ALMOST STRONGLY ω-CONTINUOUS FUNCTIONS

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Abstract. The aim of this paper is to introduce and investigate a new class of continuity, called almost strongly ω-continuous function, which contains the class of strongly θ-continuous functions and it is contained in the class of almost ω-continuous functions.
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1. Introduction

Throughout this paper, spaces always mean topological spaces with no separation axioms assumed, unless otherwise stated. Let (X, τ) be a space and A be a subset of X. The closure of A and the interior of A are denoted by $cl(A)$ and $Int(A)$, respectively. A point $x \in X$ is called a condensation point of A [4] if for each open set U containing x, the set $U - A$ is uncountable. A is said to be $\omega-$closed [5] if it contains all its condensation points. The complement of an $\omega-$closed set is said to be $\omega-$open. Note that a subset A of a space (X, τ) is $\omega-$open [2] if and only if for each $x \in A$ there exists an open set U containing x such that $U - A$ is countable. The family of all $\omega-$open subsets of a space (X, τ), forms a topology on X, denoted by τ_ω, finer than τ. The closure of A in (X, τ_ω) and the interior of A in (X, τ_ω) are denoted by $cl_\omega(A)$ and $Int_\omega(A)$. Several characterizations of $\omega-$closed subsets were proved in [5]. A subset A

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is said to be regular open [11] (resp. regular closed) if $Int(cl(A)) = A$ (resp. $cl(Int(A)) = A$).

A point $x \in X$ is called a $\delta-$cluster [12] (resp. $\theta_{o\omega}$-cluster [3]) point of A if $A \cap Int(cl(U)) \neq \phi$ (resp. $A \cap cl(U) \neq \phi$) for each open (resp. ω-open) set U containing x. The set of all $\delta-$cluster (resp. $\theta_{o\omega}$-cluster) points of A is called the δ-closure (resp. the $\theta_{o\omega}$-closure) of A and is denoted by $[A]_\delta$ (resp. $[A]_{\theta_{o\omega}}$). If $[A]_\delta = A$ (resp. $[A]_{\theta_{o\omega}} = A$), then A is said to be $\delta-$closed (resp. $\theta_{o\omega}-$closed). The complement of a $\delta-$closed (resp. $\theta_{o\omega}$-closed) set is said to be δ-open (resp. $\theta_{o\omega}$-open).

A subset A of a space X is said to be an H-set [12] or quasi H-closed relative to X [8] if for every cover $\{U_\alpha : \alpha \in \Delta\}$ of A by open sets of X, there exists a finite subset $\Delta_\circ$ of Δ such that $A \subseteq \cup\{cl(U_\alpha) : \alpha \in \Delta_\circ\}$. A space X is said to be quasi H-closed [8] if the set X is quasi H-closed relative to X. Quasi H-closed Hausdorff spaces are usually said to be $H-$closed.

For a nonempty set X, τ_{dis} will denote the discrete topology on X. $\mathbb{R}$ and $\mathbb{Q}$ denote the sets of all real numbers and rational numbers. Finally if (X, τ) and (Y, σ) are two space, then $\tau \times \sigma$ will denote the product topology on $X \times Y$.

Definition 1.1. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be δ-continuous [7] (resp. almost continuous [9] , strongly θ-continuous [7]) if for each $x \in X$ and each open set V containing $f(x)$, there exists an open set U of x such that $f(Int(cl(U))) \subseteq Int(cl(V))$ (resp. $f(U) \subseteq Int(cl(V))$, $f(cl(U)) \subseteq V$).

Definition 1.2. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be ω-continuous [6] (resp. weakly ω-continuous [1], almost ω-continuous [1]) if for each $x \in X$ and each open set V of Y containing $f(x)$ there exists an $\omega-$open set U containing x such that $f(U) \subseteq V$ (resp. $f(U) \subseteq cl(V)$, $f(U) \subseteq Int(cl(V))$).

Definition 1.3. A space (X, τ) is said to be $\omega - \mathrm{T}_2$[1] (resp. $\omega-$Uryshon [1]) if for each pair of distinct points x and y in X, there exist $\omega-$open sets U and V containing x and y, respectively, such that $U \cap V = \phi$ (resp. $cl_\omega(U) \cap cl_\omega(V) = \phi$).

Proposition 1.4. [3] *Let A be a subset of a space (X, τ), A is $\theta_{o\omega}-$open if and only if for each $x \in A$ there exists an $\omega-$open set U containing x such that $cl(U) \subseteq A$.*

Lemma 1.5. [2] *Let A be a subset of a space (X, τ). Then:*

i. $(\tau_\omega)_\omega = \tau_\omega$.

ii. $(\tau_A)_\omega = (\tau_\omega)_A$.

2. Almost strongly ω-continuous functions

Definition 2.1. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be almost strongly ω-continuous if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists an $\omega-$open set U containing x such that $f(cl(U)) \subseteq Int(cl(V))$.

Clearly, the following diagram follows immediately from the definitions and facts.

$$\begin{array}{ccccccc} \text{Continuous} & \to & \omega\text{-continuous} & \to & \text{almost } \omega\text{-continuous} & \to & \text{weakly } \omega\text{-continuous} \\ \uparrow & & & & \uparrow & & \\ \text{Strongly } \theta\text{-continuous} & \to & & & \text{almost strongly } \omega\text{-continuous} & & \end{array}$$

Note that almost strong ω-continuity and continuity (resp. ω-continuity) are independent of each other as the following examples show.

Example 2.2. Let $X=\{a,b,c,d\}$ with the topology $\tau=\{\phi,X,\{c\},\{a,b\},\{a,b,c\}\}$ and let $Y=\{p,q,r\}$ with the topology $\sigma=\{\phi,Y,\{p\},\{q\},\{p,q\}\}$. Define a function $f:(X,\tau)\to(Y,\sigma)$ as follows:

$$f(x)=\begin{cases} p & : \ x=a,b \\ r & : \ x=c,d \end{cases}.$$

Then f is continuous (hence, ω-continuous) but it is not almost strongly ω-continuous at $x=a$.

Example 2.3. Let $X=\mathbb{R}$ with the topologies $\tau=\tau_u$ and $\sigma=\{\phi,\mathbb{R},\mathbb{R}-\{0\}\}$, where τ_u is the standard topology. Let $f:(X,\tau)\to(X,\sigma)$ be the function defined by

$$f(x)=\begin{cases} 0, & x\in\mathbb{R}-\mathbb{Q} \\ 1, & x\in\mathbb{Q} \end{cases}.$$

Then f is not ω-continuous since $V=\mathbb{R}-\{0\}\in\sigma$, but $f^{-1}(V)=\mathbb{Q}\notin\tau_\omega$. On the other hand, f is almost strongly ω-continuous.

Next, several characterizations of almost strongly ω-continuous functions are obtained.

Theorem 2.4. *For a function $f:(X,\tau)\to(Y,\sigma)$, the following are equivalent:*

i. *f is almost strongly ω-continuous.*

ii. *The inverse image of a regular open set in (Y,σ) is $\theta_{o\omega}$-open in (X,τ).*

iii. *The inverse image of a regular closed set in (Y,σ) is $\theta_{o\omega}$-closed in (X,τ).*

iv. *For each $x\in X$ and each regular open set V in (Y,σ) containing $f(x)$, there exists an ω-open set U in (X,τ) containing x such that $f(cl(U))\subseteq V$.*

v. *The inverse image of a $\delta-$open set in (Y,σ) is $\theta_{o\omega}$-open in (X,τ).*

vi. *The inverse image of a δ-closed set in (Y,σ) is $\theta_{o\omega}$-closed in (X,τ).*

vii. *$f([A]_{\theta_{o\omega}})\subset[f(A)]_\delta$ for each subset A of X.*

viii. $[f^{-1}(B)]_{\theta_{o\omega}} \subset f^{-1}([B]_\delta)$ *for each subset* B *of* Y.

Proof. (i→ii) Let V be any regular open set in (Y,σ) and $x \in f^{-1}(V)$. Then $f(x) \in V$ and there exists an $\omega-$open set U in (X,τ) containing x such that $f(cl(U)) \subset V$. Thus $x \in U \subseteq cl(U) \subseteq f^{-1}(V)$ and hence, by Proposition 1.4, $f^{-1}(V)$ is $\theta_{o\omega}$-open.

(ii→iii) Let F be any regular closed set in (Y,σ). By (ii), $f^{-1}(F) = X - f^{-1}(Y-F)$ is $\theta_{o\omega}$-closed in X.

(iii→iv) Let $x \in X$ and V be any regular open set in (Y,σ) containing $f(x)$. By (iii), $f^{-1}(Y-V) = X - f^{-1}(V)$ is $\theta_{o\omega}$-closed in (X,τ). Since $f^{-1}(V)$ is a $\theta_{o\omega}$-open set containing x, by Proposition 1.4, there exists an $\omega-$open set U containing x such that $cl(U) \subseteq f^{-1}(V)$; hence $f(cl(U)) \subseteq V$.

(iv→v) Let V be a δ-open set in (Y,σ) and $x \in f^{-1}(V)$. There exists a regular open set G in (Y,σ) such that $f(x) \in G \subseteq V$. By (iv), there exists an ω-open set U containing x such that $f(cl(U)) \subseteq G$. Therefore, we obtain $x \in U \subseteq cl(U) \subseteq f^{-1}(V)$. Hence, by Proposition 1.4, $f^{-1}(V)$ is $\theta_{o\omega}$-open in (X,τ).

(v→vi) Let F be a δ-closed set in (Y,σ). By (v) we have $f^{-1}(F) = X - f^{-1}(Y-E)$ is $\theta_{o\omega}$-closed in (X,τ).

(vi→vii) Let A be a subset of X. Since $[f(A)]_\delta$ is $\delta-$closed in (Y,σ), by (vi), $f^{-1}([f(A)]_\delta)$ is $\theta_{o\omega}$-closed in (X,τ). Let $x \notin f^{-1}([f(A)]_\delta)$. Then for some ω-open set U in (X,τ) containing x, $cl(U) \cap f^{-1}([f(A)]_\delta) = \phi$ and hence $cl(U) \cap A = \phi$. So $x \notin [A]_{\theta_{o\omega}}$. Therefore, we have $f([A]_{\theta_{o\omega}}) \subseteq [f(A)]_\delta$.

(vii→viii) Let B be a subset of Y. By (vii) we have $f([f^{-1}(B)]_{\theta_{o\omega}}) \subseteq [B]_\delta$ and hence $[f^{-1}(B)]_{\theta_{o\omega}} \subseteq f^{-1}([B]_\delta)$.

(viii→i) Let $x \in X$ and V be an open set in (Y,σ) containing $f(x)$. Then $G = Y - Int(cl(V))$ is regular closed and hence δ-closed in (Y,σ). By (viii), $[f^{-1}(G)]_{\theta_{o\omega}} \subseteq f^{-1}(G)$ and hence $f^{-1}(G)$ is $\theta_{o\omega}$-closed in (X,τ). Therefore, $f^{-1}(Int(cl(V)))$ is a $\theta_{o\omega}-$open set in (X,τ) containing x. By Proposition 1.4 there exists an ω-open set U containing x such that $x \in U \subseteq cl(U) \subseteq f^{-1}(Int(cl(V)))$. Therefore, we obtain $f(cl(U)) \subseteq Int(cl(V))$. This show that f is almost strongly $\omega-$continuous. □

Note that the family of all $\theta_{o\omega}-$open [3] (resp. $\delta-$open [12]) sets in a space (X,τ) form a topology for X which is denoted by $\tau_{\theta_{o\omega}}$ (resp. τ_δ).

Theorem 2.5. *For a function* $f : (X,\tau) \to (Y,\sigma)$*, the following are equivalent:*

i. $f : (X,\tau) \to (Y,\sigma)$ *is almost strongly* $\omega-$*continuous.*

ii. $f : (X,\tau_{\theta_{o\omega}}) \to (Y,\sigma)$ *is almost continuous.*

iii. $f : (X,\tau_{\theta_{o\omega}}) \to (Y,\sigma_\delta)$ *is continuous.*

Proof. (i→ii) Let V be any regular open set in (Y,σ). By Theorem 2.4 $f^{-1}(V)$ is $\theta_{o\omega}$-open in (X,τ) and hence open in $(X,\tau_{\theta_{o\omega}})$, it follows from Theorem 2.2 of [9] that f is almost continuous.

(ii$\rightarrow$iii) Let V be an open in (Y, σ_δ). Then V is δ-open in (Y, σ) and it is the union of regular open sets in (Y, σ). By (ii), $f^{-1}(V)$ is open in $(X, \tau_{\theta_{o\omega}})$. Therefore, $f : (X, \tau_{\theta_{o\omega}}) \rightarrow (Y, \sigma_\delta)$ is continuous.

(iii$\rightarrow$i) Let V be a regular open set in (Y, σ). Since V is open in (Y, σ_δ), by (iii), $f^{-1}(V)$ is $\theta_{o\omega}$-open in (X, τ) and hence by Theorem 2.4, f is almost strongly ω-continuous. □

The composition of two almost strongly $\omega-$continuous functions need not be almost strongly $\omega-$continuous as the following examples shows.

Example 2.6. Let $X = \mathbb{R}$, $Y = \{0, 1\}$ and $Z = \{1, 2, 3\}$ with the topologies $\tau = \{\phi, X, \mathbb{Q}\}$, $\sigma = \{\phi, Y, \{0\}\}$, $\rho = \{\phi, Z, \{1\}, \{2\}, \{1, 2\}\}$ defined on X, Y and Z respectively. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be the function defined by

$$f(x) = \begin{cases} 0, & x \in \mathbb{R} - \mathbb{Q} \\ 1, & x \in \mathbb{Q} \end{cases}$$

and let $g : (Y, \sigma) \rightarrow (Z, \rho)$ be the function defined by

$$g(y) = \begin{cases} 1, & y = 1 \\ 3, & y = 0 \end{cases}.$$

Then f and g are almost strongly $\omega-$continuous. However $g \circ f$ is not almost strongly $\omega-$continuous at $x \in \mathbb{Q}$. For more clarify, let $x \in \mathbb{Q} \subseteq X$, $(g \circ f)(x) = g(f(x)) = g(1) = 1 \in V = \{1\} \in \rho$. Now for every $\omega-$open set W containing x, $cl(W) = \mathbb{R}$, therefore $(g \circ f)(cl_\rho(W)) = g(f(\mathbb{R})) = g(\{0, 1\}) = \{1, 3\} \not\subset \{1\}$.

Theorem 2.7. *Let $f : (X, \tau) \rightarrow (Y, \sigma)$ and $g : (Y, \sigma) \rightarrow (Z, \rho)$ be functions. Then the following hold:*

a. $g \circ f$ is almost strongly ω-continuous if f is almost strongly ω-continuous and g is δ-continuous.

b. $g \circ f$ is almost strongly ω-continuous if f is almost strongly ω-continuous and g is continuous and open.

c. Let $p : (X \times Y, \tau \times \sigma) \rightarrow (X, \tau)$ be the projection function. If $(f \circ p)$ is almost strongly ω-continuous, then f is almost strongly ω-continuous.

Proof. The proof of a. follows immediately from Definitions 1.1 and 2.1. Thus we prove only part b and c.

b) Let $x \in X$ and V be any open set in (Z, ρ) such that $(g \circ f)(x) \in V$. Therefore $f(x) \in g^{-1}(V)$ which is open in (Y, σ). Since f is almost strongly ω-continuous, there exists an ω-open set W in (X, τ) such that $x \in W$ and $f(cl(W)) \subseteq Int(cl(g^{-1}(V)))$. Therefore $(g \circ f)(cl(W)) = g(Int(cl(g^{-1}(V)))) \subseteq Int(cl(V))$.

c) Let $x \in X$ and V be any open set in (Y, σ) such that $f(x) \in V$. Choose $y \in Y$. Then $(f \circ p)(x, y) = f(x) \in V$. Since $(f \circ p)$ is almost strongly ω-continuous, there exists an ω-open set W in $X \times Y$ such that $(x, y) \in W$ and $(f \circ p)(cl_{\tau\times\sigma}(W)) \subseteq Int_\sigma(cl_\sigma(V))$. Since $(x, y) \in W$, choose $W_1 \in (X, \tau_\omega)$ and $W_2 \in (Y, \sigma_\omega)$ such that $x \in W_1$, $y \in W_2$ and $(x, y) \in W_1 \times W_2 \subseteq W$ and so $(f \circ p)(cl_{\tau\times\sigma}(W_1 \times W_2)) = f(cl(W_1)) \subseteq (f \circ p)(cl_{\tau\times\sigma}(W)) \subseteq Int(cl(V))$. Thus $f(cl_\tau(W_1)) \subseteq Int(cl(V))$ and so f is almost strongly $\omega-$continuous. □

To show that the assumption g is a continuous open function in part (b) of Theorem 2.7 is essential and that the projection function p in the same theorem part (c) can not be replaced by arbitrary open continuous function we consider the following examples.

Example 2.8. Let $X = \mathbb{R}$ with the topologies $\rho = \{\phi, \mathbb{R}, \mathbb{Q}\}$ and $\tau = \{U \subseteq \mathbb{R} : \mathbb{Q} \subseteq U\} \cup \{\phi\}$ and let $Y = \{0, 1, 2\}$ with the topology $\sigma = \{\phi, Y, \{0\}, \{1, 2\}\}$. Let $f : (X, \rho) \to (Y, \sigma)$ be the function defined by

$$f(x) = \begin{cases} 1, & x \in \mathbb{R} - \mathbb{Q} \\ 0, & x \in \mathbb{Q} \end{cases}$$

and $g : (X, \tau) \to (X, \rho)$ be the function defined by

$$g(x) = \begin{cases} 1, & x \in \mathbb{R} - \mathbb{Q} \\ x, & x \in \mathbb{Q} \end{cases}.$$

Then g is open and continuous but f is not almost strongly ω-continuous. Note that $g^{-1}(\mathbb{Q}) = \mathbb{Q} \in \tau$ and $g^{-1}(\mathbb{R}) = \mathbb{R}$, so we get that g is continuous and for every open set U in (X, τ), $g(U) = \mathbb{Q} \in \rho$ therefore g is open. Now $(f \circ g)(x) = 0$ for every $x \in X$ and so $(f \circ g)$ is almost strongly ω-continuous.

Example 2.9. Let $X = \mathbb{R}$ with the topology $\rho = \{\phi, \mathbb{R}, \mathbb{Q}\}$, let $Y = \{0, 1\}$ with the topology $\sigma = \{\phi, Y, \{0\}\}$ and let $Z = \{0, 1, 2\}$ with the topology $\tau = \{\phi, Z, \{0\}, \{1, 2\}\}$. Let $f : (X, \rho) \to (Y, \sigma)$ be the function defined by

$$f(x) = \begin{cases} 0, & x \in \mathbb{R} - \mathbb{Q} \\ 1, & x \in \mathbb{Q} \end{cases}$$

and let $g : (Y, \sigma) \to (Z, \tau)$ defined by

$$g(y) = \begin{cases} 2, & y = 0 \\ 1, & y = 1 \end{cases}.$$

Then f is almost strongly ω-continuous, g is continuous function but not open and $(g \circ f)$ is not almost strongly ω-continuous at $x \in \mathbb{Q}$.

Example 2.10. Let $x = \mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}\}$ and let $Y = \{0, 1\}$ with the topology $\sigma = \{\phi, Y, \{0\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the function defined by

$$f(x) = \begin{cases} 0, & x \in \mathbb{R} - \mathbb{Q} \\ 1, & x \in \mathbb{Q} \end{cases}$$

and $g : (Y, \sigma) \to (Y, \tau_{dis})$ be the identity function. Then f is almost strongly $\omega-$continuous function, g is open but not continuous and $(g \circ f)$ is not almost strongly ω-continuous. For every open set V in (Y, σ); $g(V)$ is open in (Y, τ_{dis}) and $g^{-1}(\{0\}) \notin \sigma$ and hence is not continuous. To show that $(g \circ f)$ is not almost strongly ω-continuous. Let $x \in \mathbb{Q} \subseteq X$, $(g \circ f)(x) = g(f(x)) = g(1) = 1 \in V = \{1\} \in \tau_{dis}$. Now for every ω-open set W containing x, $cl(W) = \mathbb{R}$, therefore $(g \circ f)(cl_\rho(W)) = g(f(\mathbb{R})) = g(Y) = Y \not\subset Int(cl(\{1\})) = \{1\}$.

Corollary 2.11. *Let Δ be an index set and let $f_\alpha : (X_\alpha, \tau_\alpha) \to (Y_\alpha, \sigma_\alpha)$ be a function for each $\alpha \in \Delta$. If the product function $f = \prod_{\alpha \in \Delta} f_\alpha : \prod X_\alpha \to \prod Y_\alpha$ is almost strongly ω-continuous, then f_α is almost strongly ω-continuous for each $\alpha \in \Delta$.*

Proof. For each $\beta \in \Delta$, we consider the projections $p_\beta : \prod_{\alpha \in \Delta} X_\alpha \to X_\beta$ and $q_\beta : \prod_{\alpha \in \Delta} Y_\alpha \to Y_\beta$. Then we have $q_\beta \circ f = f_\beta \circ p_\beta$ for each $\beta \in \Delta$. Since f is almost strongly ω-continuous and q_β is a continuous open function for each $\beta \in \Delta$, $q_\beta \circ f$ is almost strongly ω-continuous by Theorem 2.7 and hence $f_\beta \circ p_\beta$ is almost strongly ω-continuous. Thus f_β is almost strongly ω-continuous by Theorem 2.7 . □

Proposition 2.12. *Let $f : (X, \tau) \to (Y_1 \times Y_2, \sigma_1 \times \sigma_2)$ be a function, where (X, τ), (Y_1, σ_1) and (Y_2, σ_2) are topological spaces. Let $f_i : (X, \tau) \to (Y_i, \sigma_i)$ be defined as $f_i = p_i \circ f$ for $i = 1, 2$ where $p_i : (Y_1 \times Y_2, \sigma_1 \times \sigma_2) \to (Y_i, \sigma_i)$ is the projection function. If f is almost strongly ω-continuous, then f_i is almost strongly ω-continuous for $i = 1, 2$.*

Proof. Since p_i is a continuous open function and f is almost strongly ω-continuous, then by Theorem 2.7, $f_i = p_i \circ f$ is almost strongly ω-continuous for $i = 1, 2$. □

Theorem 2.13. *Let $f : (X, \tau) \to (Y, \sigma)$ be an almost strongly ω-continuous function. Then the restriction $f|_A : (A, \tau_A) \to (Y, \sigma)$ is almost strongly ω-continuous for any subset A of X.*

Proof. Let $a \in A$ and V be an open set in (Y, σ) containing $f(a)$. Since f is almost strongly ω-continuous, there exists an ω-open set W in (X, τ) such that $x \in W$ and $f(cl(W)) \subseteq Int(cl(V))$. Therefore by Lemma 1.5 $W \cap A \in (\tau_A)_\omega$ and $(f|_A)(cl(W \cap A)) \subseteq Int(cl(V))$. And the result follows. □

The following example shows that the converse of the previous theorem is not true in general.

Example 2.14. Let $X = \mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}\}$ and let $Y = \{0, 1, 2\}$ with the topology $\sigma = \{\phi, Y, \{0\}, \{1, 2\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the function defined by

$$f(x) = \begin{cases} 0, & x \in \mathbb{Q} \\ 1, & x \in \mathbb{R} - \mathbb{Q} \end{cases}.$$

Then f is not almost strongly ω-continuous, since if we take $x \in \mathbb{Q}$, then $f(x) = 0 \in \{0\} \in \sigma$ and for any ω-open set W containing x, $f(cl(W)) = f(\mathbb{R}) = \{0, 1\} \not\subseteq Int(cl(\{0\})) = \{0\}$. Let $A = \mathbb{Q}$. Then $A \in \tau$ and $\tau_A = \{\phi, A\}$. Note that $(f|_A)(x) = 0$ for every $x \in A$ and so $(f|_A)$ is almost strongly ω-continuous.

Note that if A is a clopen subset of a space (X, τ). Then $cl(U \cap A) = cl(U) \cap A$ for every $U \subseteq X$.

Proposition 2.15. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function and let $x \in X$. If there exists a clopen subset A of X containing x and $(f|_A)$ is almost strongly ω-continuous at x, then f is almost strongly ω-continuous at x.*

Proof. Let V be an open set in (Y, σ) containing $f(x)$. Since $(f|_A)$ is almost strongly ω-continuous at x, there exists an ω-open set W in (A, τ_A) such that $x \in W$ and $(f|_A)(cl_{\tau_A}(W)) = f(cl_{\tau_A}(W)) \subseteq Int(cl(V))$. So by Lemma 1.5 $W \in (\tau_A)_\omega = (\tau_\omega)_A$ and there exists an ω-open set U in (X, τ) such that $W = U \cap A$. Therefore W is an ω-open set in (X, τ) and $f(cl_\tau(W)) = f(cl_\tau(A \cap U)) = f(cl_\tau(U) \cap A) = f(cl_\tau(U \cap A) \cap A) = f(cl_\tau(W) \cap A) = f(cl_{\tau_A}(W)) \subseteq Int(cl(V))$ and the result follows. □

The following example shows that if the set A is ω-clopen then the result in proposition 2.15 need not be true.

Example 2.16. Let $f : (X, \tau) \to (Y, \sigma)$ be the function defined in Example 2.14. Then f is not almost strongly ω-continuous. Let $A = \mathbb{R} - \mathbb{Q}$. Then A is an ω-clopen set in (X, τ) and $(f|_A)$ is almost strongly ω-continuous. Note that $(f|_A)(x) = 1$ for every $x \in A$ so $(f|_A)(x)$ is almost strongly ω-continuous.

3. Basic properties

A space (X, τ) is said to be weakly Hausdorff [10] if each point of X is expressed by the intersection of regular closed sets of (X, τ) and it is said to be ω^*-regular if for every ω-open set U and each point $x \in U$ there exists an open set V such that $x \in U \subseteq cl(U) \subseteq V$.

Theorem 3.1. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function such that X is ω^*-regular and let $g : (X, \tau) \to (X \times Y, \tau \times \sigma)$ be the graph function of f defined by $g(x) = (x, f(x))$ for each $x \in X$. Then g is almost strongly $\omega-$continuous if and only if f is almost strongly $\omega-$continuous.*

Proof. Necessity. Suppose that g is almost strongly $\omega-$continuous. Let $x \in X$ and V be an open set in (Y, σ) containing $f(x)$. Then $X \times V$ is an open set of $X \times Y$ containing $g(x)$. Since g is almost strongly ω-continuous, there exists an ω-open set U in (X, τ) containing x such that $g(cl(U)) \subseteq Int(cl(X \times V))$. It follows $Int(cl(X \times V)) = X \times Int(cl(V))$. Therefore, we obtain $f(cl(U)) \subseteq Int(cl(V))$.

Sufficiency. Let $x \in X$ and W be any open set of $X \times Y$ containing $g(x)$. There exist open sets $U_1 \subseteq X$ and $V \subseteq Y$ such that $g(x) = (x, f(x)) \in U_1 \times V \subseteq W$. Since f is almost strongly ω-continuous, there exists an ω-open set U_2 in (X, τ) containing x such that $f(cl(U_2)) \subseteq Int(cl(V))$. Let $U = U_1 \cap U_2$, then U is an ω-open in (X, τ) containing x. Since X is ω^*-regular, there exists an open set Z such that $x \in Z \subseteq cl(Z) \subseteq U$. Therefore, we obtain $g(cl(Z)) \subseteq U_1 \times f(U_2) \subseteq Int(cl(W))$. □

Theorem 3.2. *If $f, g : (X, \tau) \to (Y, \sigma)$ are almost strongly ω-continuous functions and (Y, σ) is a Hausdorff space, then the set $E = \{x \in X : f(x) = g(x)\}$ is $\theta_{o\omega}$-closed in (X, τ).*

Proof. By Theorem 2.5 $f,\ g : (X, \tau_{\theta_{o\omega}}) \to (Y, \sigma_\delta)$ are continuous functions and hence A is closed in $(X, \tau_{\theta_{o\omega}})$. Therefore, A is $\theta_{o\omega}$-closed in (X, τ). □

Theorem 3.3. *Let $f : (X, \tau) \to (Y, \sigma)$ be an almost strongly ω-continuous injection. If Y is a Hausdorff (resp. weakly Hausdorff) space, then X is an ω-Urysohn (resp. ω-Hausdorff) space.*

Proof. Let (Y, σ) be Hausdorff and $x_1 \neq x_2$ for any $x_1, x_2 \in X$ and there exist disjoint open sets V_1 and V_2 containing $f(x_1)$ and $f(x_2)$, respectively. Since V_1 and V_2 are disjoint, we obtain $Int(cl(V_1)) \cap Int(cl(V_2)) = \phi$. Since f is almost strongly ω-continuous, for $i = 1, 2$, there exists an ω-open set U_i containing x_i such that $f(cl(U_i)) \subseteq Int(cl(V_i))$. It follows from $cl(U_1) \cap cl(U_2) = \phi$ that X is an ω-Urysohn space. Next, let Y be weakly Hausdorff and x_1, x_2 distinct points of X. Then $f(x_1) \neq f(x_2)$ and there exists a regular closed set V of Y such that $f(x_1) \notin V$ and $f(x_2) \in V$. Since f is almost strongly ω-containuous, by Theorem 2.4, there exists an ω-open set U containing x_1 such that $f(cl(U)) \subseteq Y - V$. Then we have $x_2 \in f^{-1}(V) \subseteq X - cl(U)$. This show that (X, τ) is ω-Hausdorff. □

For a function $f : (X, \tau) \to (Y, \sigma)$, the subset $\{x, f(x) : x \in X\} \subseteq X \times Y$ is called the graph of f and is denoted by $G(f)$.

The $G(f)$ is said to be $\theta_{o\omega}$-closed with respect to $X \times Y$ if for each $(x, y) \notin G(f)$, there exists an ω-open sets U and V containing x and y, respectively, such that $cl(U \times V) \cap G(f) = \phi$. It is easy to see that $G(f)$ is $\theta_{o\omega}$-closed with respect to $X \times Y$ if and only if for each $(x, y) \notin G(f)$ there exist ω-open subsets $U \subseteq X$ and $V \subseteq Y$ containing x and y, respectively, such that $f(cl(U)) \cap cl(V) = \phi$.

Definition 3.4. A subset S of a space X is said to be quasi H_ω-closed (resp. N_ω-closed) relative to X if for every cover $\{U_\alpha : \alpha \in \Delta\}$ of S by ω-open sets of

X, there exists a finite subset $\Delta_\circ$ of Δ such that $S \subseteq \cup\{cl(U_\alpha) : \alpha \in \Delta_\circ\}$(resp. $S \subseteq \cup\{Int(cl(U_\alpha)) : \alpha \in \Delta_\circ\}$). A space X is said to be quasi H_ω-closed (resp. nearly ω-compact) if the set X is quasi H_ω-closed (resp. N_ω-closed) relative to X.

Theorem 3.5. *Let $f : (X,\tau) \to (Y,\sigma)$ be a function whose graph is $\theta_{o\omega}$-closed with respect to $X \times Y$. If K is quasi H_ω-closed relative to Y, then $f^{-1}(K)$ is $\theta_{o\omega}$-closed in X.*

Proof. Let $x \in X - f^{-1}(K)$. For each $y \in K$, $(x, y) \notin G(f)$ and there exist ω-open sets U_y and V_y containing x and y, respectively, such that $f(cl(U_y)) \cap cl(V_y) = \phi$. The family $\{V_y : y \in K\}$ is a cover of K by ω-open sets of Y and $K \subseteq \cup(cl(V_y) : y \in K_0)$ for some finite subset $K_\circ$ of K. Put $U = \cap\{U_y : y \in K_\circ\}$. Then U is an ω-open set containing x and $f(cl(U)) \cap K = \phi$. Therefore, we have $cl(U) \cap f^{-1}(K) = \phi$ and hence $x \notin [f^{-1}(K)]_{\theta_{o\omega}}$. This shows that $f^{-1}(K)$ is $\theta_{o\omega}$-closed in X. □

Theorem 3.6. *If $f : (X,\tau) \to (Y,\sigma)$ almost strongly ω-continuous and K is quasi H_ω-closed relative to X, then $f(K)$ is N_ω-closed relative to Y.*

Proof. Let $\{V_\alpha : \alpha \in \Delta\}$ be a cover of $f(K)$ by ω-open sets of Y. For each $x \in K$, there exists $\alpha_x \in \Delta$ such that $f(x) \in V_{\alpha_x}$. Since f is almost strongly ω-continuous, there exists an ω-open set U_x containing x such that $f(cl(U_x)) \subseteq Int(cl(V_{\alpha_x}))$. The family $\{U_x : x \in K\}$ is a cover of K by ω-open sets of (X,τ) and hence there exists a finite subset K^* of K such that $K \subseteq \cup_{x\in K^*} cl(U_x)$. Therefore, we obtain $f(K) \subseteq f(\cup_{x\in K^*} cl(U_x)) \subseteq \cup_{x\in K^*} Int(cl(V_{\alpha_x}))$. □

Lemma 3.7. *If X is nearly ω-compact and A is regular closed in X, then A is N_ω-closed relative to X (and hence quasi H_ω-closed relative to X).*

Proof. Let $\{U_\alpha : \alpha \in \Delta\}$ be any cover of A by ω-open sets of X. Then $X = \cup\{U_\alpha : \alpha \in \Delta\} \cap (X - A)$. Since $X - A$ is regular open, it is open and hence ω-open. Since X is nearly $\omega-$compact, there exists a finite subset $\Delta_\circ$ of Δ such that $X = [\cup\{Int(cl(U_\alpha)) : \alpha \in \Delta_\circ\}] \cup Int(cl(X-A)) = (\cup\{Int(cl(U_\alpha)) : \alpha \in \Delta_\circ\}) \cup (X-A)$. Therefore, $A \subseteq \cup\{Int(cl(U_\alpha)) : \alpha \in \Delta_\circ\}$ and A is N_ω-closed relative to X. □

Theorem 3.8. *Let $f : (X,\tau) \to (Y,\sigma)$ be a function and (Y,σ) nearly ω-compact Hausdorff. Then, the following are equivalent:*

i. f is almost strongly ω-continuous.

ii. $G(f)$ is $\theta_{o\omega}$-closed with respect to $X \times Y$.

iii. If K is quasi H_ω-closed relative to Y, then $f^{-1}(K)$ is $\theta_{o\omega}$-closed in (X,τ).

Proof. (i→ii) Let $(x, y) \in X \times Y - G(f)$. Since (Y, σ) is Hausdorff, there exist two open sets V and W such that $y \in V$, $f(x) \in W$ and $V \cap W = \phi$. This gives $cl(V) \cap Int(cl(W)) = \phi$. By (i), there exists an ω-open set U containing x such that $f(cl(U)) \subseteq Int(cl(W))$. Hence $f(cl(U)) \cap cl(V) = \phi$, that is, $G(f)$ is $\theta_{o\omega}$-closed with respect to $X \times Y$.

(ii→iii) This follows from Theorem 3.5.

(iii→i) Let $x \in X$ and V be a regular open subset of Y such that $f(x) \in V$. Then $Y - V$ is a regular closed set and Y is nearly ω-compact, by Lemma 3.7 $Y - V$ is quasi H_ω-closed relative to Y. By (iii) $f^{-1}(X - V)$ is $\theta_{o\omega}$-closed in X and $x \notin f^{-1}(Y - V)$. Hence there exists an ω-open set U containing x such that $cl(U) \cap f^{-1}(Y - V) = \phi$. This implies that $f(cl(U)) \subseteq V$. Therefore, it follows from Theorem 2.4 that f is almost strongly θ-continuous. □

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