GENERALIZED NUMERICAL RADIUS INEQUALITIES FOR 2×2 OPERATOR MATRICES

Watheq Bani-Domi
Department of Mathematics
Yarmouk University
Irbed
Jordan
Watheq@yu.edu.jo

Abstract. We prove some new generalized numerical radius inequalities for 2×2 operator matrices, which improve and generalize an earlier numerical radius inequalities.
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1. Introduction

Let $B(H)$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space H with inner product $\langle .,. \rangle$. For $A \in B(H)$, let $\omega(A) = \sup\{|\langle Ax, x\rangle| : x \in H, \ \|x\| = 1\}$, $\|A\| = \sup\{\|Ax\| : x \in H, \ \|x\| = 1\}$, where $\|x\|^2 = \langle x, x\rangle$, $r(A) = \sup\{|\lambda| : \lambda \in \sigma(A)\}$, where $\sigma(A)$ is the spectrum of A, and $|A| = (A^*A)^{\frac{1}{2}}$ denote the numerical radius of A, the usual operator norm of A, the spectral radius of A, and the absolute value of A, respectively.

It is well – known that $\omega(.)$ is a norm on $B(H)$, which is equivalent to the usual operator norm $\|.\|$. In fact, for every $A \in B(H)$, we have

$$\frac{1}{2}\|A\| \leq \omega(A) \leq \|A\|. \tag{1.1}$$

These inequalities are sharp. The first inequality becomes an equality if $A^2 = 0$, and the second inequality becomes an equality if A is normal.

The inequalities in (1.1) have been improved considerably by Kittaneh in [10] and [11]. It has been shown in [10] and [11], respectively, that if $A \in (H)$, then

$$\omega(A) \leq \frac{1}{2}\||A| + |A^*|\| \leq \frac{1}{2}\left(\|A\| + \left\|A^2\right\|^{\frac{1}{2}}\right) \tag{1.2}$$

and

$$\frac{1}{4}\|A^*A + AA^*\| \leq \omega^2(A) \leq \frac{1}{2}\|A^*A + AA^*\|. \tag{1.3}$$

An important property of the numerical radius norm is its weak unitary invariance, that is , for $A \in B(H)$,

$$\omega(UAU^*) = \omega(A), \tag{1.4}$$

for every unitary $U \in B(H)$.

Several numerical radius inequalities improving the inequalities in (1.1) have been recently given in [4], [10], [11], and [12].

Let $H_1, H_2, \ldots, H_n$ be complex Hilbert spaces, and consider $H = \overset{n}{\underset{i=1}{\oplus}} \mathrm{H_i}$ with respect to this decomposition, every an $n \times n$ operator matrix representation $A = [A_{ij}]$, with entries $A_{ij} \in B(H_j, H_i)$, the space of all bounded linear operators from H_j to H_i. Operator matrices provide a useful tool for studying Hilbert space operators, which have been extensively studied in the literature (see, e.g., [5]). In [8], Hou and Du established useful estimates for the spectral radius, the numerical radius, and the usual operator norm of an $n \times n$ operator matrix $A = [A_{ij}]$. In particular, they proved that

$$r(A) \leq r([\|A_{ij}\|]), \tag{1.5}$$

$$\omega(A) \leq \omega([\|A_{ij}\|]), \tag{1.6}$$

and

$$\|A\| \leq \|[\|A_{ij}\|]\|. \tag{1.7}$$

Recent numerical radius equalities and inequalities for operator matrices can be found in [1, 2], and [6].

In this paper, we give new generalized numerical radius inequalities for 2×2 operator matrices. In section 2, we establish generalized upper bounds for the numerical radii of the off-diagonal parts of 2×2 operator matrices, i.e., operator matrices of the form $\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$. Also we establish generalized upper bounds for the numerical radii of other 2×2 operator matrices.

2. Main results

The aim of this section is to give generalized upper bounds for the numerical radius of the off- diagonal part of a 2×2 operator matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ defined on $H_1 \oplus H_2$. In order to state our results, we need the following well-known lemmas.

The first lemma is a generalization of the mixed Schwarz inequality which has been proved by kittaneh [9].

Lemma 1. *Let T be an operator in $B(H)$ and let f and g be nonnegative functions on $[0, \infty)$ which are continuous and satisfying the relation $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then* $|\langle Tx, y\rangle| \leq \|f(|T|)x\| \|g(|T^*|)y\|$, for all $x, y \in H$.

The second lemma contains two parts. Part (a) is well known and can be found in [3, p. 10]. Part (b) is also known (see, e.g., [1]).

Lemma 2. *Let $X,\ Y \in B(H)$. Then*
(a) $\omega\left(\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}\right) = \max\{\omega(X),\ \omega(Y)\}$.
(b) $\omega\left(\begin{bmatrix} X & Y \\ Y & X \end{bmatrix}\right) = \max\{\omega(X+Y),\ \omega(X-Y)\}$.
In particular, $\omega\left(\begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix}\right) = \omega(X)$.

The third lemma is very useful in computing the numerical radius for matrices (see [7]).

Lemma 3. *If $A = [a_{ij}] \in M_n(\mathbb{C})$, then*

$$\omega(A) \le \omega([|a_{ij}|]) = \frac{1}{2} r([|a_{ij}| + |a_{ji}|]).$$

Our first result is a generalization of the first inequality in (1.2).

Theorem 1. *Let $S = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$ be a 2×2 operator matrix in $B(H_1 \oplus H_2)$, and let f and g be nonnegative functions on $[0,\ \infty)$ which are continuous and satisfying the relation $f(t)\,g(t) = t$ for all $t \in [0,\ \infty)$. Then*

$$\omega(S) \le \frac{1}{2} \max\left\{\left\| f^2(|C|) + g^2(|B^*|)\right\|,\ \left\| f^2(|B|) + g^2(|C^*|)\right\|\right\}. \tag{2.1}$$

Proof. Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in (H_1 \oplus H_2)$, with $\|x\| = 1$. Then we have

$$\begin{aligned}
|\langle Sx, x\rangle| &\le \left\langle f^2(|S|)\,x, x\right\rangle^{\frac{1}{2}} \left\langle g^2(|S^*|)\,x, x\right\rangle^{\frac{1}{2}} \qquad \text{(by Lemma 1)}\\
&= \left\langle f^2\left(\begin{bmatrix} |C| & 0 \\ 0 & |B| \end{bmatrix}\right) x, x\right\rangle^{\frac{1}{2}} \left\langle g^2\left(\begin{bmatrix} |B^*| & 0 \\ 0 & |C^*| \end{bmatrix}\right) x, x\right\rangle^{\frac{1}{2}}\\
&= \left\langle \begin{bmatrix} f^2(|C|) & 0 \\ 0 & f^2(|B|) \end{bmatrix} x, x\right\rangle^{\frac{1}{2}} \left\langle \begin{bmatrix} g^2(|B^*|) & 0 \\ 0 & g^2(|C^*|) \end{bmatrix} x, x\right\rangle^{\frac{1}{2}}\\
&\le \frac{1}{2}\left(\left\langle \begin{bmatrix} f^2(|C|) & 0 \\ 0 & f^2(|B|) \end{bmatrix} x, x\right\rangle + \left\langle \begin{bmatrix} g^2(|B^*|) & 0 \\ 0 & g^2(|C^*|) \end{bmatrix} x, x\right\rangle\right)\\
&\quad \text{(by the arithmetic} - \text{geometric mean inequality)}\\
&= \frac{1}{2}\left\langle \begin{bmatrix} f^2(|C|) + g^2(|B^*|) & 0 \\ 0 & f^2(|B|) + g^2(|C^*|) \end{bmatrix} x,\ x\right\rangle.
\end{aligned}$$

Thus,

$$\begin{aligned}
\omega(S) &= \sup\{|\langle Sx, x\rangle| : x \in (H_1 \oplus H_2),\ \|x\| = 1\}\\
&\le \frac{1}{2}\max\left\{\left\| f^2(|C|) + g^2(|B^*|)\right\|,\ \left\| f^2(|B|) + g^2(|C^*|)\right\|\right\},
\end{aligned}$$

as required.

Inequality (2.1) includes several numerical radius inequalities for operator matrices. Samples of inequalities are demonstrated in the following remarks.

Remark 1. For $f(t) = t^{\propto}$ *and* $g(t) = t^{1-\propto}$, $\propto \in (0,\ 1)$, in inequality (2.1), we get the following inequality

$$\omega(S) \leq \frac{1}{2} \max\left\{ \left\| |C|^{2\propto} + |B^*|^{2(1-\propto)} \right\|,\ \ \left\| |B|^{2\propto} + |C^*|^{2(1-\propto)} \right\| \right\}.$$

Remark 2. In Remark 1, if $\propto = \frac{1}{2}$, then we get

$$\omega(S) \leq \frac{1}{2} \max\{ \||C| + |B^*|\|,\ \ \||B| + |C^*|\| \}.$$

Remark 3. By letting $H_1 = H_2$ and $B = C$ in Remark 2, and by using Lemma 2(b) it is easy to see that the inequality in Remark 2 generalizes the inequality (1.2), i.e.,

$$\omega(S) = \omega(B) \leq \frac{1}{2} \||B| + |B^*|\|.$$

In the next theorem, we employ the inequalities in (1.3) to generalize and improve the inequalities in (1.1).

Theorem 2. *Let* $S = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$ *be a* 2×2 *operator matrix in* $B(H_1 \oplus H_2)$. *Then*

$$\frac{1}{2} \max\{\ \alpha,\ \beta\ \} \leq \omega\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right) \leq \frac{1}{\sqrt{2}} \max\{\alpha, \beta\ \},$$

where $\alpha = \left\| |C|^2 + |B^*|^2 \right\|^{\frac{1}{2}}$ *and* $\beta = \left\| |B|^2 + |C^*|^2 \right\|^{\frac{1}{2}}$.

In the following results, we establish generalized upper bounds for the numerical radii of a general 2×2 operator matrices.

Theorem 3. *Let* $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ *be a* 2×2 *operator matrix in* $B(H_1 \oplus H_2)$, *and let* f *and* g *be nonnegative functions on* $[0,\ \infty)$ *which are continuous and satisfying the relation* $f(t)\, g(t) = t$, *for all* $t \in [0,\ \infty)$. *Then*

$$\omega(T) \leq \frac{1}{2} \max\{ \|a\|,\ \|b\| \},$$

where,

$$a = f^2(|A|) + g^2(|A^*|) + f^2(|C|) + g^2(|B^*|)$$

and

$$b = f^2(|D|) + g^2(|D^*|) + f^2(|B|) + g^2(|C^*|).$$

Proof. Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in (H_1 \oplus H_2)$, with $\|x\| = 1$. Then we have

$$\left|\left\langle \begin{bmatrix} A & B \\ C & D \end{bmatrix} x, x \right\rangle\right| \le \left\langle f^2\left(\left|\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right|\right) x, x \right\rangle^{\frac{1}{2}} \left\langle g^2\left(\left|\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}^*\right|\right) x, x \right\rangle^{\frac{1}{2}}$$
$$+ \left\langle f^2\left(\left|\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right|\right) x, x \right\rangle^{\frac{1}{2}} \left\langle g^2\left(\left|\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}^*\right|\right) x, x \right\rangle^{\frac{1}{2}} \text{(by Lemma 1)}$$
$$= \left\langle f^2\left(\begin{bmatrix} |A| & 0 \\ 0 & |D| \end{bmatrix}\right) x, x \right\rangle^{\frac{1}{2}} \left\langle g^2\left(\begin{bmatrix} |A^*| & 0 \\ 0 & |D^*| \end{bmatrix}\right) x, x \right\rangle^{\frac{1}{2}}$$
$$+ \left\langle f^2\left(\begin{bmatrix} |C| & 0 \\ 0 & |B| \end{bmatrix}\right) x, x \right\rangle^{\frac{1}{2}} \left\langle g^2\left(\begin{bmatrix} |B^*| & 0 \\ 0 & |C^*| \end{bmatrix}\right) x, x \right\rangle^{\frac{1}{2}}$$
$$= \left\langle \begin{bmatrix} f^2(|A|) & 0 \\ 0 & f^2(|D|) \end{bmatrix} x, x \right\rangle^{\frac{1}{2}} \left\langle \begin{bmatrix} g^2(|A^*|) & 0 \\ 0 & g^2(|D^*|) \end{bmatrix} x, x \right\rangle^{\frac{1}{2}}$$
$$+ \left\langle \begin{bmatrix} f^2(|C|) & 0 \\ 0 & f^2(|B|) \end{bmatrix} x, x \right\rangle^{\frac{1}{2}} \left\langle \begin{bmatrix} g^2(|B^*|) & 0 \\ 0 & g^2(|C^*|) \end{bmatrix} x, x \right\rangle^{\frac{1}{2}}$$
$$\le \frac{1}{2} \left(\begin{array}{c} \left\langle \begin{bmatrix} f^2(|A|) & 0 \\ 0 & f^2(|D|) \end{bmatrix} x, x \right\rangle + \left\langle \begin{bmatrix} g^2(|A^*|) & 0 \\ 0 & g^2(|D^*|) \end{bmatrix} x, x \right\rangle \\ + \left\langle \begin{bmatrix} f^2(|C|) & 0 \\ 0 & f^2(|B|) \end{bmatrix} x, x \right\rangle + \left\langle \begin{bmatrix} g^2(|B^*|) & 0 \\ 0 & g^2(|C^*|) \end{bmatrix} x, x \right\rangle \end{array} \right)$$

(by the arithmetic- geometric mean inequality)

$$= \frac{1}{2} \left\langle \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} x,\ x \right\rangle.$$

Thus,

$$\omega(T) = \sup\{|\langle Tx, x\rangle| : x \in (H_1 \oplus H_2),\ \|x\| = 1\} \le \frac{1}{2} \max\{\|a\|,\ \|b\|\},$$

as required.

Remark 4. If $f(t) = t^{\propto}$ and $g(t) = t^{1-\propto}$, $\propto \in (0,\ 1)$, in Theorem 3, then we get the following inequality

$$\omega\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \le \frac{1}{2} \max \left\{ \begin{array}{c} \left\| |A|^{2\propto} + |A^*|^{2(1-\propto)} + |C|^{2\propto} + |B^*|^{2(1-\propto)} \right\|, \\ \left\| |D|^{2\propto} + |D^*|^{2(1-\propto)} + |B|^{2\propto} + |C^*|^{2(1-\propto)} \right\| \end{array} \right\}.$$

Remark 5. From Remark 4 with $\propto = \frac{1}{2}$ and Lemma 2(b), If $A = B = C = D$, then we get the following inequality

$$\omega\left(\begin{bmatrix} A & A \\ A & A \end{bmatrix}\right) = 2\omega(A) \le \frac{1}{2} \|2|A| + 2|A^*|\| = \||A| + |A^*|\|\ ,$$

and so,

$$\omega(A) \leq \frac{1}{2}\left\| |A| + |A^*| \right\|.$$

Remark 6. From Remark 4 with $\propto = \frac{1}{2}$ and the first inequality in (1.1), If $A = C = D = 0$, then we get the following equality

$$\frac{1}{2}\|B\| = \frac{1}{2}\left\| \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix} \right\| \leq \omega\left(\begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix} \right) \leq \frac{1}{2}\max\left\{ \left\| |B^*| \right\|, \left\| |B| \right\| \right\} = \frac{1}{2}\|B\|.$$

Hence,

$$\omega\left(\begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix} \right) = \omega\left(\begin{bmatrix} 0 & 0 \\ B & 0 \end{bmatrix} \right) = \frac{1}{2}\|B\|.$$

In the following theorem, we present an improvement of the inequality (1.6) when $n = 2$.

Theorem 4. *Let* $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ *be a* 2×2 *operator matrix in* $B(H_1 \oplus H_2)$, *and let* f *and* g *be nonnegative functions on* $[0, \infty)$ *which are continuous and satisfying the relation* $f(t)\,g(t) = t$ *for all* $t \in [0, \infty)$. *Then*

$$\omega(T) \leq \omega\left(\begin{bmatrix} \omega(A) & \|f(|B|)\|\,\|g(|B^*|)\| \\ \|f(|C|)\|\,\|g(|C^*|)\| & \omega(D) \end{bmatrix} \right).$$

Proof. Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in (H_1 \oplus H_2)$, with $\|x\| = 1$. Then we have

$$\begin{aligned}
|\langle Tx, x\rangle| &= |\langle Ax_1, x_1\rangle + \langle Bx_2, x_1\rangle + \langle Cx_1, x_2\rangle + \langle Dx_2, x_2\rangle| \\
&\leq |\langle Ax_1, x_1\rangle| + |\langle Bx_2, x_1\rangle| + |\langle Cx_1, x_2\rangle| + |\langle Dx_2, x_2\rangle| \\
&\leq \omega(A)\|x_1\|^2 + \|f(|B|)\,x_2\|\,\|g(|B^*|)\,x_1\| \\
&\quad + \|f(|C|)\,x_1\|\,\|g(|C^*|)\,x_2\| + \omega(D)\|x_2\|^2 \\
&\quad (\text{by definition of } \omega(.) \text{ and Lemma 1.1}) \\
&\leq \omega(A)\|x_1\|^2 + \|f(|B|)\|\,\|g(|B^*|)\|\,\|x_1\|\,\|x_2\| \\
&\quad + \|f(|C|)\|\,\|g(|C^*|)\|\,\|x_1\|\,\|x_2\| + \omega(D)\|x_2\|^2 \\
&= \left\langle \begin{bmatrix} \omega(A) & \|f(|B|)\|\,\|g(|B^*|)\| \\ \|f(|C|)\|\,\|g(|C^*|)\| & \omega(D) \end{bmatrix} \begin{bmatrix} \|x_1\| \\ \|x_2\| \end{bmatrix}, \begin{bmatrix} \|x_1\| \\ \|x_2\| \end{bmatrix} \right\rangle.
\end{aligned}$$

Now, the result follows by taking the supremum over all unit vectors in $(H_1 \oplus H_2)$.

Here, a weaker version of Theorem 4 has been also given in [2] when $n = 2$.

Remark 7. From Theorem 4, if $f(t) = \sqrt{t} = g(t)$, then we get the following inequality

$$\begin{aligned}\omega\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) &\le \omega\left(\begin{bmatrix} \omega(A) & \left\||B|^{\frac{1}{2}}\right\|\left\||B^*|^{\frac{1}{2}}\right\| \\ \left\||C|^{\frac{1}{2}}\right\|\left\||C^*|^{\frac{1}{2}}\right\| & \omega(D) \end{bmatrix}\right) \\ &\le \omega\left(\begin{bmatrix} \omega(A) & \|B\| \\ \|C\| & \omega(D) \end{bmatrix}\right) \\ &\le \omega\left(\begin{bmatrix} \|A\| & \|B\| \\ \|C\| & \|D\| \end{bmatrix}\right).\end{aligned}$$

Now, from Remark 7 and Lemma 3 we get the inequality

$$\omega\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \le \frac{1}{2}\left(\omega(A) + \omega(D) + \sqrt{(\omega(A) - \omega(D))^2 + (\|B\| + \|C\|)^2}\right),$$

which is a generalized for the second inequality in (1.1) when we take $A = B = C = D$ and use Lemma 2(a). Also this inequality can be employed to give new bounds for the zeros of polynomials (see, e.g., [2, 10], and references therein).

Based on Lemma 2(a), the inequality (2.1) and the property $\omega(X + Y) \le \omega(X) + \omega(Y)$, we can prove the following corollary.

Corollary 1. *Let* $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ *be a* 2×2 *operator matrix in* $B(H_1 \oplus H_2)$, *and let* f *and* g *be nonnegative functions on* $[0, \infty)$ *which are continuous and satisfying the relation* $f(t)g(t) = t$ *for all* $t \in [0, \infty)$. *Then*

$$\begin{aligned}\omega(T) &\le \max\{\omega(A),\ \omega(D)\} \\ &\quad + \frac{1}{2}\max\left\{\left\|f^2(|C|) + g^2(|B^*|)\right\|,\ \left\|f^2(|B|) + g^2(|C^*|)\right\|\right\}.\end{aligned}$$

Remark 8. From Corollary 1 and Lemma 2(b), if $f(t) = \sqrt{t} = g(t)$ and if $A = B = C = D$, then we get the following inequality

$$\omega\left(\begin{bmatrix} A & A \\ A & A \end{bmatrix}\right) = 2\omega(A) \le \omega(A) + \frac{1}{2}\||A| + |A^*|\|,$$

and so,

$$\omega(A) \le \frac{1}{2}\||A| + |A^*|\|.$$

Remark 9. From Remark 4, Lemma 2(b), $\propto = \frac{1}{2}$ and If $A = D, B = C$, then we get the following

$$\max\{\omega(A - B),\ \omega(A + B)\} \le \frac{1}{2}\||A| + |A^*| + |B| + |B^*|\|.$$

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