ON FUZZY MINIMAL STRUCTURES

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Abstract. In this paper, we define fuzzy open (closed) M-set, fuzzy M- frontier, fuzzy M-semi frontier, fuzzy rarely M-set, and fuzzy rarely M-continuous functions in fuzzy minimal spaces. We will explore several interesting properties and characterizations of these newly defined notions.
Keywords: fuzzy open (closed) M-set, fuzzy M- frontier, fuzzy M-semi frontier, fuzzy rarely M-set, fuzzy rarely M-continuous function, fuzzy minimal space.

1. Introduction

Fuzziness is one of the most important and useful concepts in the modern scientific studies. The fundamental concepts of Fuzzy sets were originally initiated by Zadeh [17]. Fuzzy sets attained a very important role in the study of Fuzzy topology whose pioneer was Chang [7]. Afterwards, Lowen [10] was a mathematician who worked for fuzzy compactness in fuzzy topological spaces. With this invent a range of aspects of general topology has been investigated and carried out in the Fuzzy sense by quite a few authors.

A fuzzy set is a collection of objects with grades of membership function. Such a set is represented by a membership or characteristic function, which assigns each object a score of membership ranging between 0 and 1. Many notions like intersection, complement, inclusion, convexity, union, relation, etc. are extended to fuzzy sets and various properties of these notions in the context of fuzzy sets are established and further, it has been observed to be very helpful in solving many realistic problems. Du. et al. [8] gave the fuzzified solution to the 9-intersection Egenhofer model for depicting topological relations in Geographic Information system (GIS) query. In [13],[14] El.Naschie proved that the concept of fuzzy topology can also be related to quantum particle physics.

For a set X, we define a fuzzy set in X to be a function $\mu : X \to [0,1]$, where $\mu(x)$ represents the degree of membership of x in the fuzzy set μ. It is well

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known that the fuzzy set μ in X is a set of ordered pairs, i.e., $\mu = \{(x,\ \mu(x))\ /\ x\ \in X\}$. The family of all fuzzy sets on X is denoted by I^X, consisting of all the mappings from X to $[0,1]$. Any subset A of a set X can be identified with its characteristic function $\chi_A : X \to \{0,1\}$ defined by

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$$

and such characteristic functions are fuzzy sets in X. The characteristic functions of subsets of a set X are referred to as the crisp fuzzy sets in X. Agd. El- Monsef and Ramadam introduced the concept of fuzzy supra- topology as follows. A collection of fuzzy sets $F < I^X$ is called a fuzzy supra topology on X if $\overline{0}, \overline{1}\ \in F$ and F is closed under arbitrary union. In continuation of the study of fuzzy sets, Alimohammady and Roohi [1], [2], [3], [4], [5], introduced and studied the notions of fuzzy minimal structures and fuzzy minimal spaces. It is pertinent to mention that the concept of minimal structures and minimal spaces was introduced by Maki [11],[12] in 1996.

A subfamily m_X of $P(X)$ is called a minimal structure [12] on X, if $\varphi\ \in m_X$ and $X\ \in m_X$. Each member of m_X is said to be an m_X-open set, and the complement of an m_X-open set is said to be an m_X-closed set. A minimal structure m_X on a non-empty set X is denoted by (X, m_X) . A family M of fuzzy sets in X is said to be a fuzzy minimal structure [3] on X, if $\overline{0}\ \in M$ and $\overline{1}\ \in M$. In this case (X, M) is called a fuzzy minimal space. Each member of M is said to be a fuzzy M-open set, and the complement of a fuzzy M-open set is said to be a fuzzy M-closed set. In [16], for a fuzzy set λ on X, M-$Cl(\lambda)$ and M-$Int(\lambda)$ represents the closure and interior respectively with respect to the fuzzy minimal structure and are defined as follows :

$$M - Cl(\lambda) = \wedge\{\mu : \lambda \leq \mu; \mu^c\ \in M\},$$
$$M - Int(\lambda) = \vee\{\mu : \mu \leq \lambda; \mu\ \in M\}.$$

where $FMC(X)$ (respectively, $FMO(X)$) represents the collection of all fuzzy M-closed (respectively, fuzzy M-open) sets in X.

2. On fuzzy open-M sets

In this section, we will introduce and study the concept of fuzzy open-M set (fuzzy closed-M set).

Definition 1. *Let M be a fuzzy minimal structure on X. A fuzzy set λ on X is said to be an fuzzy open-M(respectively, fuzzy closed -M) set, if M-$Int(\lambda) = \lambda$ (respectively, M-$Cl(\lambda) = \lambda$).*

Definition 2. *A fuzzy M-open (respectively fuzzy M -closed) set is fuzzy open-M (respectively fuzzy closed-M) set, but not conversely which follows from the following example.*

Example 3. Let $X = \{a, b, c\}$, $M = \{\overline{0}, \overline{1}, \lambda, \mu, v\}$, where $\lambda = \{(a, 0.2), (b, 0.3), (c, 0)\}$, $\mu = \{(a, 0.2), (b, 0.3), (c, 0.1)\}$, $v = \{(a, 0.7), (b, 0.4), (c, 0.5)\}$.

M-$Int(v) = \vee\{\beta : \beta \leq v$ and $\beta \in M\} = \lambda \vee v \vee \mu = v$. This implies that v is fuzzy open-M set as well as fuzzy M-open set.

Again, for $M = \{\overline{0}, \overline{1}, \lambda, \mu, v\}$, where $\lambda = \{(a, 0.3)\}$, $\mu = \{(b, 0.4)\}, v = \{(c, 0.5)\}$. Let $\beta = \{(a, 0.3), (b, 0.4), (c, 0.5)\}$. Then M-$Int(\beta) = \lambda \vee \mu \vee v = \beta$. This gives that β is fuzzy open-M set, but it is not fuzzy M-open set, because $\beta \notin M$. Similarly β^c is a fuzzy closed-M set.

Lemma 4. *For a fuzzy minimal space (X, M), a fuzzy set λ on X is a fuzzy open-M set, if and only if $\lambda^c = \overline{1} - \lambda$ is a fuzzy closed-M set.*

Theorem 5. *In fuzzy minimal space (X, M), the collection of all fuzzy open-M sets forms a fuzzy supera topology on X.*

Proof. The fuzzy sets $\overline{0}, \overline{1}$ are obviously, fuzzy open-M sets. Let $\mu = \vee\{\Gamma_\alpha : \alpha \in \Delta\}$ be a join of fuzzy open-M sets, we have M-$Int(\Gamma_\alpha) = \Gamma_\alpha$, for each α. Now $\Gamma_\alpha : \alpha \leq \mu$, so $\Gamma_\alpha = M$-$Int(\Gamma_\alpha) \leq \mu$ or $\vee\{\Gamma_\alpha : \alpha \in \Delta\} = \vee\{M$-$Int(\Gamma_\alpha : \alpha \in \Delta)\} \leq \mu$, for each α. Hence $\mu = \vee\{\Gamma_\alpha : \alpha \in \Delta\} = \vee\{M$-$Int(\Gamma_\alpha) : \alpha \in \Delta\} \leq M$-$Int\{\vee\Gamma_\alpha : \alpha \in \Delta\} \leq M$-$Int(\mu)$. But we know that M-$Int(\mu) \leq \mu$. This shows that $\mu = M$-$Int(\mu)$. This proves that arbitrary union of fuzzy open -M sets is an fuzzy open -M set. □

Remark 6. Arbitrary (even finite) intersections (respectively, unions) of fuzzy open-M sets (respectively, fuzzy closed-M) sets need not to be fuzzy open -M set (respectively, fuzzy closed-M set). This is evident from the following example.

Example 7. Let $X = \{a, b, c\}$ and $M = \{\overline{0}, \overline{1}, \lambda, \mu, v\}$, where $\lambda = \{(a, 0.3), (b, 0), (c, 0)\}$, $\mu = \{(a, 0), (b, 0.4), (c, 0.5)$, $v = \{(a, 0.2), (b, 0.8), c, 0)\}$ and let $\alpha = \{(a, 0.3), (b, 0.4), (c, 0.5)\}$, $\beta = \{(a, 0.2), (b, 0.8), (c, 0.5)\}$, $\gamma = \{(a, 0.7), (b, 0.6), (c, 0.5)\}$, $\delta = \{(a, 0.8), (b, 0.2), (c, 0.5)\}$. Then M-$Int(\alpha) = \alpha$, M-$Int(\beta) = \beta$, M-$Int(\alpha \wedge \beta) \neq \alpha \wedge \beta$, M-$Cl(\gamma) = \gamma$, M-$Cl(\delta) = \delta$ and M-$Cl(\gamma \vee \delta) \neq \gamma \vee \delta$.

3. Fuzzy M-frontiers and fuzzy M-semi frontiers

In this section we will give more than a few properties of fuzzy M-frontier of a set and fuzzy M-semi frontiers of a set in fuzzy minimal spaces and these concepts are further supported by examples.

3.1 Fuzzy M-frontiers

Definition 8. *Let X be a non-empty set, which has a fuzzy minimal structure M and let Γ be a fuzzy set on X. Then fuzzy M-frontier of Γ in a fuzzy minimal space (X, M) is denoted by M-$Fr(\Gamma)$ and defined by M-$Fr(\Gamma) = M$-$Cl(\Gamma) \wedge M$-$Cl(\overline{1} - \Gamma)$.*

Theorem 9. *Let (X, M) be a fuzzy minimal space and Γ be a fuzzy set, then:*

1) $M\text{-}Fr(\Gamma) = M\text{-}Fr(\Gamma^c)$;
2) If Γ is a fuzzy M-closed set, then $M\text{-}Fr(\Gamma) \leq \Gamma$;
3) If Γ is fuzzy M-open set, then $M\text{-}Fr(\Gamma) \leq \Gamma$;
4) Let $\Gamma \leq \mu$ and $\mu \in FMC(X)$ (respectively, $\mu \in FMO(X)$). Then $M\text{-}Fr(\Gamma) \leq \mu$ (respectively, $M\text{-}Fr(\Gamma) \leq \mu^c$);
5) $(M\text{-}Fr(\Gamma))^c = M\text{-}Int(\Gamma) \vee M\text{-}Int(\Gamma^c)$ for any fuzzy set Γ in X.

Proof. Proof is simple and hence omitted. □

Theorem 10. *Let Γ be a fuzzy set in a fuzzy minimal space (X, M). Then:*

1) $M\text{-}Fr(\Gamma) = M\text{-}Cl(\Gamma) - M\text{-}Int(\Gamma)$;
2) $M\text{-}Fr(M\text{-}Int(\Gamma)) \leq M\text{-}Fr(\Gamma)$;
3) $M\text{-}Fr(M\text{-}Cl(\Gamma)) \leq M\text{-}Fr(\Gamma)$;
4) $M\text{-}Int(\Gamma) \leq \Gamma - M\text{-}Fr(\Gamma)$.

Proof. Proof is simple, therefore omitted. □

Theorem 11. *Let Γ and μ be fuzzy sets in a fuzzy minimal space (X, M).Then $M\text{-}Fr(\Gamma \vee \mu) \leq M\text{-}Fr(\Gamma) \vee M\text{-}Fr(\mu)$.*

Proof. $M\text{-}Fr(\Gamma \vee \mu) = M\text{-}Cl(\Gamma \vee \mu) \wedge M\text{-}Cl(\Gamma \vee \mu)^c$
$= (M\text{-}Cl(\Gamma) \vee M\text{-}Cl(\mu)) \wedge (M\text{-}Cl((\Gamma)^c \wedge (\mu)^c))$
$\leq (M\text{-}Cl(\Gamma) \vee M\text{-}Cl(\mu)) \wedge (M\text{-}Cl(\Gamma)^c \wedge M\text{-}Cl(\mu)^c)$
$\leq (M\text{-}Cl(\Gamma) \wedge (M\text{-}Cl(\Gamma)^c \wedge M\text{-}Cl(\mu)^c)) \vee (M\text{-}Cl(\mu) \wedge (M\text{-}Cl(\Gamma)^c \wedge M\text{-}Cl(\mu)^c))$
$= (M\text{-}Fr(\Gamma) \wedge M\text{-}Cl(\mu)^c) \vee (M\text{-}Fr(\mu) \wedge M\text{-}Cl(\Gamma)^c) \leq M\text{-}Fr(\Gamma) \vee M\text{-}Fr(\mu)$.
Therefore, $M\text{-}Fr(\Gamma \vee \mu) \leq M\text{-}Fr(\Gamma) \vee M\text{-}Fr(\mu)$. The equality does not hold in general. □

Theorem 12. *In a fuzzy minimal space (X, M), for any two fuzzy sets Γ and μ, we have $M\text{-}Fr(\Gamma \wedge \mu) \leq (M\text{-}Fr(\Gamma) \wedge M\text{-}Cl(\mu)) \vee (M\text{-}Fr(\mu) \wedge M\text{-}Cl(\Gamma))$.*

Proof. $M\text{-}Fr(\Gamma \wedge \mu) = M\text{-}Cl(\Gamma \wedge \mu) \wedge M\text{-}Cl(\Gamma \wedge \mu)^c$
$\leq (M\text{-}Cl(\Gamma) \wedge M\text{-}Cl(\mu)) \wedge (M\text{-}Cl(\Gamma)^c \vee M\text{-}Cl(\mu)^c)$
$= (M\text{-}Cl(\Gamma) \wedge (M\text{-}Cl(\mu) \wedge M\text{-}Cl(\Gamma)^c)) \vee (M\text{-}Cl(\Gamma) \wedge (M\text{-}Cl(\mu) \wedge M\text{-}Cl(\mu)^c))$
$= (M\text{-}Fr(\Gamma) \wedge M\text{-}Cl(\mu)) \vee (M\text{-}Fr(\mu) \wedge M\text{-}Cl(\Gamma))$. Therefore, $M\text{-}Fr(\Gamma \wedge \mu) \leq (M\text{-}Fr(\Gamma) \wedge M\text{-}Cl(\mu)) \vee (M\text{-}Fr(\mu) \wedge M\text{-}Cl(\Gamma))$. □

Corollary 13. *For any two fuzzy sets Γ and μ in an fuzzy minimal space (X, M), we have : $M\text{-}Fr(\Gamma \wedge \mu) \leq M\text{-}Fr(\Gamma) \vee M\text{-}Fr(\mu)$.*

Theorem 14. *For any fuzzy set λ in a fuzzy minimal space (X, M), we have:*

1) $M\text{-}Fr(M\text{-}Fr(\lambda) \leq M\text{-}Fr(\lambda)$;
2) $M\text{-}Fr(M\text{-}Fr(M\text{-}Fr(\lambda))) \leq M\text{-}Fr(M\text{-}Fr(\lambda))$.

Proof. Proof is simple therefore omitted. □

Remark 15. The example below shows that equality in (1) of Theorem 14, does not satisfy always.

Example 16. Let $X = \{e, g, h\}$ and $M = \{\overline{0}, \overline{1}, \mu_1, \mu_2, \mu_3, \mu_4, \mu_{5,}\mu_6, \mu_7\}$, where $\mu_1 = \{(e, .5), (g, .7), (h, .3)\}$, $\mu_2 = \{(e, .4), (g, .7), (h, .2)\}$, $\mu_3 = \{(e, .5), (g, .7), (h, .1)\}$, $\mu_4 = \{(e, .4), (g, .7), (h, .1)\}$, $\mu_5 = \{(e, .5), (g, .8), (h, .3)\}$, $\mu_6 = \{(e, .6), (g, .9), (h, .3)\}$, $\mu_7 = \{(e, .5), (g, .7), (h, .2)\}$.

Let $\Gamma = \{(e, .2), (g, .5), (h, .9)\}$. Then $M\text{-}Fr(\Gamma) = \overline{1}$, but $M\text{-}Fr(M\text{-}Fr(\Gamma)) = \overline{0}$.

Definition 17 ([15])**.** *A function $f : (X, M_X) \to (Y, M_Y)$ is fuzzy minimal continuous, if $f^{-1}(B) \in M_X$, for any $B \in M_Y$.*

Theorem 18. *Let $(X, M_X,)$ and (Y, M_Y) be two fuzzy minimal spaces and $f : X \to Y$ be a fuzzy M - continuous function, then fore very fuzzy set μ in Y, we have : $M_X\text{-}Fr(f^{-1}(\mu)) \le f^{-1}(M_Y\text{-}Fr(\mu))$.*

Proof. Let f be fuzzy M continuous function and μ be a fuzzy set in Y. Then $M_X\text{-}Fr(f^{-1}(\mu)) = M_X\text{-}Cl(f^{-1}(\mu)) \wedge M_X\text{-}Cl(f^{-1}(\mu))^c \le f^{-1}(M_Y\text{-}Cl(\mu)) \wedge M_X\text{-}Cl(f^{-1}(\mu)^c) = f^{-1}(M_Y\text{-}Cl(\mu)) \wedge f^{-1}(M_Y\text{-}Cl(\mu)^c) = f^{-1}(M_Y\text{-}Cl(\mu) \wedge M_Y\text{-}Cl(\mu)^c) = f^{-1}(M_Y\text{-}Fr(\mu))$. Therefore, $M_X\text{-}Fr(f^{-1}(\mu)) \le f^{-1}(M_Y\text{-}Fr(\mu))$. This proves the theorem. □

3.2 Fuzzy M-semi-frontier

In 1960, Levine [9] defined semi-open sets in topological spaces. In this contrast, we define M -semi-open sets in fuzzy minimal space (X, M). In this section we will study fuzzy M-semi frontier of fuzzy sets in fuzzy minimal space (X, M). Several interesting relations are explored.

Definition 19. *Let Γ be a fuzzy set in X. Then, Γ is called a fuzzy M-semi open set of (X, M), if there exists an M -open set v in (X, M), such that $\nu \le \Gamma \le M\text{-}Cl\ \nu$. The complement of fuzzy M-semi open set is fuzzy M-semi closed set. where $FSC(X)$ (respectively, $FMSO(X)$) denote the collection of fuzzy M-semi closed (respectively, fuzzy M-semi open) sets in X.*

Definition 20. *Let Γbe a fuzzy set in fuzzy minimal space (X, M). Then M -semi closure (briefly $M\text{-}sCl$) and M-semi interior (briefly $M\text{-}sInt$) of Γ are given as:*

$M\text{-}sCl(\Gamma) = \wedge\ \{\beta | \Gamma \le \beta, \beta \text{ is fuzzy } M\text{-semi closed in } (X, M)\}$,
$M\text{-}sInt(\Gamma) = \vee\{\beta | \beta \le \Gamma, \beta \text{ is fuzzy } M\text{-semiopenin } (X, M)\}$.

Theorem 21. *Let Γ and μ be fuzzy sets in fuzzy minimal space (X, M). Then:*

(1) $M\text{-}sInt(\Gamma \vee \mu) \ge M\text{-}sInt(\Gamma) \vee M\text{-}sInt(\mu)$;
(2) $M\text{-}sInt(\Gamma \wedge \mu) = M\text{-}sInt(\Gamma) \wedge M\text{-}sInt(\mu)$;
(3) $M\text{-}sCl(\Gamma \vee \mu) = M\text{-}sCl(\Gamma) \vee M\text{-}sCl(\mu)$;
(4) $M\text{-}sCl(\Gamma \wedge \mu) \le M\text{-}sCl(\Gamma) \wedge M\text{-}sCl(\mu)$.

Proof. 1. M-$sInt(\Gamma)$ and M-$sInt(\mu)$ are both fuzzy M- semi open sets and $\Gamma \leq \Gamma \vee \mu$, $\mu \leq \Gamma \vee \mu$ imply that M-$sInt(\Gamma) \leq M$-$sInt(\Gamma \vee \mu)$ and M-$sInt(\mu) \leq M$-$sInt(\Gamma \vee \mu)$. Combining, M-$sInt(\Gamma)\vee$ M-$sInt(\mu) \leq M$-$sInt(\Gamma \vee \mu)$ or M-$sInt(\Gamma \vee \mu) \geq M$-$sInt(\Gamma)\vee$ M-$sInt(\mu)$.

2. $\Gamma \wedge \mu \leq \Gamma$ and $\Gamma \wedge \mu \leq \mu$ imply M-$sInt(\Gamma \wedge \mu) \leq M$-$sInt(\Gamma)$, M-$sInt(\Gamma \wedge \mu) \leq M$-$sInt(\mu)$ and therefore, M-$sInt(\Gamma \wedge \mu) \leq M$-$sInt(\Gamma)\wedge$ M-$sInt(\mu)$.Conversely, M-$sInt(\Gamma) \leq \Gamma$ and M-$sInt(\mu) \leq \mu$ implies M-$sInt(\Gamma)\wedge$ M-$sInt(\mu) \leq \Gamma \wedge \mu$ and M-$sInt(\Gamma)\wedge$ M-$sInt(\mu)$ is fuzzy semi open. But M-$sInt(\Gamma \wedge \mu)$ is the largest fuzzy M-semi open set contained in $\Gamma \wedge \mu$, hence M-$sInt(\Gamma)\wedge$ M-$sInt(\mu) \leq M$-$sInt(\Gamma \wedge \mu)$.This gives the equality.

3. It follows easily from (2).

4. Since $\Gamma\wedge\mu \leq \Gamma$, $\Gamma\wedge\mu \leq \mu$ M-$sCl(\Gamma\wedge\mu) \leq M$-$sCl(\Gamma)$, also M-$sCl(\Gamma\wedge\mu) \leq M$-$sCl(\mu)$. Hence, M-$sCl(\Gamma \wedge \mu) \leq M$-$sCl(\Gamma)\wedge$ M-$sCl(\mu)$.

□

Definition 22. *Let* Γ *be a fuzzy set in fuzzy minimal space* (X, M)*. Then, the fuzzy* M*-semi frontier of* Γ *is defined as* M-$sFr(\Gamma) = M$-$sCl(\Gamma)\wedge$ M-$sCl(\Gamma)^c$*.* M-$sFr(\Gamma)$ *may or may not be a fuzzy* M*-semi closed set.*

Remark 23. We note that $\Gamma \vee M$-$sFr(\Gamma) \leq M$-$sCl(\Gamma)$, for an arbitrary fuzzy set Γ in X, the equality in general, does not hold always as it is clear by the following example.

Example 24. Let $X = \{e, g\}$ and fuzzy minimal structure defined on X be $M = \{\overline{0}, \{e.4, g.8\}, \{e.6, g.9\}, \{e.5, g.7\}, \{e.5, g.2\}, \{e.8, g.7\}, \{e.3, g.2\},\{e.4, g.7\}, \{e.4, g.2\}, \{e.6, g.7\}, \{e.5, g.8\}, \{e.8, g.8\}, \{e.6, g.8\}, \{e.8, g.9\}, 1\}$.

Then M-semi open sets are $\overline{0}$,$\{e.3, g.2\}$, $\{e.4, g.2\}$, $\{e.4, g.7\}$, $\{e.5, g.2\}$, $\{e.5, g.7\}$, $\{e.5, g.8\}$, $\{e.6, g.7\}$, $\{e.6, g8\}$, $\{e.6, g.9\}$, $\{e.6, g.1\}$, $\{e.7, g.9\}$, $\{e.7, g1\}$, $\{e.8, g.8\}$, $\{e.8, g.9\}$, $\{e.8, g1\}$, $\{e.9, g.9\}$, $\{e.9, g1\}$, $\overline{1}$.

Choose $\Gamma = \{e.4, g.7\}$, then by computation we get: M-$sFr(\Gamma) = M$-$sCl(\Gamma)\wedge$ M-$sCl(\Gamma^c) = \{a.5, b.3\}$,

$\Gamma\vee$ M-$sFr(\Gamma) = \{a.5, b.7\}$, M-$sCl(\Gamma) = \{a.5, b.8\} \neq \{a.5, b.7\} = \Gamma\vee$ M-$sFr(\Gamma)$.

Theorem 25. *Let* Γ *be a fuzzy set in fuzzy minimal space* (X, M)*, then the following hold:*

(1) M-$sFr(\Gamma) = M$-$sFr(\Gamma^c)$;
(2) If Γ is fuzzy M-semi closed, then M-$sFr(\Gamma) \leq \Gamma$;
(3) If Γ is fuzzy M- semi open, then M-$sFr(\Gamma) \leq \Gamma^c$;
(4) Let $\Gamma \leq \mu$ and $\mu \in FMSC(X)$ (respectively, $\mu \in FMSO(X)$). Then, M-$sFr(\Gamma) \leq \mu$ (respectively, M-$sFr(\Gamma) \leq \mu^c$);
(5) $(M$-$sFr(\Gamma))^c = M$-$sInt(\Gamma)\vee$ M-$sInt(\Gamma^c)$;

(6) M-$sFr(\Gamma) \leq M$-$Fr(\Gamma)$;
(7) M-$sCl(M$-$sFr(\Gamma)) \leq M$-$Fr\ (\Gamma)$.

Theorem 26. *Let* Γ *be a fuzzy set in an fuzzy minimal space* X*. Then:*

(1) M-$sFr(\Gamma) = (M$-$sCl(\Gamma)) - (M$-$sInt(\Gamma))$;
(2) M-$sFr(M$-$sInt(\Gamma)) \leq M$-$sFr(\Gamma)$;
(3) M-$sFr(M$-$sCl(\Gamma)) \leq M$-$sFr(\Gamma)$;
(4) M-$sInt(\Gamma) \leq \lambda - (M$-$sFr(\Gamma))$.

Theorem 27. *Let* λ *and* μ *be fuzzy* M*- sets in a fuzzy minimal structure space* (X, M)*. Then* M-$sFr(\lambda \vee \mu) \leq M$-$sFr(\lambda)\vee$ M-$sFr(\mu)$.

The reverse inequality, in general is not true as shown in the following example.

Example 28. In the fuzzy minimal structure M of Example 24, we select fuzzy set $\alpha = \{e.4, g.7\}$ and $\beta = \{e.6, g.2\}$, then computations gives

M-$sFr(\alpha) = M$-$sCl(\alpha)\wedge$ M-$sCl(\alpha^c) = \{e.5, g.8\} \wedge \{e.6, g.3\} = \{e.5, g.3\}$
M-$sFr(\beta) = M$-$sCl(\beta)\wedge$ M-$sCl(\beta^c) = \{e.6, g.2\} \wedge \{e.5, g.8\} = \{e.5, g.2\}$
M-$sFr(\alpha \vee \beta) = M$-$sCl(\alpha \vee \beta) \wedge M$-$sCl(\alpha \vee \beta)^c = \{e.6, g.8\} \wedge \{e.4, g.3\} = \{e.4, g.3\}$
M-$sFr\alpha \vee M$-$sFr\beta = \{e.5, g.3\} \vee \{e.5, g.2\} = \{e.5, g.3\} \not\leq \{e.4, g.3\} = M$-$sFr(\alpha \vee \beta)$
Now we show that M-$sFr\gamma\wedge$ M-$sFr\delta \not\leq M$-$sFr(\gamma \wedge \delta)$
For this choose $\gamma = \{e.4, g.8\}$ and $\delta = \{e.6, g.3\}$. Computations gives
M-$sFr(\gamma) = M$-$sCl(\gamma)\wedge$ M-$sCl(\gamma^c) = \{e.5, g.8\} \wedge \{e.6, g.3\} = \{e.5, g.3\}$
M-$sFr(\delta) = M$-$sCl(\delta\ )\wedge$ M-$sCl(\delta^c) = \{e.6, g.3\} \wedge\{e.5, g.8\} = \{e.5, g.3\}$
M-$sFr(\gamma \wedge \delta) = M$-$sCl(\gamma \wedge \delta) \wedge M$-$sCl(\gamma \wedge \delta)^c = \{e.4, g.3\} \wedge \{e.6, g.8\} = \{e.4, g.3\}$
M-$sFr(\gamma)\wedge$ M-$sFr(\delta) = \{e.5, g.3\} \wedge \{e.5, g.3\} = \{e.5, g.3\}$
M-$sFr(\gamma)\wedge$ M-$sFr(\delta) = \{e.5, g.3\} \not\leq \{e.4, g.3\} = M$-$sFr(\gamma \wedge \delta)$

Theorem 29. *For any fuzzy* M*-sets* λ *and* μ *in a fuzzy minimal space* (X, M)*.* M-$sFr(\lambda \wedge \mu) \leq (M$-$sFr(\lambda)\wedge$ M-$sCl(\mu)) \vee (M$-$sFr(\mu)\wedge$ M-$sCl(\lambda))$

Proof. M-$sFr(\lambda \wedge \mu) = M$-$sCl(\lambda \wedge \mu) \wedge M$-$sCl(\lambda \wedge \mu)^c$
$\leq (M$-$sCl(\lambda)\wedge$ M-$sCl(\mu)) \wedge (M$-$sCl(\lambda)^c\vee$ M-$sCl(\mu^c)$
$= M$-$sCl(\lambda)\wedge(M$-$sCl(\mu)\wedge$ M-$sCl(\lambda^c)$ $\vee(M$-$sCl(\lambda)\wedge(M$-$sCl(\mu)\wedge$ M-$sCl(\mu^c))$
$= (M$-$sFr(\lambda)\wedge$ M-$Cl(\mu)) \vee (M$-$sFr(\mu)\wedge$ M-$sCl(\lambda))$.
Therefore, M-$sFr(\lambda \wedge \mu) \leq (M$-$sFr(\lambda)\wedge$ M-$sCl(\mu)) \vee (M$-$sFr(\mu)\wedge$ M-$sCl(\lambda))$ □

The reverse inequality in general is not true as shown by the following example:

Example 30. In the fuzzy minimal structure M of Example 24, we choose fuzzy set $\gamma = \{e.4, g.8\}$ *and* $\delta = \{e.6, g.3\}$, then computations gives

$(M\text{-}sFr(\gamma) \wedge M\text{-}sCl(\delta)) = \{e.5, g.8\} \wedge \{e.6, g.3\} = \{e.5, g.3\}$
$(M\text{-}sFr(\delta) \wedge M\text{-}sCl(\gamma)) = \{e.5, g.3\} \wedge \{e.5, g.8\} = \{e.5, g.3\}$
$(M\text{-}sFr(\gamma) \wedge M\text{-}sCl(\delta)) \vee (M\text{-}sFr(\gamma) \wedge M\text{-}sCl(\delta)) = \{e.5, g.3\} \wedge \{e.5, g.3\}$
$= \{e.5, g.3\} \not\leq \{e.4, g.3\} = M\text{-}sFr(\gamma \wedge \delta)$

Theorem 31. *Let $f : X \to Y$ be a fuzzy M- continuous function, then M-$sFr(f^{-1}(\mu)) \leq f^{-1}(M\text{-}sFr(\mu))$*

Proof. Since f is fuzzy M-continuous function and μ be a fuzzy M set in Y. Then

$$\begin{aligned} M_X\text{-}sFr(f^{-1}(\mu)) &= M_X\text{-}sCl(f^{-1}(\mu)) \wedge M_X\text{-}sCl(f^{-1}(\mu^c)) \\ &\leq f^{-1}(M_Y\text{-}sCl(\mu)) \wedge M_X\text{-}sCl(f^{-1}(\mu)^c) \\ &= f^{-1}(M_Y\text{-}sCl(\mu)) \wedge f^{-1}(M_Y\text{-}sCl(\mu^c) \\ &= f^{-1}(M_Y\text{-}sCl(\mu) \wedge M_Y\text{-}sCl(\mu^c) \\ &= f^{-1}(M_Y\text{-}sFr(\mu)). \end{aligned}$$

Therefore $M_X\text{-}sFr(f^{-1}(\mu)) \leq f^{-1}(M_Y\text{-}sFr(\mu))$. □

4. Fuzzy rarely-M sets

The aim of this sectionis to define fuzzy rarely- M sets and fuzzy dense-M sets. Some properties of these sets are studied. It is also shown that the non-zero (respectively, non universal) fuzzy rare- M set (respectively, fuzzy dense-M set) is not fuzzy open-M set (respectively, fuzzy closed-M set).

Definition 32. *A fuzzy set λ on X is said to be a fuzzy rare-M set, if M-$Int(\lambda) = \overline{0}$.*

Definition 33. *Let λ be a fuzzy set on X. It is said to be a fuzzy dense-M set, if M-$Cl(\lambda) = \overline{1}$.*

Example 34. Let $X = \{e, g, h\}, M = \{\overline{0}, \overline{1}, \Gamma, \mu, \nu\}$, where $\Gamma = \{(e, 0.3), (g, 0.2), (h, 0)\}$ $\mu = \{(e, 0.1), (g, 0), (h, 0.2)\}, \nu = \{(e, 0), (g, 0.1), (h, 0.1)\}$. Let $\alpha = \{(e, 0), (g, 0), (h, 0.1)\}, \beta = \{(e, 1), (g, 0.9), (h, 1)\}, \delta = \{(e, 0.3), (g, 0.2), (h, 0.1)\}$. Then M-$Int(\alpha) = \overline{0}$ gives α is a fuzzy rare M -set. Again M-$Cl(\beta) = \overline{1}$ gives β is fuzzy dense M -set. M-$Int(\delta) \neq \overline{0}$ and M-$Cl(\delta) \neq \overline{1}$. This gives that δ is neither fuzzy rarely-M set and nor fuzzy dense- M set.

Theorem 35. *A fuzzy set λ on X is a fuzzy rare-M set, if and only if λ^c is a fuzzy dense -M set.*

Proof. Let λ be fuzzy rare-M set.Then M-$Int(\lambda) = \overline{0}$ or $\overline{1}-$ M-$Int(\lambda) = \overline{1} - \overline{0}$. M-$Cl(\overline{1} - \lambda) = \overline{1}$ or M-$Cl(\lambda^c) = \overline{1}$ and conversely, let λ^c be fuzzy rare-M set.Then M-$Cl(\lambda^c) = \overline{1}$ or $\overline{1}-$ M-$Cl(\lambda) = \overline{1} - \overline{1}$. That is M-$Int(\overline{1} - \lambda) = \overline{0}$ or M-$Int(\lambda^c) = \overline{0}$. □

Theorem 36. *A fuzzy set λ on X is both a fuzzy open -M set and a fuzzy rare - set, if and only if it is the $\overline{0}$ set.*

Proof. Let λ be a fuzzy set which is a fuzzy open -M set and a fuzzy rare-M set, if and only if $M\text{-}Int(\lambda) = \lambda = \overline{0}$. □

Corollary 37. *A fuzzy set λ on X is both an fuzzy closed M-set and fuzzy dense M-set, if and only if it is $\overline{1}$ set.*

Remark 38. Since fuzzy M-open sets are fuzzy open-M sets, a fuzzy M-open set other than $\overline{0}$ would be a fuzzy rare- M set gives a contradiction. Hence no fuzzy M-open set other than $\overline{0}$ is a fuzzy rare-M set. Similarly the fuzzy M -closed set, which is also a fuzzy dense -M set is $\overline{1}$.

Remark 39. A fuzzy set λ on X can be both a fuzzy rare-M set and a fuzzy dense -M set.This follows from the following example.

Example 40. Let $X = \{a, b, c\}$ and $M = \{\overline{0}, \overline{1}, \lambda\}$, where $\lambda = \{(a, 0.1), (b, 0.2), (c, 0.6)\}$. Consider the set $\mu = \{(a, 0.1), (b, 0.1), (c, 0.6)\}$. Then $M\text{-}Cl(\mu) = \overline{1}$, $M\text{-}Int(\mu) = \overline{0}$. So μ is both fuzzy rare-M set and a fuzzy dense -M set.

Theorem 41. *A fuzzy set λ of (X, M) is both a fuzzy rare-M set and a fuzzy dense-M set, if and only if there exists neither a fuzzy M- open set contained in λ nor a fuzzy M-closed set containing λ , except for $\overline{0}$ and $\overline{1}$ respectively.*

Proof. A fuzzy set λ of (X, M) is both a fuzzy rare- M set and a fuzzy dense-M set,then by definition $M\text{-}Cl(\lambda) = \overline{1}$ and $M\text{-}Int(\lambda) = \overline{0}$. This implies that λ contains neither fuzzy M- open set except $\overline{0}$ nor contained in fuzzy M - closed set except $\overline{1}$ and conversely. □

Remark 42. If a fuzzy set λ of (X, M) is both a fuzzy rare- M set and a fuzzy dense- M set. Then λ is neither an fuzzy M-open set nor a fuzzy M-closed set. Converse needs not be true as follows from the following example.

Example 43. Let $X = \{a, b.c\}, M = \{\overline{0}, \overline{1}, \lambda\}$. $\lambda = \{(a, 0.2), (b, 0), (c, 0)\}$. Let $\mu = \{(a, 0), (b, 0.5), (c, 0)\}$. Then μ is neither fuzzy M-open set nor fuzzy M-closed set. But $M\text{-}Int(\mu) = \overline{0}$, $M\text{-}Cl(\mu) = \{(a, 0.8), (b, 1), (c, 1)\} \neq \overline{1}$, that is μ is not a fuzzy dense-M set.

Theorem 44. *In a fuzzy minimal space*(X, M) :

1. $\overline{1}$ is a fuzzy dense-M set, but it is not a fuzzy rare-M set.
2. $\overline{0}$ is a fuzzy rare-M set,but it is not a fuzzy dense-M set.
3. Arbitrary intersection (respectively union) of fuzzy rare-M (respectively fuzzy dense-M) set is fuzzy rare-M (respectively fuzzy dense-M) set.

Proof. (1) and (2) are obvious . (3) Let $\lambda = \wedge\{\Gamma_\alpha : \alpha \in \Delta$ be an arbitrary intersection of fuzzy rare M sets, that is M-$Int(\Gamma_\alpha) = \overline{0}$, for each $\alpha \in \Delta$.Then $\wedge\{M\text{-}Int(\Gamma_\alpha) : \alpha \in \Delta\} = \overline{0}$. We know that $\overline{0} = \wedge\{M\text{-}Int(\Gamma_\alpha) : \alpha \in \Delta\} \geq M\text{-}Int \wedge \{(\Gamma_\alpha) : \alpha \in \Delta\}$. This implies that $\overline{0} \geq M\text{-}Int(\Gamma)$ or $M\text{-}Int(\Gamma) = \overline{0}$. Similarly, it can be shown that arbitraryunion of fuzzy dense -M sets is fuzzy dense -M set. □

Remark 45. Finite unions of fuzzy rare- Msets need not be fuzzy rare- M set. This follows from the following example:

Example 46. Let $X = \{a, b, c\}$, $M = \{\overline{0}, \overline{1}, \lambda, \mu, \nu\}$, where $\lambda = \{(a, .4), (b, .8), (c, .4)\}$, $\mu = \{(a, .1), (b, .4), (c, .1)\}$, $\nu = \{(a, .2), (b, .5), (c, .6)\}$

Let $\alpha = \{(a, .1), (b, .4), (c, 0)\}$ $\beta = \{(a, .1), (b, .3), (c, .5)\}, \gamma = \{(a, 0), (b, 0), (c, .2)\}$. Then M-$Int(\alpha) = 0$, M-$Int(\beta) = \overline{0}$. Here $\alpha \vee \beta = \{(a, .1), (b, 0.4), (c, .5)\}$ gives M-$Int(\alpha \vee \beta) \neq \overline{0}$

Theorem 47. *A fuzzy set λ of (X, M) is a fuzzy dense - M (respectively fuzzy rare-M) set, if and only if for e very fuzzy open- M (respecti vely fuzzy closed-M) set μ satisfying $\lambda \leq \mu$ (respectively $\mu \leq \lambda$) we have M-$Cl(\lambda) \geq \mu$ (respectively M-$Int(\lambda) \leq \mu$).*

Proof. First, assume that λ is a fuzzy dense-M set and take a fuzzy open -M set μ with $\lambda \leq \mu$, then M-$Cl(\lambda) = \overline{1} \geq \mu$. Conversely, let the given condition hold and take $\mu = \overline{1}$.Then μ is an fuzzy open -M setand $\lambda \leq \mu$, so M-$Cl(\lambda) \geq \mu = \overline{1}$, that is, M-$Cl(\lambda) = \overline{1}$. Hence λ is a fuzzy dense- M set. The other part can be proved similarly. □

Remark 48. A fuzzy set λ of (X, M) is a fuzzy rare -M set, if there exist no fuzzy open-M set other $\overline{0}$ contained in λ.

Theorem 49. *The union (respectively, intersection) of fuzzy dense-M (respectively fuzzy rare-M) sets and fuzzy closed-M(respectively fuzzy open-M)sets is fuzzy dense-M (respectively fuzzy rare-M) set.*

Proof. 1) Let λ be a fuzzy dense- M set and μ is a fuzzy closed - M set. If v is a fuzzy open- M set with $\lambda \vee \mu \leq v$, then $\lambda \leq v$ and so M-$Cl(\lambda) \geq v$ by Theorem 47. Now M-$Cl(\lambda \vee \mu) \geq M$-$Cl(\lambda)\vee$ M-$Cl(\mu) \geq \mu \vee v \geq v$. That is the union of a fuzzy dense-M set and a fuzzy closed- M set is a fuzzy dense - M set by Theorem 47. Now we prove the result for fuzzy rarely- M set. 2) Let λbe a fuzzy rare- M set and μ is a fuzzy open- M set. If v is a fuzzy closed-M set with$v \leq \lambda \wedge \mu$, then$v \leq \lambda$ and so M-$Int(\lambda) \leq v$ by Theorem 47. Now M-$Int(\lambda \wedge \mu) = M$-$Int(\lambda)\wedge$ M-$Int(\mu) \leq \mu \wedge v \leq v$. That is the intersection of a fuzzy rare- M set and a fuzzy open- M set is a fuzzy rare - M set by Theorem 47. □

Theorem 50. *M-$Cl(\lambda)$(resp. M-$Int(\lambda)$)is a fuzzy dense-M(respectively fuzzy rare- M) set whene ver λ is a fuzzy dense-M (respectively fuzzy rare- M) set.*

Proof. Let λ be a fuzzy dense-M set. Hence $M\text{-}Cl(\lambda) = \overline{1}$.

This implies that $M\text{-}Cl(M\text{-}Cl(\lambda)) = M\text{-}Cl(1) = \overline{1}$, this proves that $M\text{-}Cl(\lambda)$ is be a fuzzy dense-M set. □

Remark 51. Let (X, M) be a fuzzy minimal structure formed from the fuzzy topological space (X, T).Then, if a subset λ of (X, M) is a fuzzy dense set, it is a fuzzy dense M-set, since $Cl(\lambda) \leq M\text{-}Cl(\lambda)$.

Similarly, if a fuzzy set λ of $(X, M)$is a fuzzy rare set, then it is a fuzzy rare-M set.

Remark 52. A fuzzy set λ of (X, M) is said to be a fuzzy closed rare -M (respectively, fuzzy open dense-M) set if the fuzzy set λ is both a fuzzy closed -M set and a fuzzy rare-M set (respecti vely a fuzzy open-M set and a fuzzy dense-M set).

Example 53. Let $X = \{a, b, c\}$, $M = \{\overline{0}, \overline{1}, \lambda, \mu\}$, where $\lambda = \{(a, 0.7), (b, 0.4), (c, 0)\}$, $\mu = \{(a, 0.5), (b, 1), (c, 0)\}$.

Let $\alpha = \{(a, 0.3), (b, 0), (c, 1)\}$, $M\text{-}Int(\alpha) = \overline{0}$ and $M\text{-}Cl(\alpha) = \{(a, 0.3), (b, 0), (c, 1)\} = \alpha$. Then α is fuzzy closed rare- M set.

Remark 54. Let λ be a fuzzy dense-M set that is $M\text{-}Cl(\lambda) = \overline{1}$.Then there exists an fuzzy open dense -M set μ containing λ.

Example 55. Let $X = \{a, b, c\}$, $M = \{\overline{0}, \overline{1}, \lambda, \mu, \nu\}$, Where $\lambda = \{(a, 0.7), (b, 0.4), (c, 1)\}$, $\mu = \{(a, 0.5), (b, 1), (c, 1)\}$, $\nu = \{(a, 0.6), (b, 1), (c, 1)\}$.

Let $\alpha = \{(a, 0.7), (b, 1), (c, 1)\}$, $\beta = \{(a, 0.6), (b, 0.7), (c, 1)\}$. Then $M\text{-}Int(\alpha) = \alpha$ and $M\text{-}Cl(\alpha) = \overline{1}$, $M\text{-}Cl(\beta) = \overline{1}$ implies $\beta \leq \alpha$.

Theorem 56. *A fuzzy set Γ of (X, M) is a fuzzy closed rare-M set, if and only if Γ is a fuzzy closed -M set which does not contain any fuzzy open -M set other than $\overline{0}$.*

Proof. Let Γ be a fuzzy closed rare -M set, then by definition $M\text{-}Int(\Gamma) = \overline{0}$ and $M\text{-}Cl(\Gamma) = \Gamma$.This shows Γ is a fuzzy closed -M set which does not contain any fuzzy open -M set other than $\overline{0}$. □

Theorem 57. *A fuzzy set λ of (X, M) is a fuzzy open dense-M set, if and only if λ is a fuzzy closed - M set which does not contain any fuzzy open M-set other than $\overline{1}$.*

Proof. Let λ be a fuzzy open dense -M set,then by definition $M\text{-}Int(\lambda) = \lambda$ and $M\text{-}Cl(\lambda) = \overline{1}$. This shows that Γ is a fuzzy open-M set, which does not contain any fuzzy closed-M set other than $\overline{1}$. □

Theorem 58. *If a fuzzy set λ of (X, M) is a fuzzy dense-M set, then $M\text{-}Fr(\lambda) = \overline{1} -$ $M\text{-}Int(\lambda)$.*

Proof. Let λ be a fuzzy dense M set that is $M\text{-}Cl(\lambda) = \overline{1}$, then $M\text{-}Fr(\lambda) = M\text{-}Cl(\lambda) \wedge M\text{-}Cl(\overline{1} - \lambda) = \overline{1} \wedge M\text{-}Cl(\overline{1} - \lambda) = M\text{-}Cl(\overline{1} - \lambda) = \overline{1} - M\text{-}Int(\lambda)$ □

Theorem 59. *If fuzzy set λ of (X, M) is a fuzzy rare M set, then $M\text{-}Fr(\lambda) = M\text{-}Cl(\lambda)$.*

Proof. Let λ be a fuzzy rare M set.Then $M\text{-}Int(\lambda) = \overline{0}$. Now $M\text{-}Fr(\lambda) = M\text{-}Cl(\lambda) \wedge M\text{-}Cl(\overline{1} - \lambda) = M\text{-}Cl(\lambda) \wedge (\overline{1} - M\text{-}Int(\lambda)) = M\text{-}Cl(\lambda) \wedge (\overline{1} - \overline{0}) = M\text{-}Cl(\lambda)$ □

Theorem 60. *A fuzzy set λ of (X, M) is a both fuzzy dense -M set and a fuzzy rare-M set, if and only if $M\text{-}Fr(\lambda) = \overline{1}$.*

Proof. Necessity follows from the Theorem 58. Conversely , let $M\text{-}Fr(\Gamma) = \overline{1}$, then $M\text{-}Cl(\lambda) \wedge M\text{-}Cl(\overline{1} - \lambda) = \overline{1}$ i.e $M\text{-}Cl(\lambda) = \overline{1} \rightarrow (i)$ and $M\text{-}Cl(\overline{1} - \lambda) = \overline{1} \rightarrow (ii)$ .By (i) we get λ is fuzzy dense M set and from (ii) $M\text{-}Cl(\overline{1} - \lambda) = \overline{1} - M\text{-}Int(\lambda) = \overline{1}$ i.e $M\text{-}Int(\lambda) = \overline{0}$, hence λ is fuzzy rare-M set. This completes the proof. □

Theorem 61. *In fuzzy minimal structure (X, M)*

1. $\overline{0}$ is a fuzzy closed rare-M set;
2. Arbitrary intersections of fuzzy closed rare -M sets are fuzzy closed rare -M set.

Proof. (1) is obvious. (2) Let $\{\lambda_i : i \in I\}$ be a collection of fuzzy closed rare -M sets, i.e $M\text{-}Cl(\lambda_i) = \lambda_i$ and $M\text{-}Int(\lambda_i) = \overline{0}\ \forall\ i = 1, 2, 3 \ldots$..Consider $M\text{-}Cl \wedge \{\lambda_i : i \in I\} = \wedge\{M\text{-}Cl(\lambda_i) : i \in I\} = \wedge\{\lambda_i : i \in I\}$.This proves that arbitrary intersection of fuzzy closed- Msets is fuzzyclosed-M set. Again, since each λ_i is rare- M set, therefore, $M\text{-}Int(\lambda_i) = \overline{0}$. Consider $M\text{-}Int \wedge \{\lambda_i : i \in I\} \wedge \{M - Int(\lambda_i) : i \in I\} = \wedge\{\overline{0}\} = \overline{0}$. This completes the proof. □

Remark 62. $\overline{1}$ is not fuzzy closed rare- M set, since $\overline{1}$ is a fuzzy closed -M set but not a fuzzy rare-M set. Also arbitrary unions of fuzzy closed rare-M sets need not be fuzzy closed rare M sets.

5. Fuzzy rarely-M continuous functions

In this section we will define fuzzy rarely-M continuous function and give some characterizations of such functions.

Definition 63. *A function $f : (X, M_X) \rightarrow (Y, M_Y)$ is called fuzzy rarely-M continuous function at x_α, if for a fuzzy open -M set μ containing $f(x_\alpha)$, there exists a fuzzy rare-M set $R_\mu$with $\mu \wedge M\text{-}Cl(R_\mu) = \overline{0}$ and a fuzzy open-M set Γ containing x_α, such that $f(\Gamma) \leq \mu \vee R_\mu$.*

Remark 64. Every fuzzy M- continuous function is fuzzy rarely M- continuous. However converse is not true in general.

Theorem 65. *Let $g : (X, M_X) \to (Y, M_Y)$ be a fuzzy M continuous and injection, then g preserves fuzzy rare-M sets.*

Proof. Suppose that Γ_u is a fuzzy rare-M set in (X, M_X), but $g(\Gamma_u)$ is not a fuzzy rare-M set in (Y, M_Y). Then $M\text{-}Int(g(\Gamma_u)) \neq \overline{0}$.

So, there exist a fuzzy open -M set v other than $\overline{0}$ contained in $g(\Gamma_u)$. Since g is injective, $g^{-1}(v) \leq \Gamma_u$. Since g is fuzzy M continuous function, therefore, $g^{-1}(v)$ is a fuzzy open-M set.But this contradicts the fact that Γ_u is fuzzy rare-M set. Hence g preserves fuzzy rare - M sets. □

Theorem 66. *The following statements are a like for a function $f : (X, M_X) \to (Y, M_Y)$.*

1. The function f is fuzzy rarely -M continuous at a fuzzy point x_α of (X, M_X);
2. For each fuzzy open -M set μ containing $f(x_\alpha)$, there exists a fuzzy open -M set Γ containing x_α, such that $M\text{-}Int(f(\Gamma) \wedge (\overline{1} - \mu)) = \overline{0}$;
3. For each fuzzy open -M set μ containing $f(x_\alpha)$, there exists a fuzzy open -M set Γ containing x_α such that $M\text{-}Int(f(\Gamma)) \leq M\text{-}Cl(\mu)$.

Proof. 1. $\to$ **2**. Let μ be a fuzzy open -M set containing $f(x_\alpha)$ in (Y, M_Y). Since μ is fuzzy open -M set, therefore, $\mu = M\text{-}Int_Y(\mu) < M\text{-}Int_Y(M\text{-}Cl_Y(\mu))$. Also $M\text{-}Int_Y(M\text{-}Cl_Y(\mu))$ is a fuzzy open -M set containing $f(x_\alpha)$. Since f is fuzzy rarely M-continuous function,there exists a fuzzy rare-M set R_μ with $M\text{-}Int_Y(M\text{-}Cl_Y(\mu))\wedge$ $M\text{-}Cl_Y(R_\mu) = \overline{0}$ and a fuzzy open -M set Γ containing x_α, such that $f(\Gamma) \leq M\text{-}Int_Y(M\text{-}Cl_Y(\mu)) \vee R\mu$. We have

$$\begin{aligned} M\text{-}Int_Y(f(\Gamma) \wedge (\overline{1} - \mu)) &= M\text{-}Int_Y(f(\Gamma))\wedge\ M\text{-}Int_Y(\overline{1} - \mu) \\ &\leq M\text{-}Int_Y(M\text{-}Cl_Y(\mu) \vee R_\mu)\ \wedge\ M\text{-}Int_Y(\overline{1} - \mu) \\ &= M\text{-}IntY_Y(M\text{-}Cl(\mu)) \vee (M\text{-}Int(R_\mu)) \wedge (\overline{1}-\ M\text{-}Cl_Y(\mu)) \\ &= M\text{-}Int_Y(M\text{-}Cl_Y(\mu)) \wedge (\overline{1}-\ M\text{-}Cl_Y(\mu)) = \overline{0}. \end{aligned}$$

2. $\to$ **3**. Consider (2) $M\text{-}Int(f(\Gamma))\wedge$ $M\text{-}Int(\overline{1} - \mu) = \overline{0}$. This implies $M\text{-}Int(f(\Gamma)) \wedge (\overline{1}-\ M\text{-}Cl(\mu)) = \overline{0}$. This implies $M\text{-}Int(f(\Gamma)) \leq M\text{-}Cl(\mu)$, this proves (3).

3. $\to$ **1**. Let μ be a fuzzy open -M set containing $f(x_\alpha)$. Then by (3), there exists a fuzzy open- M set Γ containing x_α, such that $M\text{-}Int(f(\Gamma)) \leq M\text{-}Cl(\mu))$. We consider $f(\Gamma) = (f(\Gamma)-\ M\text{-}Int(f(\Gamma))\vee\ M\text{-}Int(f(\Gamma))$

   $\leq (f(\Gamma)-\ M\text{-}Int(f(\Gamma)))\vee\ M\text{-}Cl(\mu)$

$= (f(\Gamma)- M\text{-}Int(f(\Gamma))) \vee \mu \vee (M\text{-}Cl(\mu) - \mu)$

$= ((f(\Gamma)- M\text{-}Int(f(\Gamma))) \wedge (\overline{1} - \mu)) \vee \mu \vee (M\text{-}Cl(\mu) - \mu)$.

put $R_1 = (f(\Gamma)- M\text{-}Int(f(\Gamma))) \wedge (\overline{1} - \mu)$ and $R_2 = (M\text{-}Cl(\mu) - \mu)$.Then R_1 and R_2 are fuzzy rare-M sets. Moreover $R_\mu = R_1 \vee R_2$ is a rare -M set, such that $M\text{-}Cl(R_\mu) \wedge \mu = \overline{0}$ and $f(\Gamma) \leq \mu \vee R_\mu$. This shows that f is rarely- M continuous.

□

Theorem 67. *Let $f : (X, M_X) \to (Y, M_Y)$ be a fuzzy rarely- M continuous function and g: $(Y, M_Y) \to (Z, M_Z)$ a one to one fuzzy M continuous function. Then $g \circ f : (X, M_X) \to (Z, M_Z)$ is fuzzy rarely-M continuous.*

Proof. Let $x_\alpha$be a fuzzy point of $(X, M_X)$and $g \circ f(x_\alpha) \in \mu$, where μ is an open M-set in Z. By hypothesis, g is fuzzy M-continuous, therefore there exists a fuzzy open M-set ν in Y containing $f(x_\alpha)$ such that $g(\nu) \leq \mu$. Since f is fuzzy rarely- M continuous therefore there exists a fuzzy rare -M-set R_G in Y with $M\text{-}Cl(R_G) \wedge \nu = \overline{0}$, and a fuzzy open-$M$ set λ containing x_α such that $f(\lambda) \leq \nu \vee R_G$. Since fuzzy rare M-set are preserved under fuzzy M continuous function, therefore $g(R_G)$ is fuzzy rare M-set in Z. Now $g \circ f(\lambda) \leq g(\nu \vee R_G) \leq g(\nu) \vee g(R_G) \leq \mu \vee g(R_G)$. This proves that $g \circ f$ is fuzzy rarely- M continuous function. This completes the proof. □

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