before, we assume that $23 \in \pi(S_1)$. Then by [5], we get that S_1 is isomorphic to M_{24}, M_{23} or $L_2(23)$. Similarly, we can rule out $L_2(23)$. If $S_1 \simeq M_{23}$, then $|Z(G)| = 2^3 \cdot 3$ and $G \simeq G' \times Z(G)$, where $G' \simeq M_{23}$. By [5], we know that there is an involution in G' such that $|C_G(y)| > |C_G(x)|$, a contradiction. Hence, S_1 must be isomorphic to M_{24} and so $G \simeq M_{24}$. □

Lemma 3.5. *If $N = J_1$, then $G \simeq J_1$.*

Proof. By the hypothesis, $|G| = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$ and $scs(G) = scs(J_1) = 7 \cdot 11 \cdot 19$. Clearly, $Z(G)$ is a proper subgroup of G and $7, 11, 19 \notin \pi(Z(G))$. By Lemma 2.4, we get that $G/Z(G)$ has a minimal normal subgroup S which is non-abelian simple and $19 \in \pi(S)$. By [5], S must be isomorphic to J_1 and so $G \simeq J_1$. □

Lemma 3.6. *If $N = J_3$, then $G \simeq J_3$.*

Proof. In this case, $|G| = 2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$ and $scs(G) = 3^4 \cdot 17 \cdot 19$. Then, for some $x \in G$, we have that $|x^G| = scs(G)$ and $|C_G(x)| = 2^7 \cdot 3 \cdot 5$. Therefore, $17, 19 \notin \pi(Z(G))$ and by Lemma 2.4, we have that $19 \in \pi(M)$. Without loss of generality, we suppose that $19 \in \pi(S_1)$. Then, $S_1 \simeq L_2(19)$ or J_3 by [5]. If $S_1 \simeq L_2(19)$, then M may be isomorphic to one of the following groups:

$$L_2(19) \times L_2(17),\ L_2(19) \times L_3(3).$$

In any case, we have that $\pi(Z(G)) \subseteq \{2, 3\}$. Let $y \in G$ with order 19. Then $|C_G(x)| < |C_G(y)| < |G|$ and so $1 < |y^G| < |x^G|$, a contradiction. Therefore, S_1 must be J_3, which implies that $G \simeq J_3$. □

Lemma 3.7. *If $N = J_4$, then $G \simeq J_4$.*

Proof. By the hypothesis, we have that $|G| = 2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$ and $scs(G) = 11^2 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$.

Let $x \in G$ such that $|x^G| = scs(G)$. Discussing as above, we see that M is a non-abelian simple group, isomorphic to $L_2(43)$ or J_4. If $M \simeq L_2(43)$, then $G/Z(G) \simeq L_2(43)$ or $L_2(43).2$. Let $y \in G$ with order 43. Then $y \notin Z(G)$ and $|C_G(y)| > |C_G(x)|$, a contradiction. Hence, $S_1 \simeq J_4$ and consequently $G \simeq J_4$. □

Lemma 3.8. *If $N \simeq McL$, then $G \simeq McL$.*

Proof. In this case, $|G| = 2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$ and $scs(G) = 3^4 \cdot 5^2 \cdot 11$. Suppose that $x \in G$ such that $|x^G| = scs(G)$. Then, clearly, $Z(G)$ is a proper subgroup of G and $11 \notin \pi(Z(G))$. Moreover, by Lemma 2.4, $11 \in \pi(M)$. Suppose that $11 \in \pi(S_1)$. Then, by [5], we get that S_1 may be isomorphic to one of the following groups:

$$L_2(11),\ M_{11},\ M_{12},\ M_{22},\ A_{11},\ McL.$$

First, assume $S_1 \simeq L_2(11)$. By Lemma 2.4, we have that 7 does not divide $|Out(M)|$. Let $y \in G$ with order 11. Then, y is not contained in $Z(G)$ and furthermore, we have that in fact $|C_G(y)| > |C_G(x)|$, a contradiction.

Suppose that $S_1 \simeq M_{11}$. Then, by Lemma 2.4, we see that $7 \notin \pi(Out(M))$. If 7 does not divide $|Z(G)|$, then for some y in G of order 7, $1 < |y^G| < |x^G|$, a contradiction. If $7 \in \pi(Z(G))$, then $M \simeq M_{11} \times A_5$ or $M_{11} \times A_6$.

If $M \simeq M_{11} \times A_5$, then, for some $z \in G$ of order 11, we get that $1 < |z^G| < |x^G|$, a contradiction. If $M \simeq M_{11} \times A_6$, then for some involution $g \in G$ such that $gZ(G) \in A_6$, we have that $1 < |g^G| < |x^G|$, also a contradiction. Hence, S_1 can not be isomorphic M_{11}.

Now, assume that $S_1 \simeq M_{12}$. Then it is easy to see that $M = M_{12}$. Therefore $G/Z(G) \simeq M_{12}$ or $M_{12}.2$. If the former case occurs, then, by [5], one can choose an element y in G such that $1 < |y^G| < |x^G|$, a contradiction. If the second case happens, then $|Z(G)| = 3^3 \cdot 5^2 \cdot 7$. By [5], one can pick $z \in G$ of order 2 such that $\overline{z}$ in $\overline{G}$ is a involution and $C_{\overline{G}}(\overline{z}) = 240$. Since $Z(G)$ is a $2'$-group, we have that

$$C_{\overline{G}}(\overline{z}) = C_G(z)Z(G)/Z(G) = C_G(z)/Z(G),$$

and so $|C_G(z)| > |C_G(x)|$, a contradiction.

If $S_1 \simeq M_{22}$, then $M \simeq M_{22}$. In fact, we have $G/Z(G) \simeq M_{22}$. Then $|Z(G)| = 3^4 \cdot 5^2$. Since there is some involution y in G such that $C_{\overline{G}}(\overline{y}) = 384$, we get that $|C_G(y)| > |C_G(x)|$, a contradiction.

If $S_1 \simeq A_{11}$, then $M = S_1 = A_{11}$ and therefore $G/Z(G) \simeq A_{11}$. It follows that $G'/Z(G') \simeq A_{11}$. Hence, $G' \simeq A_{11}$ or $G' \simeq 2.A_{11}$.

Since 2^8 does not divide $|G|$, we have that G' must be isomorphic to A_{11}, which shows that $G = G' \times Z(G)$. By [5], we can choose an element $y \in G'$ of order 3 such that $C_{G'}(y) = 60840$, which leads to that $|C_G(y)| > C_G(x)$, a contradiction.

Thus, S_1 must be isomorphic to McL and so $G \simeq McL$, as desired. □

Lemma 3.9. *If $N \simeq Co_3$, then $G \simeq Co_3$.*

Proof. By the hypothesis, $|G| = |Co_3| = 2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$ and $scs(G) = 3^3 \cdot 5^2 \cdot 11 \cdot 23$. Then, for some $x \in G$, $|x^G| = scs(G)$ and $|C_G(x)| = 2^{10} \cdot 3^4 \cdot 5 \cdot 7$. Obviously, $Z(G)$ is a proper subgroup of G and $11, 23 \notin \pi(Z(G))$. By Lemma 2.4, $11, 23 \in \pi(M)$. Without loss of generality, we assume that $23 \in \pi(S_1)$. Then S_1 may be isomorphic to one of the following groups:

$$Co_3,\ M_{24},\ M_{23},\ L_2(23).$$

In order to show $S_1 \simeq Co_3$, we should rule out other 3 cases.

(I) S_1 is not isomorphic to $L_2(23)$.

First, assume that $S_1 \simeq L_2(23)$. By the hypothesis and Lemma 2.4, we have that 5^2 divides the order of M. From now on, we distinguish the following 4 cases.

(1) Both 5 and 7 are in $\pi(Z(G))$.

If this case occurs, then M is isomorphic to one of

$$L_2(23) \times A_5 \times A_5,\ \ L_2(23) \times A_5 \times A_6,\ \ L_2(23) \times A_6 \times A_6.$$

In all these 3 cases, G has an element y of order 23 such that $1 < |y^G| < |x^G|$, a contradiction.

(2) $7 \in \pi(Z(G))$ but $5 \notin \pi(Z(G))$.

In this case, we have that 5^3 divides the order of M. Then M is isomorphic to one of the following:

$$L_2(23) \times A_5 \times A_5 \times A_5,\ \ L_2(23) \times A_5 \times A_5 \times A_6.$$

In any case, we have that $|Z(G)| \geq 3^2 \cdot 7$. Now, as above, one can derive a contradiction.

(3) $5 \in \pi(Z(G))$ but $7 \notin \pi(Z(G))$.

If this case occurs, one can show that M is isomorphic to one of the following groups:

$$L_2(23) \times L_2(7) \times A_5 \times A_5,\ \ L_2(23) \times L_2(8) \times A_5 \times A_5.$$

Then, in any case, we obtain that $|Z(G)| \geq 3^2 \cdot 5$. Then, similarly as above, one have a contradiction.

(4) Both 5 and 7 are not in $\pi(Z(G))$.

If this case happens, we see that M may be one of the following:

$$L_2(23) \times A_7 \times A_5 \times A_5,\ \ L_2(23) \times U_3(5).$$

Again, we get a contradiction as the foregoing cases.

Hence, we conclude that S_1 can not be isomorphic to $L_2(23)$.

(II) S_1 is not isomorphic to M_{23}.

Suppose that $S_1 \simeq M_{23}$. Then we can conclude that $M \simeq M_{23} \times A_5$. It follows that $G/Z(G)$ is isomorphic to one of the following groups:

$$M_{23} \times A_5,\ \ M_{23} \times S_5.$$

If $G/Z(G) \simeq M_{23} \times S_5$, then $|Z(G)|$ is an odd number. Let y be an involution in G such that $\overline{y}$ is also an involution of S_5. Then one can check that $1 < |y^G| < |x^G|$, a contradiction.

If $G/Z(G) \simeq M_{23} \times A_5$, then one can pick a non-central element z of G such that $\overline{z}$ is contained in A_5. Then, it is clear that $|y^G| < |x^G|$, a contradiction.

(III) S_1 can not be isomorphic to M_{24}.

Note that $|M_{24}| = 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$. Hence, if $S_1 \simeq M_{24}$, then $2 \notin \pi(Z(G))$. Then one can choose an involution y of G such that $|C_G(y)| > |C_G(x)|$, a final contradiction.

Finally, we see that S_1 must be isomorphic to Co_3 and therefore $G \simeq Co_3$. Thus, our proof is complete. □

Proof of Theorem 1.1. By Lemmas 3.2-3.9, we have that our assertion holds for $M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, J_1, J_3, J_4, Co_3, McL$. The remaining groups can be dealt with similar methods. Therefore their proofs are omitted.

4. Proof of Theorem 1.2

We divide the proof of Theorem 1.2 into several lemmas. Throughout this section, we assume that G is a group satisfying the condition of Theorem 1.2.

Lemma 4.1. *The prime graph of G is not connected and, in particular, for some $p \in \pi(G)$, $\{p\}$ is a component of the prime graph of G.*

Proof. By [5], we have that for any group N under consideration, $lcs(N) = |x^G|$ with x is of order $p \in \pi(N)$ and $C_G(x) = \langle x \rangle$. Thus, p is an isolated point in $\Gamma(G)$. □

Lemma 4.2. *If $N = J_2$, then $G \simeq J_2$.*

Proof. By [5], we have that $p = 7$. If G is a Frobenius group or a 2-Frobenius group, then G has an element of order 35 in view of Lemmas 2.2 and 2.3, which is a contradiction. Then, by Lemma 2.1, we have that G has a normal series $1 \subseteq H \subseteq K \subseteq G$, where H is a nilpotent π_1-group, K/H is a non-abelian simple group and G/K is a π_1-group such that $|G/K|$ divides the order of the outer automorphism group of K/H. Especially, $\{7\}$ is a component of $\Gamma(K/H)$. By [5], K/H may be isomorphic to one of the following groups:

$$L_3(2),\ L_2(8),\ A_7,\ U_3(3),\ A_8,\ L_3(4),\ J_2.$$

If K/H is isomorphic to the former 6 groups, then 5 and 7 are connected in $\Gamma(G)$. Hence, K/H must be isomorphic to J_2 and so $G \simeq J_2$, as desired. □

Lemma 4.3. *If $N = HS$, then $G \simeq HS$.*

Proof. By [5], we know that $p = 7$. By Lemmas 2.2 and 2.3, we see that G is neither a Frobenius group nor a 2-Frobenius group, since, otherwise, 7 and 11 are connected in $\Gamma(G)$. Therefore G has a normal series $1 \subseteq H \subseteq K \subseteq G$, where H is a nilpotent π_1-group, K/H is a non-abelian simple group and G/K is a π_1-group such that $|G/K|$ divides the order of the outer automorphism group of K/H. Especially, $\{7\}$ is a component of $\Gamma(K/H)$. By [5], K/H is isomorphic to one of $L_3(2), L_2(8), A_7, A_8, L_3(4), U_3(5), M_{22}, HS$. If K/H is isomorphic to the former 6 groups, then $77 \in \pi(G)$, a contradiction. If $K/H \simeq M_{22}$, then $35 \in \pi(G)$, a contradiction again. Thus, $K/H \simeq HS$ and so $G \simeq HS$, completing the proof. □

Lemma 4.4. *If $N = Suz$, then $G \simeq Suz$.*

Proof. By [5], we have that $p = 11$. By Lemmas 2.2 and 2.3, we obtain that G is neither a Frobenius group nor a 2-Frobenius group. Otherwise, 11 and 13 are connected in $\Gamma(G)$, a contradiction. Now, according to Lemma 2.1, G has a normal series $1 \subseteq H \subseteq K \subseteq G$, where H is a nilpotent π_1-group, K/H is a non-abelian simple group and G/K is a π_1-group such that $|G/K|$ divides the order of the outer automorphism group of K/H. Furthermore, $\{11\}$ is a prime graph component of G. By [5], K/H may be isomorphic to one of the following groups $L_2(11), M_{11}, M_{12}, M_{22}, A_{11}, A_{12}, A_{13}, Suz$. If K/H is isomorphic to the former 6 groups, then 11 is connected to 13 in $\Gamma(G)$. If $K/H \simeq A_{13}$, then $|H_3| = 3^2$, where H_3 is a Sylow 3-subgroup of H. It is easy to see that 11 does not divide the order of $Aut(H_3)$, which implies that $33 \in \pi(G)$, a final contradiction. Thus, K/H must be isomorphic to Suz and so $G \simeq Suz$. □

Lemma 4.5. *If $N = Fi_{22}$, then $G \simeq Fi_{22}$.*

Proof. By [5], $p = 13$. Invoking Lemmas 2.2 and 2.3, one can conclude that G can neither be Frobenius group nor a 2-Frobenius group. Hence, by Lemma 2.1, G has a normal series $1 \subseteq H \subseteq K \subseteq G$, where H is a nilpotent π_1-group, K/H is a non-abelian simple group and G/K is a π_1-group such that $|G/K|$ divides the order of the outer automorphism group of K/H. Furthermore, $\{13\}$ is a prime graph component of G. It follows from [5] that K/H may be isomorphic to one of the following groups:

$$L_2(13),\ L_3(3),\ L_2(25),\ L_2(27),\ Sz(8),\ U_3(4),\ L_2(64),\ G_2(3),$$

$$L_4(3),\ {}^2F_4(2)',\ L_3(9),\ G_2(4),\ A_{13},\ S_6(2),\ O_7(3),\ Suz,\ Fi_{22}.$$

If K/H is isomorphic to one of the former 14 groups, then 11 and 13 are connected in $\Gamma(G)$, a contradiction.

If K/H is isomorphic to A_{13} or Suz, then G has an element of order 26, since 13 does not divide the order of $GL(n, 2)$ with $n \leq 8$.

Thus, K/H must be isomorphic to Fi_{22} and consequently $G \simeq Fi_{22}$. □

Proof of Theorem 1.2. By lemmas 4.2-4.5, our statement holds for J_2, HS, Suz and Fi_{22}. Analogously, the remaining cases can be verified and so the proof is omitted.

5. Proof of Proposition 1.3

By the hypothesis,

$$|G| = 2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47,$$
$$scs(G) = 2^3 \cdot 3^4 \cdot 5^4 \cdot 23 \cdot 31 \cdot 47,$$
$$sscs(G) = 3^7 \cdot 5^3 \cdot 7 \cdot 13 \cdot 17 \cdot 19 \cdot 31 \cdot 47.$$

Clearly, $Z(G)$ is a proper subgroup of G and $47 \notin \pi(Z(G))$. Let $\overline{G} = G/Z(G)$. Then we claim that every minimal normal subgroup of $\overline{G}$ is non-soluble. If not, assume by way of contradiction that some ninimal normal subgroup S of $\overline{G}$ is soluble. Then S is an elementary abelian group. Let $|S| = r^t$ and the pre-image of S in G is T. Then T is a nilpotent group and the Sylow r-subgroup R of T is normal in G. Note that R is not contained in $Z(G)$. By our hypothesis, $r \notin \{3, 5, 7, 11, 13, 17, 19, 23, 31, 47\}$. Now, assume that $r = 2$. Then it is possible that $scs(G) < |R|$. If this occurs, then since $|G_2| < sscs(G)$

with G_2 is a Sylow 2-subgroup of G, we conclude that $scs(G)$ divides $|R| - 1$. However, by direct computation, we see that this is impossible. Hence, S can not be soluble, and so every minimal normal subgroup of $\overline{G}$ is non-soluble, as claimed.

Now let $M = S_1 \times \cdots \times S_k$ be the product of all minimal normal subgroups of $\overline{G}$, where S_i is a non-abelian simple group. It is obvious that $M \leq \overline{G} \leq Aut(M)$. We suppose that $47 \in \pi(S_1)$. then, by [5], we get that S_1 is isomorphic to $L_2(47)$ or B (Baby Monster). It is easy to rule out the case $S_1 \simeq L_2(47)$. Hence, we obtain that $S_1 \simeq B$ and therefore $G \simeq B$, as desired.

Acknowledgements

This work was partially supported by National Natural Science Foundation of China (11501071, 11671324, 11671063), the Scientific Research Foundation of Chongqing Municipal Science and Technology Commission (cstc2015jcyjA00052, cstc2016jcyjA0065) and the Scientific and Technological Research Program of Chongqing Municipal Education Commission (KJ1711273).

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Accepted: 8.07.2016