SCHUR m-POWER CONVEXITY OF GEOMETRIC BONFERRONI MEAN

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Abstract. In this paper the Schur m-power convexity of the geometric Bonferroni mean for n variables is discussed.
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1. Introduction

Throughout the paper $\mathbb{R}$ denotes the set of real numbers, $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ denotes n-tuple (n-dimensional real vectors), the set of vectors can be written as

$$\mathbb{R}^n = \{\boldsymbol{x} = (x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R}, i = 1, 2, \ldots, n\},$$

$$\mathbb{R}^n_+ = \{\boldsymbol{x} = (x_1, x_2, \ldots, x_n) : x_i \geq 0, i = 1, 2, \ldots, n\},$$

$$\mathbb{R}^n_{++} = \{\boldsymbol{x} = (x_1, x_2, \ldots, x_n) : x_i > 0, i = 1, 2, \ldots, n\}.$$

In particular, the notations $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{R}_{++}$ denote $\mathbb{R}^1$, $\mathbb{R}^1_+$ and $\mathbb{R}^1_{++}$, respectively.

The Bonferroni mean has important application in multi criteria decision-making (see[2–7]), it was initially proposed by Bonferroni [1], as follows

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Definition 1. *Let* $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}_+^n$ *and* $p, q \geq 0, p + q \neq 0$*. The Bonferroni mean is defined by*

$$B^{p,q}(\boldsymbol{x}) = \left(\frac{1}{n(n-1)} \sum_{i,j=1, i \neq j}^{n} x_i^p x_j^q \right)^{\frac{1}{p+q}}. \tag{1}$$

Motivated by the Bonferroni mean $B^{p,q}(\boldsymbol{x})$ and the geometric mean $G(\boldsymbol{x}) = \prod_{i=1}^{n} (x_i)^{\frac{1}{n}}$, Xia et al. [4] introduced a new mean which is called the geometric Bonferroni mean, i.e.,

Definition 2. *Let* $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}_{++}^n$ *and* $(p, q) \in \mathbb{R}_{++}^2$*. The geometric Bonferroni mean is defined by*

$$GB^{p,q}(\boldsymbol{x}) = \frac{1}{p+q} \prod_{i,j=1, i \neq j}^{n} (px_i + qx_j)^{\frac{1}{n(n-1)}}. \tag{2}$$

Obviously, the geometric Bonferroni mean has the following properties:

(*i*) $GB^{p,q}(0, 0, \ldots, 0) = 0$.

(*ii*) $GB^{p,q}(x, x, \ldots, x) = x$, if $x_i = x$ for $i = 1, 2, \ldots, n$.

(*iii*) $GB^{p,q}(\boldsymbol{x}) \geq GB^{p,q}(\boldsymbol{y})$, if $x_i \geq y_i$ for $i = 1, 2, \ldots, n$.

(*iv*) $\min\{x_1, x_2, \ldots, x_n\} \leq GB^{p,q}(\boldsymbol{x}) \leq \max\{x_1, x_2, \ldots, x_n\}$.

Furthermore, if $q = 0$, then the geometric Bonferroni mean reduces to the geometric mean, i.e.,

$$GB^{p,0}(\boldsymbol{x}) = \frac{1}{p} \prod_{i,j=1, i \neq j}^{n} (px_i)^{\frac{1}{n(n-1)}} = \prod_{i=1}^{n} (x_i)^{\frac{1}{n}} = G(\boldsymbol{x}).$$

In recent years, the theory of majorization has been used as an important tool in studying the properties of the mean. Yang [8],[9],[10] generalized the notion of Schur convexity to Schur f-convexity and discussed the Schur m-power convexity of Stolarsky means [8], Gini means [9] and Daróczy means [10]. Subsequently, the Schur m-power convexity has evoked the interest of many researchers (see [11], [12], [13], [14]).

In this paper, we discuss the Schur m-power convexity of the geometric Bonferroni mean $GB^{p,q}(\boldsymbol{x})$, Our main results are stated in the following theorem.

Theorem 1. *Let* $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}_{++}^n$*. For fixed* $(p, q) \in \mathbb{R}_{++}^2$ *and* $n \geq 3$,

(*i*) *if* $m < 0$*, then* $GB^{p,q}(\boldsymbol{x})$ *is Schur* m*-power convex;*

(*ii*) *if* $m = 0$*, then* $GB^{p,q}(\boldsymbol{x})$ *is Schur* m*-power convex;*

(*iii*) *if* $m = 1$*, then* $GB^{p,q}(\boldsymbol{x})$ *is Schur* m*-power concave;*

(*iv*) *if* $m \geq 2$*, then* $GB^{p,q}(\boldsymbol{x})$ *is Schur* m*-power concave.*

2. Preliminaries

We begin with recalling some basic concepts and notations in the theory of majorization. For more details, we refer the reader to [15, 16].

Definition 3. *Let* $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ *and* $\boldsymbol{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$.

(*i*) $\boldsymbol{x} \geq \boldsymbol{y}$ *means* $x_i \geq y_i$ *for all* $i = 1, 2, \ldots, n$.

(*ii*) *Let* $\Omega \subset \mathbb{R}^n$, $\varphi\colon \Omega \to \mathbb{R}$ *is said to be increasing if* $\boldsymbol{x} \geq \boldsymbol{y}$ *implies* $\varphi(\boldsymbol{x}) \geq \varphi(\boldsymbol{y})$. φ *is said to be decreasing if and only if* $-\varphi$ *is increasing.*

Definition 4. *Let* $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ *and* $\boldsymbol{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$.

(*i*) $\boldsymbol{x}$ *is said to be majorized by* $\boldsymbol{y}$ *(in symbols* $\boldsymbol{x} \prec \boldsymbol{y}$*) if* $\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}$ *for* $k = 1, 2, \ldots, n-1$ *and* $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$, *where* $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}$ *and* $y_{[1]} \geq y_{[2]} \geq \cdots \geq y_{[n]}$ *are rearrangements of* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *in a descending order.*

(*ii*) *Let* $\Omega \subset \mathbb{R}^n$*,the function* $\varphi\colon \Omega \to \mathbb{R}$ *is said to be Schur convex on* Ω *if* $\boldsymbol{x} \prec \boldsymbol{y}$ *on* Ω *implies* $\varphi(\boldsymbol{x}) \leq \varphi(\boldsymbol{y})$. φ *is said to be Schur concave function on* Ω *if and only if* $-\varphi$ *is Schur convex function on* Ω.

Definition 5. *Let* $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ *and* $\boldsymbol{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$.

(*i*) $\Omega \subset \mathbb{R}^n$ *is said to be a convex set if* $\boldsymbol{x}, \boldsymbol{y} \in \Omega$, $0 \leq \alpha \leq 1$ *implies*

$$\alpha\boldsymbol{x}+(1-\alpha)\boldsymbol{y}= (\alpha x_1+(1-\alpha)y_1, \alpha x_2+(1-\alpha)y_2, \ldots, \alpha x_n + (1-\alpha)y_n) \in\Omega.$$

(*ii*) *Let* $\Omega \subset \mathbb{R}^n$ *be convex set. A function* $\varphi\colon \Omega \to \mathbb{R}$ *is said to be convex on* Ω *if*

$$\varphi(\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \leq \alpha\varphi(\boldsymbol{x}) + (1-\alpha)\varphi(\boldsymbol{y})$$

for all $\boldsymbol{x}, \boldsymbol{y} \in \Omega$, *and all* $\alpha \in [0, 1]$. *The function* φ *is said to be concave on* Ω *if and only if* $-\varphi$ *is convex function on* Ω.

Definition 6. (*i*) *A set* $\Omega \subset \mathbb{R}^n$ *is called symmetric, if* $\boldsymbol{x} \in \Omega$ *implies* $\boldsymbol{x}P \in \Omega$ *for every* $n \times n$ *permutation matrix* P.

(*ii*) *A function* $\varphi : \Omega \to \mathbb{R}$ *is called symmetric if for every permutation matrix* P, $\varphi(\boldsymbol{x}P) = \varphi(\boldsymbol{x})$ *for all* $\boldsymbol{x} \in \Omega$.

The first systematical study of the functions preserving the ordering of majorization was made by I. Schur in 1923. In Schur's honor, such functions are said to be Schur convex (see [15]). It can be used extensively in analytic inequalities, combinatorial optimization, quantum physics, information theory, and other related fields.

The following proposition is called Schur's condition (see[15]). It provides an approach for testing whether a vector valued function is Schur convex or not.

Proposition 1. *Let $\Omega \subset \mathbb{R}^n$ be symmetric and have a nonempty interior convex set. Ω^0 is the interior of Ω. $\varphi : \Omega \to \mathbb{R}$ is continuous on Ω and differentiable in Ω^0. Then φ is the Schur convex function (Schur concave function) if and only if φ is symmetric on Ω and*

$$(x_1 - x_2)\left[\frac{\partial\varphi(\boldsymbol{x})}{\partial x_1} - \frac{\partial\varphi(\boldsymbol{x})}{\partial x_2}\right] \geq 0 \ (\leq 0) \tag{3}$$

holds for any $\boldsymbol{x} \in \Omega^0$.

A generalization of Schur convex functions was introduced by Yang [8], as follows

Definition 7. *Let $f : \mathbb{R}_{++} \to \mathbb{R}$ be defined by*

$$f(x) = \begin{cases} \dfrac{x^m - 1}{m}, & m \neq 0; \\ \ln x, & m = 0. \end{cases} \tag{4}$$

Then a function $\varphi : \Omega \subset \mathbb{R}^n_{++} \to \mathbb{R}$ is said to be Schur m-power convex on Ω if

$$(f(x_1), f(x_2), \ldots, f(x_n)) \prec (f(y_1), f(y_2), \ldots, f(y_n))$$

for all $(x_1, x_2, \ldots, x_n) \in \Omega$ and $(y_1, y_2, \ldots, y_n) \in \Omega$ implies $\phi(x) \leq \phi(y)$.

If $-\varphi$ is Schur m-power convex, then we say that φ is Schur m-power concave.

Similarly to the Schur's condition mentioned above, Yang [8] gave a method of determining the Schur m-power convex functions, i.e.,

Proposition 2. *Let $\Omega \subset \mathbb{R}^n_{++}$ be a symmetric set with nonempty interior Ω° and $\varphi : \Omega \to \mathbb{R}$ be continuous on Ω and differentiable in Ω°. Then φ is Schur m-power convex (Schur m-power concave) on Ω if and only if φ is symmetric on Ω and*

$$\frac{x_1^m - x_2^m}{m}\left[x_1^{1-m}\frac{\partial\varphi(\boldsymbol{x})}{\partial x_1} - x_2^{1-m}\frac{\partial\varphi(\boldsymbol{x})}{\partial x_2}\right] \geq 0 \ (\leq 0), \quad \text{if } \ m \neq 0 \tag{5}$$

and

$$(\log x_1 - \log x_2)\left[x_1\frac{\partial\varphi(\boldsymbol{x})}{\partial x_1} - x_2\frac{\partial\varphi(\boldsymbol{x})}{\partial x_2}\right] \geq 0 \ (\leq 0), \quad \text{if } \ m = 0 \tag{6}$$

for all $\boldsymbol{x} \in \Omega^\circ$.

3. Proof of Theorem 1

Proof. Note that the geometric Bonferroni mean is defined by

$$GB^{p,q}(\boldsymbol{x}) = \frac{1}{p+q}\prod_{i,j=1,i\neq j}^{n}(px_i+qx_j)^{\frac{1}{n(n-1)}}.$$

Taking the natural logarithm gives

$$\log GB^{p,q}(\boldsymbol{x}) = \log\frac{1}{p+q} + \frac{1}{n(n-1)}Q$$

where

$$\begin{aligned} Q = &\sum_{j=3}^{n}[\log(px_1+qx_j)+\log(px_2+qx_j)] + \sum_{i=3}^{n}[\log(px_i+qx_1)+\log(px_i+qx_2)] \\ &+\log(px_1+qx_2)+\log(px_2+qx_1)+\sum_{i,j=3,i\neq j}^{n}\log(px_i+qx_j). \end{aligned}$$

Hence, we have

$$\begin{aligned} \frac{\partial GB^{p,q}(\boldsymbol{x})}{\partial x_1} &= \frac{GB^{p,q}(\boldsymbol{x})}{n(n-1)}\left(\sum_{j=3}^{n}\frac{p}{px_1+qx_j}+\sum_{i=3}^{n}\frac{q}{px_i+qx_1}\right. \\ &\quad \left.+\frac{p}{px_1+qx_2}+\frac{q}{px_2+qx_1}\right) \\ \frac{\partial GB^{p,q}(\boldsymbol{x})}{\partial x_2} &= \frac{GB^{p,q}(\boldsymbol{x})}{n(n-1)}\left(\sum_{j=3}^{n}\frac{p}{px_2+qx_j}+\sum_{i=3}^{n}\frac{q}{px_i+qx_2}\right. \\ &\quad \left.+\frac{q}{px_1+qx_2}+\frac{p}{px_2+qx_1}\right) \end{aligned}$$

It is easy to see that $GB^{p,q}(\boldsymbol{x})$ is symmetric on $\mathbb{R}^n_+$. Without loss of generality, we may assume that $x_1 \geq x_2$.

Direct computation gives

$$\begin{aligned} \Delta := &\frac{x_1^m-x_2^m}{m}\left(x_1^{1-m}\frac{GB^{p,q}(\boldsymbol{x})}{\partial x_1} - x_2^{1-m}\frac{\partial GB^{p,q}(\boldsymbol{x})}{\partial a_2}\right) \\ = &\frac{(x_1^m-x_2^m)GB^{p,q}(\boldsymbol{x})}{mn(n-1)}\Big[p\sum_{j=3}^{n}(\frac{x_1^{1-m}}{px_1+qx_j}-\frac{x_2^{1-m}}{px_2+qx_j}) \end{aligned}$$

$$
\begin{aligned}
&+q\sum_{i=3}^{n}\Big(\frac{x_1^{1-m}}{px_i+qx_1}-\frac{x_2^{1-m}}{px_i+qx_2}\Big)\\
&+\frac{px_1^{1-m}-qx_2^{1-m}}{px_1+qx_2}+\frac{qx_1^{1-m}-px_2^{1-m}}{px_2+qx_1}\Big]\\
&=\frac{(x_1^m-x_2^m)GB^{p,q}(\boldsymbol{x})}{mn(n-1)}\Big[p\sum_{j=3}^{n}\frac{px_1x_2(x_1^{-m}-x_2^{-m})+qx_j(x_1^{1-m}-x_2^{1-m})}{(px_1+qx_j)(px_2+qx_j)}\\
&+q\sum_{i=3}^{n}\frac{qx_1x_2(x_1^{-m}-x_2^{-m})+px_i(x_1^{1-m}-x_2^{1-m})}{(px_i+qx_1)(px_i+qx_2)}\\
&+\frac{x_1x_2(p^2+q^2)(x_1^{-m}-x_2^{-m})+2pq(x_1^{2-m}-x_2^{2-m})}{(px_1+qx_2)(px_2+qx_1)}\Big].
\end{aligned}
$$

If $m<0$, then $x_1^m-x_2^m\le 0$, $x_1^{-m}-x_2^{-m}\ge 0$, $x_1^{1-m}-x_2^{1-m}\ge 0$ and $x_1^{2-m}-x_2^{2-m}\ge 0$. Thus, $\Delta\ge 0$. From Proposition 2, it follows that $GB^{p,q}(\boldsymbol{x})$ is Schur m-power convex for $\boldsymbol{x}\in\mathbb{R}_{++}^n$.

If $m\ge 2$, then $x_1^m-x_2^m\ge 0$, $x_1^{-m}-x_2^{-m}\le 0$, $x_1^{1-m}-x_2^{1-m}\le 0$ and $x_1^{2-m}-x_2^{2-m}\le 0$. Thus, $\Delta\le 0$. By Proposition 2, we conclude that $GB^{p,q}(\boldsymbol{x})$ is Schur m-power concave for $\boldsymbol{x}\in\mathbb{R}_{++}^n$.

If $m=1$, then

$$
\begin{aligned}
\Delta=&-\frac{(x_1-x_2)^2GB^{p,q}(\boldsymbol{x})}{n(n-1)}\Big[\sum_{j=3}^{n}\frac{p^2}{(px_1+qx_j)(px_2+qx_j)}\\
&+\sum_{i=3}^{n}\frac{q^2}{(px_i+qx_1)(px_i+qx_2)}+\frac{(p-q)^2}{(px_1+qx_2)(px_2+qx_1)}\Big]\\
&\le 0.
\end{aligned}
$$

By using Proposition 2, we deduce that $GB^{p,q}(\boldsymbol{x})$ is Schur m-power concave for $\boldsymbol{x}\in\mathbb{R}_{++}^n$.

If $m=0$, then

$$
\begin{aligned}
\Delta&=(\log x_1-\log x_2)\left(x_1\frac{GB^{p,q}(\boldsymbol{x})}{\partial x_1}-x_2\frac{\partial GB^{p,q}(\boldsymbol{x})}{\partial x_2}\right)\\
&=\frac{(\log x_1-\log x_2)GB^{p,q}(\boldsymbol{x})}{n(n-1)}\Big[p\sum_{j=3}^{n}\Big(\frac{x_1}{px_1+qx_j}-\frac{x_2}{px_2+qx_j}\Big)\\
&+q\sum_{i=3}^{n}\Big(\frac{x_1}{px_i+qx_1}-\frac{x_2}{px_i+qx_2}\Big)+\frac{px_1-qx_2}{px_1+qx_2}+\frac{qx_1-px_2}{px_2+qx_1}\Big]
\end{aligned}
$$

$$= \frac{(x_1 - x_2)(\log x_1 - \log x_2)GB^{p,q}(\boldsymbol{x})}{n(n-1)}\Big[p\sum_{j=3}^{n}\frac{qx_j}{(px_1+qx_j)(px_2+qx_j)}$$
$$+ q\sum_{i=3}^{n}\frac{px_i}{(px_i+qx_1)(px_i+qx_2)} + \frac{2pq(x_1+x_2)}{(px_1+qx_2)(px_2+qx_1)}\Big]$$
$$\geq 0.$$

From Proposition 2, we conclude that $GB^{p,q}(\boldsymbol{x})$ is Schur m-power convex for $\boldsymbol{x} \in \mathbb{R}^n_{++}$.

The proof of Theorem 1 is completed.

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