ON SOFT LA-MODULES AND EXACT SEQUENCES

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Abstract. We introduce the concepts of soft LA-modules, Soft homomorphisms and the exact sequences of soft LA-modules. Some properties concerning the exact sequences of the LA- modules are investigated. A functional approach to study the soft LA-modules is adopted so that a characterization theorem of soft LA-modules is obtained.
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1. Introduction

To deal with uncertainties, though many theories have been developed, yet difficulties are seem to be there. In order to cope with these uncertainties, the theory of soft sets was first proposed by D. Molodtsov [12] in 1999. Nowadays, the theory of soft sets has been successfully applied into a number of mathematical disciplines and scientific fields.

We notice that P. K. Maji, R. Biswas and R. Roy [10], [11] have recently applied the concept soft sets in decision making problems. We also observe that soft set parameterization reduction and a comparison of soft sets with attributes reduction in rough set theory was given by Chen [5]. Later on, A. Sezgin et al., [16], introduced the union soft subnear-rings and union soft ideals of a near-ring.

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H. Aktas and N. Çağman [2], first applied the theory of soft sets in algebraic structures. In addition, X. Liu et al., [8], established some fuzzy isomorphism theorems of soft rings. They also considered the fuzzy ideals of soft rings. In [9], X. Liu et al., have provided the proof of isomorphism theorems of soft rings. In [24], Q. M. Sun et al., have discussed and investigated some of its basic properties of soft modules.

The Left Almost Ring (LA-ring) is actually an off shoot of LA-semigroup and LA-group. In fact, an LA-ring is a non-commutative and non-associative algebraic structure. Due to its atypical characteristics, the class of LA-rings has been emerging as a useful non-associative class which intuitively would have reasonable contribution to enhance non-associative ring theory. By an LA-ring, we mean a non-empty set R together two binary operations "+" and "·", such that under "+" it is an LA-group and under "·" it is an LA-semigroup and distributive laws hold both from left and right .

In [21], T. Shah et al., generalized a commutative semigroup ring and established several basic results of left almost ring (LA-ring) of finitely nonzero functions. Moreover, T. Shah and I. Rehman [22], have considered ideals in LA-rings. Consequently, the theory of ideals of LA-rings was applied in fuzzy sets. Moreover, T. Shah et al., also considered intuitionistics fuzzy sets and soft sets in LA-rings. For example, T. Shah et al., [19], have applied the concept of intuitionistic fuzzy sets and established some useful results. In [15], some computational research through Mace4, has been obtained and some interesting characteristics of LA-rings are explored. It is noted that in [17], T. Shah et al., have introduced a new approach to apply the Molodtsov's soft set theory to a class of non-associative rings. And in [18], T. Shah and Asima Razzaque have further discussed some basic properties regarding soft M-systems, soft P-systems and soft I-systems in a non-associative left almost rings. T. Shah et al., [20], have promoted the concept of LA-modules and establish some isomorphism theorems and the direct sum of LA-modules. For more information of LA-rings, the readers are referred to the papers ([14], [23]).

In this paper, we first consider the soft LA-modules. Then we established some results of substructures of soft LA-modules, homomorphisms and finally, we consider exact sequence of soft LA-modules and provide some useful results of exact sequences.

2. Basic definitions and some preliminary results

In this section, we first cite some basic definitions which are closely related with the soft sets and LA-modules.

Definition 1. *[12, Def: 2.1] Suppose that U be an initial universe. Let E be a set of parameters. Then we denote the power set of U by $P(U)$ and a non-empty subset α of E. We now call a pair (F, α) a soft set over an initial universe U, where F is a mapping given by $F : \alpha \to P(U)$.*

Now, we call a soft set over the initial universe U, a parametrized family of subsets of the universe U. For $\varepsilon \in \alpha$, $F(\varepsilon)$ may be considered as the set of $\varepsilon-$approximate elements of the soft set (F, α). Clearly, a soft set is not a set.

Definition 2. *[13, Def: 2] Let (F, α) and (G, β) be the two non-empty soft sets over a common universe U. Then we call that (F, α) is a soft subset of (G, β) if*
(i) $\alpha \subseteq \beta$ and
(ii) for all $\varepsilon \in \alpha$, $F(\varepsilon) \subseteq G(\varepsilon)$.

The above soft sets can be written as $(F, \alpha)\widetilde{\subset}(G, \alpha)$. If a soft set (G, β) is a soft subset of a soft set (F, α). Then (F, α) is said to be a soft super set of (G, β) and it can be defined as $(F, \alpha)\widetilde{\supset}(G, \beta)$.

Definition 3. *[11, Def: 2.4] Suppose that (F, α) and (G, β) are two non-empty soft sets over a common universe U. Then these soft sets are said to be soft equal if $(F, \alpha)\widetilde{\subset}(G, \beta)$ and $(G, \beta)\widetilde{\subset}(F, \alpha)$.*

Definition 4. *[11, Def: 2.7] A soft set (F, A) over an initial universe U is said to be a $NULL$ soft set and is denoted by Φ if for every $\varepsilon \in \alpha$, $F(\varepsilon) = \emptyset$ (null set).*

Definition 5. *[1, Def: 2.4] (F, α) and (G, β) be two soft sets over a common universe U. Then the bi-intersection of these two soft sets (F, α) and (G, β) is defined as the soft set (K, γ) which satisfies the following conditions:*
(i) $\gamma = \alpha \cap \beta$ and
(ii) for every $x \in \gamma$, $K(x) = F(x) \cap G(x)$.

Now we denote the bi-intersection of these two soft sets by $(F, \alpha)\,\widetilde{\sqcap}\,(G, \beta) = (K, \gamma)$.

Definition 6. *[6, Def: 3.5] Let $(F_i, \alpha_i)_{i \in I}$ be a non-empty family of soft sets over a common universe U. Then the bi-intersection of the family of soft sets is defined to be the soft set (K, γ), such that $\gamma = \cap_{i \in I}\alpha_i$, and $H(x) = \cap_{i \in I}F_i(x)$ for all $x \in \gamma$. This intersection can be written as $\widetilde{\sqcap}_{i \in I}(F_i, \alpha_i) = (K, \gamma)$.*

Definition 7. *[1, Def: 2.7] Consider two non-null soft sets (F, α) and (G, β) over a common universe U, then "AND" of these two soft sets is denoted by $(F, \alpha)\,\widetilde{\wedge}\,(G, \beta)$. It is defined as $(K, \gamma) = (F, \alpha)\,\widetilde{\wedge}\,(G, \beta)$, where $\gamma = \alpha \times \beta$ and for all $(x, y) \in \gamma$, $K(x, y) = F(x) \cap G(y)$.*

Definition 8. *[1, Def: 2.3] Let (F, α) and (G, β) be two non-empty soft sets over a common universe U. Then the intersection of (F, α) and (G, β) is defined as the soft set (K, γ) satisfying the following conditions:*
(i) $\gamma = \alpha \cap \beta$ and
(ii) for all $x \in \gamma$, $H(x) = F(x)$ or $G(x)$.

Hence the intersection of two soft sets (F, α) and (G, β) can be written as $(K, \gamma) = (F, \alpha)\,\widetilde{\cap}\,(G, \beta)$.

Definition 9. *[1, Def: 2.5] Suppose that (F, α) and (G, β) be two non-empty soft sets over U. Then the union of these two soft sets is defined as (K, γ) which is also a soft set, satisfying the following conditions:*
(i) $\gamma = \alpha \cup \beta$ and
(ii) for all $x \in \gamma$,

$$K(x) = \begin{cases} F(x), & \text{if } x \in \alpha - \beta, \\ G(x), & \text{if } x \in \beta - \alpha, \\ F(x) \cup G(x), & \text{if } x \in \alpha \cap \beta. \end{cases}$$

The above union of two soft sets can be written as $(F, \alpha) \widetilde{\cup} (G, \beta) = (K, \gamma)$.

Definition 10. *[1, Def: 2.9] If (F, α) is a soft set over a universe U, then the set supp $(F, \alpha) = \{\varepsilon \in \alpha \mid F(\varepsilon) \neq \phi\}$ is called the support of the soft set (F, α). We now call a soft set to be non null if its support is not equal to the empty set.*

Definition 11. *[21, Def: 1] Consider an LA-ring $(R, +, .)$ with left identity e. An LA-group $(M, +)$ is said to be an LA-module over R if $R \times M \to M$ defined as $(a, n) \mapsto an \in M$, where $a \in R$, $n \in M$ satisfies the followinh conditions:*
(i) $(a + b)n = an + bn$,
(ii) $a(m + n) = am + an$,
(iii) $a(bn) = b(an)$,
(iv) $1.n = n$,
where for all $a, b \in R$, $m, n \in M$.

${}_RM$ is used for left R-LA-module or simply M. Right R-LA-module can be defined in similar manner and is denoted by M_R.

Definition 12. *[20, Def: 2] Consider a left R-LA-module M, a sub LA-group N of M over an LA-ring R is called left R-sub LA-module of M, if $RN \subseteq N$, i.e., $rn \in N$ for all $n \in N$ and $r \in R$. We denote this sub LA-module by $N \leq M$.*

Definition 13. *[20, Def: 3] Let M be an LA-module and $A \subset M$ is a sub LA-module. We define quotient module or factor module M/A by $M/A = \{A + m : m \in M\}$. That is, M/A is the set of equivalence classes of elements of M. An equivalence class is denoted by $A + m$ or by $[m]$. Each element in the class $A + m$ is called a representative of the class.*

For the intersection of LA-modules, sums, products of modules, the Jacobson radical of modules and the LA-module homomorphisms, we have the following results and definitions.

Theorem 1. *[20, Theorem: 2] If A and B are two sub LA-modules of an LA-module M over an LA-ring R, then the intersection i.e., $A \cap B$ is also a sub LA-module of M.*

Corollary 1. *Intersection of any number of sub LA-modules of an LA-module is a sub LA-module.*

Definition 14. *[24, Def: 7] Consider a non-empty family of R-modules i.e.,* $\{M_i \mid i \in I\}$. *Then* $P = \prod\limits_{i\in I} M_i = \{(m_i) \mid m_i \in M_i\}$ *is a set of direct product, if* $(m_i) + (n_i) = (m_i + n_i)$ *and* $r(m_i) = (rm_i)$ *are the operators given on the product, then* P *induces* $\{M_i \mid i \in I\}$, *which is denoted by* $\prod\limits_{i\in I} M_i$*,a left R-module structure called direct product.*

Proposition 1. *[24, Prop: 2] Consider a non-empty family of sub modules* $\{M_i \mid i \in I\}$ *of* M. *Then* $\bigcap\limits_{i\in I} M_i$ *and* $\sum\limits_{i\in I} M_i$ *are all sub modules of* M.

Definition 15. *[24, Def: 8] In the direct product* $\prod\limits_{i\in I} M_i$ *all elements* (m_i), *where* m_i *is zero for almost all* $i \in I$ *except finite one, a sub module of* $\prod\limits_{i\in I} M_i$ *is said to be direct sum of* $\{M_i \mid i \in I\}$, *which can be written as* $\coprod\limits_{i\in I} M_i$ *or* $\bigoplus\limits_{i\in I} M_i$.

Definition 16. *[20, Def: 4(a)] Let* M, N *be the LA-modules over an LA-ring* R. *Then the map* $\varphi : M \longrightarrow N$ *is called an LA-module homomorphism(or simply R-homomorphism) if, for all* $r \in R$ *and* $n, m \in M$

(*i*) $\varphi(n + m) = \varphi(n) + \varphi(m)$

(*ii*) $\varphi(rn) = r\varphi(n)$

Theorem 2. *[20, Theorem: 5] Let* $\varphi : M \longrightarrow N$ *be an LA-module homomorphism from an LA-module* M *to an LA-module* N, *then*

(1) *If* A *is a sub LA-module of* M, *then* $\varphi(A)$ *is a sub LA-module of* N.

(2) *If* B *is a sub LA-module of* N, *then* $\varphi^{-1}(B)$ *is a sub LA-module of* M.

Lemma 1. *[20, Lemma: 2] With the canonical operations, by choosing representatives,* $(A+n)+(A+m) = A+(n+m)$, *the set* M/A *is an LA-group.* A, *the equivalence class of* $0 \in M$ *is the left identity of* M/A. *The map* $\pi : M \longrightarrow M/A$, $\pi(n) = A + n$ *is a surjective LA-group homomorphism.*

Definition 17. *[24, Def: 6] Consider a non-empty family of sub modules* $\{M_i \mid i \in I\}$ *of* M. *Then we the following statements:*

(*i*) $\bigcap\limits_{i\in I} M_i$ *is a sub module of* M *if* $\{M_i \mid i \in I\}$ *be a non-empty family of maximal sub modules of* M. *Then it is called Jacobson radical of the module. It is written as* $radM$.

(*ii*) $\sum\limits_{i\in I} M_i$ *is a sub module of* M *if* $\{M_i \mid i \in I\}$ *be a non-empty family of minimal sub modules of* M. *Then it is said to be Socle of module and is written as* $socM$.

Definition 18. *[24, Def: 9] A sequence of R-homomorphisms and R-modules* $\cdots \longrightarrow P_{n-1} \xrightarrow{f_{n-1}} P_n \xrightarrow{f_n} P_{n+1} \longrightarrow \cdots$ *is said be an exact sequence if* $Imf_{n-1} = kerf_n$ *for all* $n \in \mathbb{N}$. *An exact sequence of the form* $0 \longrightarrow P' \xrightarrow{f} P \xrightarrow{g} P'' \longrightarrow 0$ *is said to be a short exact sequence.*

Proposition 2. *[25, 51] Let $f : M \longrightarrow N$ be R-homomorphism for R-modules M and N. Then we have the following properties:*

(1) $0 \longrightarrow M \xrightarrow{f} N$ *is exact if and only if f is monomorphism.*

(2) $M \xrightarrow{f} N \longrightarrow 0$ *is exact if and only if f is an epimorphism.*

(3) $0 \longrightarrow M \xrightarrow{f} N \longrightarrow 0$ *is exact if and only if f is an isomorphism.*

3. Soft LA-modules and their related properties

In this section, we study the properties of the Soft LA-modules. We first state the following definitions and results.

Definition 19. *Consider a soft set (F, α) over a left R-LA module M. The pair (F, α) is said to be soft LA-module over M if $F(a)$ is a sub LA-module of M for all $\varepsilon \in \alpha$. It can be written as $F(\varepsilon) \leq M$.*

Throughout this paper, M is considered to be a left R-LA module, and α be any non-empty set. The pair (F, α) refers to a soft set over M and $F : \alpha \longrightarrow P(M)$ considered to be a set valued function.

Proposition 3. *Consider two soft LA-modules (F, α) and (G, β) over M. Then the following statements hold:*

(1) $(F, \alpha) \widetilde{\cap} (G, \beta)$ *is a soft LA-module over M.*

(2) *if α and β are disjoint i.e., $A \cap B = \phi$ then $(F, \alpha) \widetilde{\cup} (G, \beta)$ is a soft LA-module over M*

Proof. (1) By definition 8, $(F, \alpha) \widetilde{\cap} (G, \beta) = (K, \gamma)$ is a soft set over soft LA-module M, where $\alpha \cap \beta = \gamma$. Since (F, α) and (G, β) are two soft LA-modules over M, so the $K(x) = F(x) \leq M$ and also $K(x) = G(x) \leq M$ for every $x \in \gamma$. Therefore, it follows that $(F, \alpha) \widetilde{\cap} (G, \beta)$ is a soft LA-module over M.

(2) Using the definition 9, we can write $(F, \alpha) \widetilde{\cup} (G, \beta) = (K, \gamma)$ which is a soft set, where $\gamma = \alpha \cup \beta$ and for every $x \in \gamma$,

$$K(x) = \begin{cases} F(x), & \text{if } x \in \alpha - \beta, \\ G(x), & \text{if } x \in \beta - \alpha, \\ F(x) \cup G(x), & \text{if } x \in \alpha \cap \beta. \end{cases}$$

Given that α and β are disjoint, therefore either $x \in \alpha - \beta$ or $x \in \beta - \alpha$. If $x \in \alpha - \beta$, then $K(x) = F(x) \leq M$ as (F, α) is a soft LA-module. Similarly if $x \in \beta - \alpha$, then $K(x) = G(x) \leq M$ as (G, β) is a soft LA-module. Hence $(F, \alpha) \widetilde{\cup} (G, \beta) = (K, \gamma)$ is a soft LA-module over M. □

Theorem 3. *Let (F, α) and (G, β) be two soft LA-modules over M. Then the following conditions hold:*

(1) *If $(F, \alpha) \widetilde{\wedge} (G, \beta)$ is non-null then is a soft LA-module over M.*

(2) *If the bi-intersection is non-null then $(F, \alpha) \widetilde{\sqcap} (G, \beta)$ is a soft LA-module over M.*

Proof. (1) By definition 7, let $(F,\alpha)\,\widetilde{\wedge}\,(G,\beta)=(K,\gamma)$, where $\gamma=\alpha\times\beta$ and $K(x,y)=F(x)\cap G(y)$, for all $(x,y)\in\gamma$. Since by hypothesis (K,γ) is non-null soft set over M and $(x,y)\in \mathrm{supp}(K,\gamma)$, so $K(x,y)=F(x)\cap G(y)\neq\emptyset$. As both $F(x)$ and $G(y)$ are sub LA-modules, hence this result implies that $K(x,y)$ is also a sub LA-modules. Thus we deduce that $(K,\gamma)=(F,\alpha)\,\widetilde{\wedge}\,(G,\beta)$ is a soft LA-module over M.

(2) By definition 5, we write $(F,\alpha)\,\widetilde{\sqcap}\,(G,\beta)=(K,\gamma)$, where $\gamma=\alpha\cap\beta$ and $K(x)=F(x)\cap G(x)\neq\emptyset$ for some $x\in\alpha\cap\beta$. Since the non-empty sets $F(x)$ and $G(x)$ both are sub LA-modules over M and hence, we see that $F(x)\cap G(x)$ is a sub LA-module over M. Consequently $(F,\alpha)\,\widetilde{\sqcap}\,(G,\beta)=(K,\gamma)$ is a soft LA-module over M. □

Definition 20. *Consider (F,α) and (G,β) be two soft LA-modules over M. Then $(F,\alpha)+(G,\beta)=(K,\gamma)$, where $\gamma=\alpha\times\beta$ and $K(x,y)=F(x)+G(y)$ for every $(x,y)\in\gamma$.*

Based on above definition, we have the following theorem:

Theorem 4. *Consider (F,A) and (G,B) be two soft LA-modules over M. Then $(F,\alpha)+(G,\beta)=(K,\gamma)$, where $\gamma=\alpha\times\beta$ is a soft LA-module over M.*

Proof. By definition 20, we have $(F,\alpha)+(G,\beta)=(K,\gamma)$, where $\gamma=\alpha\times\beta$ and $K(x,y)=F(x)+G(y)$ for every $(x,y)\in\gamma$. Since (F,α) and (G,β) are two soft LA-modules over M, therefore we have $F(x)\leq M$ and $G(y)\leq M$ for every $(x,y)\in\gamma$ respectively. Then $F(x)+G(y)\leq M$ for all $(x,y)\in\gamma$. Hence it follows that $K(x,y)\leq M$ for all $(x,y)\in\gamma$. Thus $(F,\alpha)+(G,\beta)=(K,\gamma)$ is a soft LA-module over M. □

Definition 21. *Let (F,α) and (G,β) be two soft LA-modules over M and N respectively. Then $(F,\alpha)\times(G,\beta)=(K,\gamma)$, where $\gamma=\alpha\times\beta$ is defined as $K(x,y)=F(x)\times G(y)$ for every $(x,y)\in\gamma$.*

Theorem 5. *Let (F,α) and (G,β) be two soft LA-modules over M and N respectively. Then $(F,\alpha)\times(G,\beta)=(K,\gamma)$, where $\gamma=\alpha\times\beta$ is a soft LA-module over $M\times N$.*

Proof. From definition 21, we have $(F,\alpha)\times(G,\beta)=(K,\gamma)$, where $\gamma=\alpha\times\beta$ and $K(x,y)=F(x)\times G(y)$ for every $(x,y)\in\gamma$. Since (F,α) and (G,β) are two soft LA-modules over M and N respectively, therefore $F(x)\leq M$ and $G(y)\leq N$ for every $(x,y)\in\gamma$. Hence by definition 15, it can be easily verified that $K(x,y)=F(x)\times G(y)\leq M\times N$ for every $(x,y)\in\gamma$. Thus $(F,\alpha)\times(G,\beta)=(K,\gamma)$ is a soft LA-module over $M\times N$. □

Definition 22. *Consider (F,α) and (G,β) be two soft LA-modules over M. Then (G,β) is called a soft sub LA-module of (F,α), if the following conditions are satisfied:*

(i) $\beta \subseteq \alpha$ *and*

(ii) $G(x) \leq F(x)$ *for every* $x \in \beta$.

The above soft sub LA-module can be written as $(G, B) \mathrel{\tilde{\leq}} (F, A)$.

If $(G, \beta) \mathrel{\tilde{\leq}} (F, \alpha)$ and $(F, \alpha) \mathrel{\tilde{\leq}} (G, \beta)$, then these two soft sets over M are called soft equal sub LA-modules and it is written as $(F, \alpha) = (G, \beta)$.

Theorem 6. *Consider* (F, α) *and* (G, α) *be two soft LA-modules over* M*. The soft set* (G, α) *is a soft sub LA-module of* (F, α) *in case if* $G(\varepsilon)$ *is a sub LA-module of* $F(\varepsilon)$ *i.e.,*$G(\varepsilon) \subseteq F(\varepsilon)$ *for all* $\varepsilon \in \alpha$.

Proof. By definition 22, the proof is straightforward. □

Definition 23. *Every soft LA-module* (F, α) *over* M *has at least two soft sub LA-modules* (F, α) *and* (F, E)*, which are called the trivial soft sub LA-module over* M*, where* $E = \{e\}$*, and* e *is a unit of* α.

Theorem 7. *Suppose that* (F, α) *is a soft LA-module over* M*, and* $\{(G_i, \beta_i) \mid i \in I\}$ *be a family of non-empty soft sub LA-modules of* (F, α)*. Then the following assertions hold:*

(1) *The* $\sum\limits_{i \in I} (G_i, \beta_i)$ *is a soft sub LA-module of* (F, α).

(2) *The* $\mathop{\tilde{\sqcap}}\limits_{i \in I} (G_i, \beta_i)$ *is a soft sub LA-module of* (F, α).

(3) *The* $\mathop{\tilde{\cup}}\limits_{i \in I} (G_i, \beta_i)$ *is soft a sub LA-module of* (F, α)*, if* $\beta_i \cap \beta_j = \emptyset$ *for every* $i, j \in I$.

Proof. (1) By proposition 1, the proof is straightforward.

(2) By definition 6, let $\mathop{\tilde{\sqcap}}\limits_{i \in I} (G_i, \beta_i) = (K, \gamma)$, where $\gamma = \bigcap\limits_{i \in I} \beta_i$ and $K(x) = \bigcap\limits_{i \in I} G_i(x)$ for all $x \in \gamma$. Since (K, γ) is non-null soft set and if $x \in \text{supp}(K, \gamma)$, then $K(x) = \bigcap\limits_{i \in I} G_i(x) \neq \emptyset$. As the intersection of any number of sub LA-modules over M is a sub LA-module, hence for all $i \in I$, the non-empty set $G_i(x)$ is a sub LA-module of (F, α) over M. Hence (G_i, β_i) is a soft sub LA-module of (F, α) over M and therefore $K(x)$ is a sub LA-module over M for all $x \in \text{supp}(K, \gamma)$. Thus it follows that $\tilde{\sqcap}_{i \in I} (G_i, \beta_i) = (K, \gamma)$ is a soft sub LA-module of (F, α) over M.

(3) Straightforward by proposition 3, part (2). □

Definition 24. *Suppose that* (F, α) *and* (G, β) *be two soft LA-modules over M. Let* $(G, \beta) \mathrel{\tilde{\leq}} (F, \alpha)$*. Then* (G, β) *is called a maximal soft sub LA-module of* (F, α) *if* $G(\eta)$ *is maximal sub LA-module of* $F(\eta)$ *for all* $\eta \in \beta$*. Similarly* (G, β) *is called a minimal soft sub LA-module of* (F, α) *if* $G(\eta)$ *is minimal sub LA-module of* $F(\eta)$ *for all* $\eta \in \beta$.

Proposition 4. *Consider a soft LA-module (F, α) over M. Then the following conditions are true:*

(1) *The* $\widetilde{\bigcap}_{i \in I}(G_i, \beta_i)$ *is a maximal soft sub LA-module of* (F, α), *if* $\{(G_i, \beta_i) \mid i \in I\}$ *be a family of non-empty maximal soft sub LA-modules of* (F, α).

(2) *The* $\sum_{i \in I}(G_i, \beta_i)$ *is a minimal soft sub LA-module of* (F, α), *if* $\{(G_i, \beta_i) \mid i \in I\}$ *be a family of non-empty minimal soft sub LA-modules of* (F, α).

Proof. The proof follows from the definition 17. □

4. Soft LA modules and exact sequences

In this section, we consider the LA-homomorphisms and the quotient soft sub LA-modules. We will establish a theorem concerning the short exact sequence of LA modules. Throughout this section, the homomorphism always means an LA-module homomorphism.

Definition 25. *Consider (F, α) and (G, β) be two soft LA-modules over M and N respectively. Then (θ, φ) is a soft homomorphism if the following assertions are true:*

(*i*) $\theta : M \longrightarrow N$ *is a homomorphism*

(*ii*) $\varphi : \alpha \longrightarrow \beta$ *is a mapping*

(*iii*) $\theta(F(\varepsilon)) = G(\varphi(\varepsilon))$ *for all* $\varepsilon \in \alpha$.

We now call (F, α) is soft homomorpic to (G, β) if the above definition is satisfied. It can be written as $(F, \alpha) \simeq (G, \beta)$.

If $\theta : M \longrightarrow N$ is a bijective homomorphism (isomorphism) and $\varphi : \alpha \longrightarrow \beta$ is a one-one and onto mapping, then (θ, φ) is called a soft bijective morphism (isomorphism). Moreover, we also call (F, α) is soft isomorphic to (G, β), and it is defined by $(F, \alpha) \cong (G, \beta)$.

Definition 26. *Consider (F, α) be a soft LA-module over M, then the following statements hold:*

(*i*) *If $F(\varepsilon) = \{0\}$ for all $\varepsilon \in \alpha$, where 0 is zero element of M, then (F, α) is said to be trivial (null) soft LA-module over M.*

(*ii*) *If $F(\varepsilon) = M$ for all $\varepsilon \in \alpha$, then (F, α) is said to be whole (absolute) soft LA-module over M.*

Proposition 5. *Consider two soft LA-modules (F, α) and (G, β) over M, and (G, β) be a soft sub LA-module of (F, α). If $\theta : M \longrightarrow N$ is a homomorphism then $(\theta(F), \alpha)$ and $(\theta(G), \beta)$ are two soft LA-modules over N. Also $(\theta(G), \beta)$ is a soft sub LA-module of $(\theta(F), \alpha)$.*

Proof. Since $\theta : M \longrightarrow N$ is a homomorphism and we know that homomorphic image of sub LA-module is a sub LA-module by proposition 2. Therefore $\theta(F(x))$ and $\theta(G(y))$ are soft LA-modules over N for every $\varepsilon \in \alpha$ and $\eta \in \beta$. Hence $(\theta(F), \alpha)$ and $(\theta(G), \beta)$ are two soft LA-modules over N. If (G, β) is a soft sub LA-module of (F,α), then $G(\eta)$ is a sub LA-module of $F(\eta)$ for every $\eta \in \beta$ and θ is a homomorphism so intuitively $\theta(G(\eta))$ is a sub LA-module of $\theta(F(\eta))$ for every $\eta \in \beta$. Therefore, we can say that $(\theta(G),\beta)$ is a soft sub LA-module of $(\theta(F),\alpha)$. □

Proposition 6. (1) *Consider a soft LA-module (F,α) over M and a homomorphism $\theta : M \longrightarrow N$, if $F(\varepsilon) = ker\theta$ for all $\varepsilon \in \alpha$, then $(\theta(F),\alpha)$ is a trivial (null) soft LA-module over N.*

(2) *Consider (F,α) to be an absolute soft LA-module over M and $\theta : M \longrightarrow N$ be an onto homomorhism (epimorphism), if $F(\varepsilon) = M$ for all $\varepsilon \in \alpha$, then $(\theta(F),\alpha)$ is a whole (absolute) soft LA-module over N.*

Proof. (1) Since $F(\varepsilon) = ker\theta$ for all $\varepsilon \in \alpha$, and $\theta : M \longrightarrow N$ is a homomorphism, so this implies that $\theta(F(\varepsilon)) = \theta(ker\theta) = \{0_N\}$ for all $\varepsilon \in \alpha$. Therefore, by definition 26, $(\theta(F),\alpha)$ is a trivial (null) soft LA-module over N.

(2) Since (F,α) is an absolute soft LA-module over M, then clearly $F(\varepsilon) = M$ for all $\varepsilon \in \alpha$. Since θ is an epimorphism, therefore it follows that $\theta(F(\varepsilon)) = \theta(M) = N$ for all $\varepsilon \in \alpha$. Hence it is proved that $(\theta(F),\alpha)$ is a whole (absolute) soft LA-module over N. □

Definition 27. *A sequence of R-homomorphisms and R-LA-modules $\cdots \longrightarrow P_{n-1} \overset{\theta_{n-1}}{\longrightarrow} P_n \overset{\theta_n}{\longrightarrow} P_{n+1} \longrightarrow \cdots$ is called an exact sequence if $Im\theta_{n-1} = ker\theta_n$ for all $n \in \mathbb{N}$. An exact sequence of the form $0 \longrightarrow P' \overset{\theta}{\longrightarrow} P \overset{\varphi}{\longrightarrow} P'' \longrightarrow 0$ is called a short exact sequence.*

Example 1. Consider a sequence

$$(*) \qquad 0 \to \langle h(s)\rangle \overset{i}{\to} R[X;S] \overset{\pi}{\to} \frac{R[X;S]}{\langle h(s)\rangle} \to 0$$

where $i : \langle h(s)\rangle \to R[X;S]$ is an inclusion map defined as $i[\langle h(s)\rangle] = \langle h(s)\rangle$, where $\langle h(s)\rangle = \{g(X).h(s) : g(X) \in R[X;S]\}$. Let $\pi : R[X;S] \to \frac{R[X;S]}{\langle h(s)\rangle}$ defined by $\pi[g(X)] = \langle h(s)\rangle + g(X)$ for every $g(X) \in R[X;S]$, where $h(s)$ is an irreducible polynomial in $R[X;S]$. Here $\langle h(s)\rangle$, $R[X;S]$ and $\frac{R[X;S]}{\langle h(s)\rangle}$ being LA-rings are LA-modules over an LA-ring R and S is a commutative semigroup and every commutative semigroup is an LA-semigroup. It is important to note that the additive identity in $\frac{R[X;S]}{\langle h(s)\rangle}$ is $\langle h(s)\rangle$.So if i is monomorphism, π is an epimorphism and $Imi = ker\pi$. Then the sequence $(*)$ is an exact sequence of LA-modules over an LA-ring R.

First we show that $i : \langle h(s)\rangle \to R[X;S]$ defined by $i[\langle h(s)\rangle] = \langle h(s)\rangle$, where $\langle h(s)\rangle = \{g(X).h(s) : g(X) \in R[X;S]\}$, is a homomorphism. Let $h_1(s), h_2(s) \in \langle h(s)\rangle$ where $h_1(s) = g_1(X).h(s)$ and $h_2(s) = g_2(X).h(s)$

$$\begin{aligned} i\{h_1(s)+h_2(s)\} &= i\{g_1(X).h(s)+g_2(X).h(s)\} \\ &= \{g_1(X).h(s)+g_2(X).h(s)\} \\ &= i\{h_1(s)\}+i\{h_2(s)\} \\ i\{rh_1(s)\} &= i\{rg_1(X).h(s)\} \\ &= rg_1(X).h(s) = r\{g_1(X).h(s)\} = ri\{h_1(s)\} \end{aligned}$$

so i is an LA-module homomorphism. Now we show that it is monic, as we know that $f_1(s), f_2(s) \in \langle f(s)\rangle$ where $f_1(s) = g_1(X).f(s)$ and $f_2(s) = g_2(X).f(s)$, so

$$\begin{aligned} i\{f_1(s)\} &\neq i\{f_1(s)\} \\ &\Longrightarrow i\{g_1(X).f(s)\} \neq i\{g_2(X).f(s)\} \\ &\Longrightarrow g_1(X).f(s) \neq g_2(X).f(s) \\ &\Longrightarrow f_1(s) \neq f_2(s) \end{aligned}$$

Hence we have proved that i is monic and hence it is monomorphism.

Now we have to show that $\pi : R[X;S] \to \frac{R[X;S]}{\langle h(s)\rangle}$ is an epimorphism defined as $\pi\{g(X)\} = \langle h(s)\rangle + g(X)$ for all $g(X) \in R[X;S]$, where $h(s)$ is an irreducible polynomial in $R[X;S]$.

Let

$$\begin{aligned} \pi\{g_1(X)+g_2(X)\} &= \langle h(s)\rangle + \{g_1(X)+g_2(X)\} \\ &= \{\langle h(s)\rangle + g_1(X)\} + \{\langle h(s)\rangle + g_2(X)\} \\ &= \pi\{g_1(X)\} + \pi\{g_2(X)\} \\ \pi\{rg(X)\} &= \{\langle h(s)\rangle + rg(X)\} \\ &= r\{\langle h(s)\rangle + g(X)\} = r\pi\{g(X)\} \end{aligned}$$

Hence π is an LA-module homomorphism. So by definition it is obviously onto and hence π is an epimorphism.

Finally, we proceed to prove that $Imi = ker\pi$

As it has been known that $\pi : R[X;S] \to \frac{R[X;S]}{\langle h(s)\rangle}$ which is defined by $\pi\{g(X)\} = \langle h(s)\rangle + g(X)$ for every $g(X) \in R[X;S]$, where $h(s)$ is an irreducible polynomial in $R[X;S]$. Now $ker\pi = \{g(X) \in R[X;S] : \pi\{g(X)\} = \langle h(s)\rangle\}$. Let $g(X) \in ker\pi$. As $\pi\{g(X)\} = \langle h(s)\rangle \Longrightarrow \langle h(s)\rangle + g(X) = \langle h(s)\rangle \Longrightarrow g(X) \in \langle h(s)\rangle \Longrightarrow ker\pi \subseteq \langle h(s)\rangle = Imi \Longrightarrow ker\pi \subseteq Imi$ $(**)$. Let $g(X) \in Imi$, this implies that there exist an element $g(X) \in \langle h(s)\rangle$ such that $g(X) = i[g(X)] \Longrightarrow \pi\{g(X)\} = \pi[i\{g(X)\}] \Longrightarrow \langle h(s)\rangle + g(X) = \pi\{g(X)\} \Longrightarrow \langle h(s)\rangle = \pi\{g(X)\} \Longrightarrow g(X) \in ker\pi \Longrightarrow Imi \subseteq ker\pi$ $(***)$. Hence from $(**)$ and $(***)$ it is proved that that $ker\pi = Imi$. Thus the sequence $(*)$ is an exact sequence of LA-modules over an LA-ring R.

Definition 28. *Let (F_n, α) be a soft LA-module over M, where $n \in \mathbb{N}$. Then a sequence of R-homomorphisms and soft LA-modules* $\cdots \longrightarrow F_{n-1}(\varepsilon) \overset{\widetilde{\theta}_{n-1}}{\longrightarrow}$

$F_n(\varepsilon) \xrightarrow{\widetilde{\theta}_n} F_{n+1}(\varepsilon) \longrightarrow \cdots$ is called an exact sequence if $Im\widetilde{\theta}_{n-1} = ker\widetilde{\theta}_n$ for every $n \in \mathbb{N}$ and for all $\varepsilon \in \alpha$. An exact sequence of the form $0 \longrightarrow F'(\varepsilon) \xrightarrow{\widetilde{\theta}} F(\varepsilon) \xrightarrow{\widetilde{\varphi}} F''(\varepsilon) \longrightarrow 0$ is called a short exact sequence.

The followings results are the main results of this paper.

Theorem 8. *Let (F,α) to be a trivial (null) soft LA-module over V and let (G,β) be an absolute soft LA-module over W. If $0 \longrightarrow V \xrightarrow{\theta} M \xrightarrow{\varphi} W \longrightarrow 0$ is a short exact sequence, then $0 \longrightarrow F(\varepsilon) \xrightarrow{\widetilde{\theta}} M \xrightarrow{\widetilde{\varphi}} G(\eta) \longrightarrow 0$ is a short exact sequence for every $\varepsilon \in \alpha$ and $\eta \in \beta$.*

Proof. It has been already known that the sequence $0 \xrightarrow{\theta'} V \xrightarrow{\theta} M \xrightarrow{\varphi} W \xrightarrow{\varphi'} 0$ is a short exact sequence, hence by definition 18 and proposition 2, we deduce that $Im\theta' = \{0\} = ker\theta$. Given that (F,α) is a trivial (null) soft LA-module over V, therefore, by definition 26, $F(\varepsilon) = \{0\}$ for all $\varepsilon \in \alpha$. Then from the triplet $0 \xrightarrow{\widetilde{\widetilde{\theta}}} F(\varepsilon) = \{0\} \xrightarrow{\widetilde{\theta}} M$, we have $Im\widetilde{\widetilde{\theta}} = \{0\} = ker\widetilde{\theta}$. This implies that $\widetilde{\theta}$ is a one-one and hence it is a monomorphism. Conversely if $\widetilde{\theta}$ is a monomorphism, then obviously the triplet $0 \xrightarrow{\widetilde{\widetilde{\theta}}} F(\varepsilon) = \{0\} \xrightarrow{\widetilde{\theta}} M$ becomes an exact sequence. From the hypothesis we have $Im\varphi = ker\varphi'$. Now consider the triplet $M \xrightarrow{\widetilde{\varphi}} G(\eta) \xrightarrow{\widetilde{\widetilde{\varphi}}} 0$, also given that (G,β) is an absolute soft LA-module over W, so $G(\eta) = W$ for all $\eta \in \beta$. From triplet $M \xrightarrow{\widetilde{\varphi}} G(\eta) \xrightarrow{\widetilde{\widetilde{\varphi}}} 0$ it is obvious that $ker\widetilde{\widetilde{\varphi}} = G(\eta)$for all $\eta \in \beta$. This implies that $Im\widetilde{\varphi} = G(\eta) = ker\widetilde{\widetilde{\varphi}}$ for all $\eta \in \beta$. It follows that $\widetilde{\varphi}$ is an onto and hence it is an epimorphism. Conversely, if $\widetilde{\varphi}$ is an epimorphism, then obviously the triplet $M \xrightarrow{\widetilde{\varphi}} G(\eta) \xrightarrow{\widetilde{\widetilde{\varphi}}} 0$ becomes an exact sequence. Now from the proposition 2, it follows that $0 \longrightarrow F(\varepsilon) \xrightarrow{\widetilde{\theta}} M \xrightarrow{\widetilde{\varphi}} G(\eta) \longrightarrow 0$ is a short exact sequence for all $\varepsilon \in \alpha$ and $\eta \in \beta$. □

Proposition 7. *Let (F,α) to be a trivial (null) soft LA-module over V and (G,β) be an absolute soft LA-module over W. If $0 \longrightarrow V \xrightarrow{\theta} M \xrightarrow{\varphi} W \longrightarrow 0$ is a short exact sequence, then $0 \longrightarrow \theta(F(\varepsilon)) \xrightarrow{\widetilde{\theta}} M \xrightarrow{\widetilde{\varphi}} \varphi(G(\eta)) \longrightarrow 0$ is a short exact sequence for every $\varepsilon \in \alpha$ and $\eta \in \beta$.*

Proof. The proof follows from theorem 8 and proposition 2. □

Theorem 9. *Let a soft LA-module (F,α) over V and (G,β) is a soft LA-module over W. Then for a short exact sequence $0 \longrightarrow F(\varepsilon) \xrightarrow{\widetilde{\theta}} M \xrightarrow{\widetilde{\varphi}} G(\eta) \longrightarrow 0$*

of soft R-LA-modules and R-homomorphisms, the following assertions are equivalent:

(1) *There exists an R-homomorphism* $i : M \longrightarrow F(\varepsilon)$ *such that* $i\widetilde{\theta} = 1_{F(\varepsilon)}$ *for all* $\varepsilon \in \alpha$.

(2) *There exists an R-homomorphism* $\pi : G(\eta) \longrightarrow M$ *such that* $\widetilde{\varphi}\pi = 1_{G(\eta)}$ *for all* $\eta \in \beta$.

(3) $Im\widetilde{\theta}$ *is direct summand of* M.

Proof. (1) $\Longrightarrow$ (3) Suppose that there exists a homomorphism $i : M \longrightarrow F(x)$ such that $i\widetilde{\theta} = 1_{F(\varepsilon)}$ for all $\varepsilon \in \alpha$. Let $m \in M$. Then $i(m) \in F(\varepsilon)$. We have $i\widetilde{\theta} = 1_{F(\varepsilon)}$for all $\varepsilon \in \alpha$, then $i\widetilde{\theta}(i(m)) = i(m) \Longrightarrow i\widetilde{\theta}(i(m)-i(m)) = 0 \Longrightarrow i(\widetilde{\theta}(i(m)-m))) = 0$ or $i(m - \widetilde{\theta}(i(m))) = 0$. It follows that $m - \widetilde{\theta}(i(m)) \in keri = K$ (say) $\Longrightarrow m-\widetilde{\theta}(i(m)) \in K \Longrightarrow m-\widetilde{\theta}(i(m)) = k$ for some $k \in K \Longrightarrow m = k+\widetilde{\theta}(i(m))$ for some $k \in K$. Thus $M = K + Im\widetilde{\theta}$. Suppose that there is an another element $k' \in K$ and an element $a \in F(\varepsilon)$ such that $m = k + \widetilde{\theta}(i(m)) = k' + \widetilde{\theta}(a)$.......(∗). Then $i(k + \widetilde{\theta}(i(m))) = i(k' + \widetilde{\theta}(a)) \Longrightarrow i(k) + i(\widetilde{\theta}(i(m))) = i(k') + i(\widetilde{\theta}(a)) \Longrightarrow 0 + i(\widetilde{\theta}(i(m))) = 0 + i(\widetilde{f}(a))$, since $k, k' \in K, \Longrightarrow i(\widetilde{\theta}(i(m))) = i(\widetilde{\theta}(a)) \Longrightarrow (i\widetilde{\theta})(i(m)) = (i\widetilde{\theta})(a) \Longrightarrow 1_{F(\varepsilon)}(i(m)) = 1_{F(\varepsilon)}(a) \Longrightarrow i(m) = a$. Thus, the relation (∗) becomes $k+\widetilde{\theta}(a) = k'+\widetilde{\theta}(a) \Longrightarrow k = k'$. Hence every element of M can be uniquely expressed as $k + \theta(a)$ for some $k \in K$ and $a \in F(\varepsilon)$. Therefore $M = K \oplus Im\widetilde{\theta}$.

(3) $\Longrightarrow$ (2) Let K be a soft sub LA-module of M such that $M = K \oplus Im\widetilde{\theta}$. Let $a' \in G(\eta)$ and $\widetilde{\varphi}$ is an epimorphism so $\widetilde{\varphi}(m) = a'$. Let $m = k+\widetilde{\theta}(a)$ for some $k \in K$ and $a \in F(\varepsilon)$. Then $\widetilde{\varphi}(m) = \widetilde{\varphi}(k + \widetilde{\theta}(a)) \Longrightarrow a' = \widetilde{\varphi}(k) + \widetilde{\varphi}(\widetilde{\theta}(a)) \Longrightarrow a' = \widetilde{\varphi}(k) + 0 \Longrightarrow a' = \widetilde{\varphi}(k)$. If k' is another element such that $a' = \widetilde{\varphi}(k)$, then $\widetilde{\varphi}(k') = \widetilde{\varphi}(k) \Longrightarrow \widetilde{\varphi}(k') - \widetilde{\varphi}(k) = 0 \Longrightarrow \widetilde{\varphi}(k') + \widetilde{\varphi}(-k) = 0 \Longrightarrow \widetilde{\varphi}(k' - k) = 0 \Longrightarrow k' - k \in ker\widetilde{\varphi} \Longrightarrow k' - k \in Im\widetilde{\theta}$. This implies that there exists an element $a'' \in F(\varepsilon)$ such that $\widetilde{\theta}(a'') = k' - k$. The direct summand property of M shows that $k' - k = \widetilde{\theta}(a'') = 0 \Longrightarrow k = k'$. Thus there exists a unique $k \in K$ such that $a' = \widetilde{\varphi}(k)$. Define $\pi : G(\eta) \longrightarrow M$ by $\pi(a') = k$, where k is a unique element such that $\widetilde{\varphi}(k) = a'$. Let $a'_1, a'_2 \in G(\eta) \Longrightarrow$ there exist unique elements $k_1, k_2 \in K$ such that $\widetilde{\varphi}(k_1) = a'_1$ and $\widetilde{\varphi}(k_2) = a'_2 \Longrightarrow \pi(a'_1) = k_1$ and $\pi(a'_2) = k_2$. Since $\widetilde{\varphi}$ is homomorphism so $\widetilde{\varphi}(k_1+k_2) = \widetilde{\varphi}(k_1)+\widetilde{g}(k_2) = a'_1+a'_2 \Longrightarrow \widetilde{\varphi}(k_1 + k_2) = a'_1 + a'_2 \Longrightarrow \pi(a'_1 + a'_2) = k_1 + k_2 = \pi(a'_1) + \pi(a'_2)$. Let $r \in R$ and $a' \in G(\eta) \Longrightarrow ra' \in G(\eta)$. Consequently, there exists a unique element $k \in K$ such that $\widetilde{\varphi}(k) = a' \Longrightarrow \pi(a') = k$. Also $\widetilde{\varphi}(rk) = r\widetilde{\varphi}(k) = ra' \Longrightarrow \pi(ra') = rk = r\pi(a')$. Thus π is an R-homomorphism and since $\widetilde{\varphi}\pi(a') = \widetilde{\varphi}(k) = a'$for all $a' \in G(\eta) \Longrightarrow \widetilde{\varphi}\pi = 1_{G(\eta)}$ for all $\eta \in \beta$. (2) $\Longrightarrow$ (1) Suppose that there exists

an R-homomorphism $\pi : G(\eta) \longrightarrow M$ such that $\widetilde{\varphi}\pi = 1_{G(\eta)}$ for all $\eta \in \beta$. Let $m \in M$ and $\widetilde{\varphi}(m) = a'$. Then $\widetilde{\varphi}(m) = a' = \widetilde{\varphi}\pi(a') \Longrightarrow \widetilde{\varphi}(m) - \widetilde{\varphi}\pi(a') = 0 \Longrightarrow \widetilde{\varphi}(m - \pi(a')) = 0 \Longrightarrow m - \pi(a') \in ker\widetilde{\varphi} \Longrightarrow m - \pi(a') \in Im\widetilde{\theta}$. Intuitively, there exists an element $a \in F(\varepsilon)$ such that $\widetilde{\theta}(a) = m - \pi(a') \Longrightarrow m = \pi(a') + \widetilde{\theta}(a)$. To prove the uniqueness, let $\pi(a^{''}) + \widetilde{\theta}(a') = \pi(a') + \widetilde{\theta}(a)$. Then $\pi(a^{''}) - \pi(a') = \widetilde{\theta}(a) - \widetilde{\theta}(a') \Longrightarrow \pi(a^{''} - a') = \widetilde{\theta}(a - a') \Longrightarrow \widetilde{\varphi}\pi(a^{''} - a') = \widetilde{\varphi}\widetilde{\theta}(a - a') = 0 \Longrightarrow a^{''} - a' = 0 \Longrightarrow a^{''} = a'$. Now we have $\pi(a') + \widetilde{\theta}(a') = \pi(a') + \widetilde{\theta}(a) \Longrightarrow \widetilde{\theta}(a') - \widetilde{\theta}(a) = 0 \Longrightarrow \widetilde{\theta}(a' - a) = 0 \Longrightarrow a' - a \in ker\widetilde{\theta} \Longrightarrow a' - a \in \{0\}$ since $\widetilde{\theta}$ is monic so $ker\widetilde{\theta} = \{0\} \Longrightarrow a' = a$. Hence every element of M can be written as $\pi(a') + \widetilde{\theta}(a)$, $a' \in G(\eta)$ and $a \in F(\varepsilon)$. Now we define a map $i : M \longrightarrow F(\varepsilon)$ by $i(\pi(a') + \widetilde{\theta}(a)) = a$. Obviously i is well-defined homomorphism and $i\widetilde{\theta} = 1_{F(\varepsilon)}$ for all $\varepsilon \in \alpha$. Let $\pi(a_1') + \widetilde{\theta}(a_1)$, $\pi(a_2') + \widetilde{\theta}(a_2) \in M$, then $i((\pi(a_1') + \widetilde{\theta}(a_1)) + (\pi(a_2') + \widetilde{\theta}(a_2))) = i((\pi(a_1') + \pi(a_2')) + (\widetilde{\theta}(a_1) + \widetilde{\theta}(a_2))) = i(\pi(a_1' + a_2') + \widetilde{\theta}(a_1 + a_2)) = a_1 + a_2 = i(\pi(a_1') + \widetilde{\theta}(a_1)) + i(\pi(a_2') + \widetilde{\theta}(a_2))$ and $i(r(\pi(a') + \widetilde{\theta}(a)) = i(r\pi(a') + r\widetilde{\theta}(a)) = i(\pi(ra') + \widetilde{\theta}(ra)) = ra = r(i(\pi(a') + \widetilde{\theta}(a))$.

Hence i is a homomorphism. Now the $\widetilde{\theta}(a) = m - \pi(a') = i\widetilde{\theta}(a) = i(m - \pi(a')) = i(m) - i(\pi(a')) = a - 0 = a \Longrightarrow i\widetilde{\theta}(a) = a \Longrightarrow i\widetilde{\theta} = 1_{F(\varepsilon)}$ for all $\varepsilon \in \alpha$. □

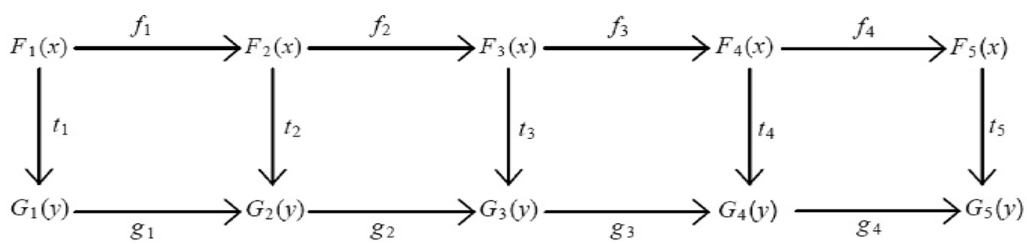
Definition 29. *Let (F, α) to be a soft LA-module over M and (G, β) be a soft sub LA-module of (F, α). Then the soft quotient LA-module is defined as $(F, \alpha)/(G, \beta) = \{(G, \beta) + F(\varepsilon) \mid$ for every $\varepsilon \in \alpha\}$.*

Theorem 10. *Suppose (F, α) is a soft LA-module over M. (G, β) be a soft sub LA-module of (F, α) and $\theta : (G, \beta) \longrightarrow (F, \alpha)$ is the inclusion map defined by $\theta(G(\eta)) = G(\eta)$ for all $\eta \in \beta$ and $\varphi : (F, \alpha) \longrightarrow (F, \alpha)/(G, \beta)$ is the natural projection i.e., $\varphi(F(\varepsilon)) = (G, \beta) + F(\varepsilon)$ for all $\varepsilon \in \alpha$. Then θ is a monomorphism, φ is an epimorphism and $Im\theta = (G, \beta) = ker\varphi$. Then the sequence $0 \longrightarrow G(\eta) \xrightarrow{\theta} F(\varepsilon) \xrightarrow{\varphi} F(\varepsilon)/G(\eta) \longrightarrow 0$ is a short exact sequence.*

Proof. Define $\theta : (G, \beta) \longrightarrow (F, \alpha)$ by $\theta(G(\eta)) = G(\eta)$ for all $\eta \in \beta$. For all $\eta_1, \eta_2 \in \beta$, $G(\eta_1) = G(\eta_2) \Longrightarrow \theta(G(\eta_1)) = \theta(G(\eta_2))$ by definition, so θ is well defined. Let $\theta(G(\eta_1)) = \theta(G(\eta_2))$ for all $\eta_1, \eta_2 \in \beta \Longrightarrow G(\eta_1) = G(\eta_2)$ by definition. Hence θ is one-one. Now for all $\eta_1, \eta_2 \in \beta$ $\theta(G(\eta_1) + G(\eta_2)) = G(\eta_1) + G(\eta_2) = \theta(G(\eta_1)) + \theta(G(\eta_2))$, and for all $\eta \in \beta$ and $r \in R$, $\theta(r \cdot G(\eta)) = r \cdot G(\eta) = r \cdot \theta(G(\eta))$. So it follows that θ is a homomorphism and hence monomorphism. Define $\varphi : (F, \alpha) \longrightarrow (F, \alpha)/(G, \beta)$ by $\varphi(F(\varepsilon)) = (G, \beta) + F(\varepsilon)$ for all $\varepsilon \in \alpha$. For all $\varepsilon_1, \varepsilon_2 \in \alpha$, $F(\varepsilon_1) = F(\varepsilon_2) \Longrightarrow (G, \beta) + F(\varepsilon_1) = (G, \beta) + F(\varepsilon_2) \Longrightarrow \varphi(F(\varepsilon_1)) = \varphi(F(\varepsilon_2))$ by definition, so φ is well defined. Now to show φ an is onto map, it is obvious that for every $(G, \beta) + F(\varepsilon) \in (F, \alpha)/(G, \beta)$ there exists an element $F(\varepsilon) \in (F, \alpha)$ such that $\varphi(F(\varepsilon)) = (G, \beta) + F(\varepsilon)$ for all $\varepsilon \in \alpha$, hence φ is an onto. Now $\varphi(F(\varepsilon_1) + F(\varepsilon_2)) = (G, \beta) + (F(\varepsilon_1) + F(\varepsilon_2)) = ((G, \beta) + F(\varepsilon_1)) + ((G, \beta) + F(\varepsilon_2)) = \varphi(F(\varepsilon_1)) + \varphi(F(\varepsilon_2))$. Also for all $r \in R$ and for all $\varepsilon \in \alpha$, $\varphi(r \cdot F(\varepsilon)) = (G, \beta) + r \cdot F(\varepsilon) = r \cdot (G, \beta) + r \cdot F(\varepsilon) = r \cdot [(G, \beta) + F(\varepsilon)] = r \cdot \varphi(F(\varepsilon))$ because (G, β) is zero of $(F, \alpha)/(G, \beta)$ so we can write $(G, \beta) = r \cdot (G, \beta)$. Therefore φ is homomorphism and hence epimorphism. Now we show that $Im\theta = (G, \beta) = ker\varphi$. So for this, let $F(\varepsilon) \in ker\varphi \Longrightarrow \varphi(F(\varepsilon)) = (G, \beta) \Longrightarrow (G, \beta) + F(\varepsilon) = (G, \beta) \Longrightarrow F(\varepsilon) \in (G, \beta)$. Since θ is an inclusion map so $\theta(F(\varepsilon)) = F(\varepsilon) \Longrightarrow F(\varepsilon) \in \theta(G, \beta) = Im\theta \Longrightarrow F(\varepsilon) \in Im\theta$. Hence it follows that $ker\varphi \subseteq Im\theta$.................$(*)$. Let $F(\varepsilon) \in Im\theta$. Then there exist an element $F(\varepsilon) \in (G, \beta)$ such that $F(\varepsilon) = \theta(F(\varepsilon)) \Longrightarrow \varphi(F(\varepsilon)) = \varphi(\theta(F(\varepsilon))) \Longrightarrow (G, \beta) + F(\varepsilon) = \varphi(F(\varepsilon)) \Longrightarrow (G, \beta) = \varphi(F(\varepsilon)) \Longrightarrow F(\varepsilon) \in ker\varphi \Longrightarrow Im\theta \subseteq ker\varphi$.................$(**)$. Hence from $(*)$ and $(**)$ we have $Im\theta = ker\varphi$. Now we proceed to show $Im\theta = (G, \beta)$, let $F(\varepsilon) \in Im\theta$. Then there exists an element $G(\eta) \in (G, \beta)$ such that $\theta(G(\eta)) = F(\varepsilon)$, where $\eta \in \beta$. Now as θ is an inclusion map so $G(\eta) = F(\varepsilon) \Longrightarrow F(\varepsilon) \in (G, \beta) \Longrightarrow Im\theta \subseteq (G, \beta)$......$(***)$. For all $\eta \in \beta$, let $G(\eta) \in (G, \beta) \Longrightarrow \theta(G(\eta)) = G(\eta) \Longrightarrow G(\eta) \in Im\theta$. Hence $(G, \beta) \subseteq Im\theta$......$(****)$. Thus from $(***)$ and $(****)$ we have $Im\theta = (G, \beta)$. Finally, let $F(\varepsilon) \in ker\varphi \Longleftrightarrow \varphi(F(\varepsilon)) = (G, \beta) \Longleftrightarrow (G, \beta) + F(\varepsilon) = (G, \beta) \Longleftrightarrow F(\varepsilon) \in (G, \beta) \Longleftrightarrow (G, \beta) = ker\varphi$. Hence $Im\theta = (G, \beta) = ker\varphi$. Consequently, the sequence $0 \longrightarrow G(\eta) \xrightarrow{\theta} F(\varepsilon) \xrightarrow{\varphi} F(\varepsilon)/G(\eta) \longrightarrow 0$ is a short exact sequence. □

In closing this paper, we extend the well known "Five lemma" in exact sequence in terms of the soft LA modules.

Lemma 2. *Let (F_i, α_i) and (G_i, β_i) be soft LA-modules over M, where $i = 1, 2, 3, 4, 5$. Consider a commutative diagram*



of R- LA-modules and homomorphisms with exact rows

(1) *If $t_2$and t_4 are epimorphisms and t_5 is a monomorphism, then t_3 is an epimorphism.*

(2) *If $t_2$and t_4 are monomorphisms and t_1 is an epimorphism, then t_3 is a monomorphism.*

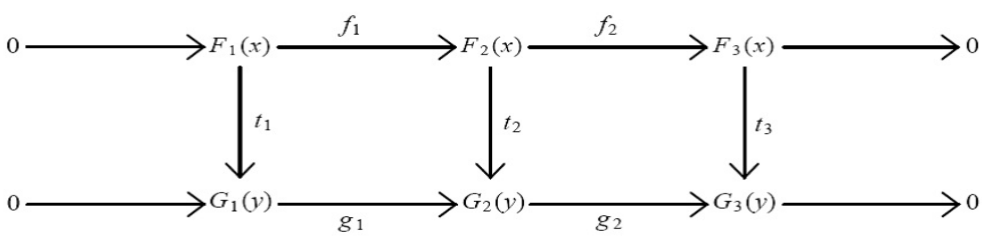
Proof. (1) Suppose that $t_2$and t_4 are epimorphisms and t_5 is a monomorphism. For all $y \in \beta_3$, let $\varepsilon_3 \in G_3(y)$. Then $g_3(\varepsilon_3) \in G_4(y)$ for all $y \in \beta_4$. Since

t_4 is an epimorphism, then there exists an element $\varepsilon_4' \in F_4(x)$ for all $x \in \alpha_4$ such that $g_3(\varepsilon_3) = t_4(\varepsilon_4')$ for all $y \in \beta_3$. By commutativity of diagram, $t_5 f_4(\varepsilon_4') = g_4 t_4(\varepsilon_4') \Longrightarrow t_5 f_4(\varepsilon_4') = g_4(g_3(\varepsilon_3)) = 0$ because $Img_3 = kerg_4$. As t_5 is a monomorphism thus $f_4(\varepsilon_4') = 0$. The row being exact there exists an element $\varepsilon_3^{'} \in F_3(x)$ for all $x \in \alpha_3$, such that $f_3(\varepsilon_3') = \varepsilon_4'$. Then $g_3(\varepsilon_3) = t_4(\varepsilon_4') = t_4(f_3(\varepsilon_3')) = g_3 t_3(\varepsilon_3')$ and so $g_3(\varepsilon_3) - g_3 t_3(\varepsilon_3') = 0 \Longrightarrow g_3(\varepsilon_3 - t_3(\varepsilon_3')) = 0 \Longrightarrow \varepsilon_3 - t_3(\varepsilon_3') \in kerg_3 \Longrightarrow \varepsilon_3 - t_3(\varepsilon_3') \in Img_2$ because $Img_2 = kerg_3$. Therefore there exists an element $\varepsilon_2 \in G_2(y)$ for all $y \in \beta_2$ such that $g_2(\varepsilon_2) = \varepsilon_3 - t_3(\varepsilon_3')$. The homomorphism t_2 being an epimorphism, there exists an element $\varepsilon_2' \in F_2(x)$ for all $x \in \alpha_2$ such that $t_2(\varepsilon_2') = \varepsilon_2$. But then $g_2(\varepsilon_2) = g_2(t_2(\varepsilon_2')) \Longrightarrow \varepsilon_3 - t_3(\varepsilon_3') = g_2(t_2(\varepsilon_2')) = t_3 f_2(\varepsilon_2')$ by commutativity of diagram. It follows that $\varepsilon_3 = t_3 f_2(\varepsilon_2') + t_3(\varepsilon_3') = t_3(f_2(\varepsilon_2') + \varepsilon_3')$. Hence t_3 is epimorphism.

(2) Suppose $t_2$and t_4 are monomorphisms and t_1 is an epimorphism. For all $x \in \alpha_3$, let $\varepsilon_3' \in F_3(x)$ such that $t_3(\varepsilon_3') = 0$. Then $0 = g_3(0) = g_3 t_3(\varepsilon_3') = t_4(f_3(\varepsilon_3'))$ because diagram is commutative, $\Longrightarrow 0 = t_4(f_3(\varepsilon_3'))$. Now t_4 being a monomorphism we have $f_3(\varepsilon_3') = 0$. It follows that $\varepsilon_3' \in kerf_3$ but due to exact row $kerf_3 = Imf_2 \Longrightarrow \varepsilon_3' \in Imf_2$. Then there exists an element $\varepsilon_2' \in F_2(x)$ for all $x \in \alpha_2$ such that $f_2(\varepsilon_2') = \varepsilon_3'$. But from commutativity of diagram $g_2 t_2(\varepsilon_2') = t_3 f_2(\varepsilon_2')$. Then it follows that $g_2 t_2(\varepsilon_2') = t_3 f_2(\varepsilon_2') = t_3(\varepsilon_3') = 0$. This implies that $g_2 t_2(\varepsilon_2') = 0 \Longrightarrow t_2(\varepsilon_2') \in kerg_2$ and hence $t_2(\varepsilon_2') \in Img_1$. Then there exists an element $\varepsilon_1 \in G_1(y)$ for all $y \in \beta_1$ such that $g_1(\varepsilon_1) = t_2(\varepsilon_2')$. Since t_1 is an epimorphism, then there exists an element $\varepsilon_1' \in F_1(x)$ for all $x \in \alpha_1$ such that $t_1(\varepsilon_1') = \varepsilon_1$. Then $t_2(\varepsilon_2') = g_1(\varepsilon_1) = g_1(t_1(\varepsilon_1')) = t_2 f_1(\varepsilon_1') \Longrightarrow t_2(\varepsilon_2') = t_2 f_1(\varepsilon_1')$. Now t_2 being a monomorphism, we have $\varepsilon_2' = f_1(\varepsilon_1')$ and hence $\varepsilon_3' = f_2(\varepsilon_2') = f_2(f_1(\varepsilon_1')) = 0 \Longrightarrow \varepsilon_3' = 0$. Hence proved that t_3 is a monomorphism. □

The following corollary is the consequence of lemma 2.

Corollary 2. *Consider a commutative diagram with exact rows*



If any two t_1, t_2, t_3 are isomorphism, then so is the third.

Proof. The proof follows from lemma 2. □

5. Conclusion

The research done in this paper can be regarded as a systematic study of soft LA-modules. We have presented a detailed theoretical study of the homomorphism together with the exact sequence of soft LA-modules. A functional approach has been developed to undertake a characterization of soft LA-modules which leads to a determination of some of its interesting theoretic properties. We have established some useful results regarding soft LA-modules, homomorphisms, soft quotient LA-modules and finally the exact sequence. This study can be enhanced by defining projective soft LA-modules, injective soft LA-modules, tensor product of soft LA-modules and their related properties.

Compliance with ethical standards

This study was not funded by any organization, institution or any other person.

Ethical approval: This article does not contain any studies with human participants or animals performed by any of the authors. The authors declare that they have no con‡ict of interest regarding the publication of this article.

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