HEPTAVALENT SYMMETRIC GRAPHS OF ORDER $6p$

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Abstract. A graph is symmetric if its automorphism group acts transitively on the set of arcs of the graph. In this paper, we classify connected heptavalent symmetric graphs of order $6p$ for each prime p. As a result, there are three sporadic such graphs: one for $p = 5$ and two for $p = 13$.

Keywords: Symmetric graph, s-transitive graph, Cayley graph.

1. Introduction

Throughout this paper graphs are assumed to be finite, simple, connected and undirected. For group-theoretic concepts or graph-theoretic terms not defined here we refer the reader to [22, 25] and [1, 2], respectively. Let G be a permutation group on a set Ω and $v \in \Omega$. Denote by G_v the stabilizer of v in G, that is, the subgroup of G fixing the point v. We say that G is *semiregular* on Ω if $G_v = 1$ for every $v \in \Omega$ and *regular* if G is transitive and semiregular.

For a graph X, denote by $V(X)$, $E(X)$ and $\mathrm{Aut}(X)$ its vertex set, its edge set and its full automorphism group, respectively. A graph X is said to be G-*vertex-transitive* if $G \leq \mathrm{Aut}(X)$ acts transitively on $V(X)$. X is simply called *vertex-transitive* if it is $\mathrm{Aut}(X)$-vertex-transitive. An s-*arc* in a graph is an ordered $(s+1)$-tuple $(v_0, v_1, \cdots, v_{s-1}, v_s)$ of vertices of the graph X such that v_{i-1} is adjacent to v_i for $1 \leq i \leq s$, and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$. In particular, a 1-arc is just an arc and a 0-arc is a vertex. For a subgroup $G \leq \mathrm{Aut}(X)$, a graph X is said to be (G, s)-*arc-transitive* and (G, s)-*regular* if G is transitive and regular on the set of s-arcs in X, respectively. A (G, s)-arc-transitive graph is said to be (G, s)-*transitive* if it is not $(G, s+1)$-arc-transitive. In particular, a $(G, 1)$-arc-transitive graph is called G-*symmetric*. A graph X is simply called s-*arc-transitive*, s-*regular* and s-*transitive* if it is $(\mathrm{Aut}(X), s)$-arc-transitive, $(\mathrm{Aut}(X), s)$-regular and $(\mathrm{Aut}(X), s)$-transitive, respectively.

As we all known that the structure of the vertex stabilizers of symmetric graphs is very useful to classify such graphs, and this structure of the cubic or tetravalent case was given by Miller [18] and Potočnik [21]. Thus, classifying symmetric graphs with valency 3 or 4 has received considerable attention and a lot of results have been achieved, see [8, 28, 29]. Guo [10] determined the exact structure of pentavalent case. Following this structure, a series of pentavalent symmetric graphs was classified in [15, 19, 20, 26, 27, 12]. Recently, Guo [11] gave the exact structure of heptavalent case. Thus, as an application, we classify connected heptavalent symmetric graphs of order $6p$ for each prime p in this paper.

2. Preliminary results

Let X be a connected G-symmetric graph with $G \leq \mathrm{Aut}(X)$, and let N be a normal subgroup of G. The *quotient graph* X_N of X relative to N is defined as the graph with vertices the orbits of N on $V(X)$ and with two orbits adjacent if there is an edge in X between those two orbits. In view of [16, Theorem 9], we have the following:

Proposition 2.1. *Let X be a connected heptavalent G-symmetric graph with $G \leq \mathrm{Aut}(X)$, and let N be a normal subgroup of G. Then one of the following holds:*

(1) *N is transitive on $V(X)$;*
(2) *X is bipartite and N is transitive on each part of the bipartition;*
(3) *N has $r \geq 3$ orbits on $V(X)$, N acts semiregularly on $V(X)$, the quotient graph X_N is a connected heptavalent G/N-symmetric graph.*

The following proposition characterizes the vertex stabilizers of connected heptavalent s-transitive graphs (see [11, Theorem 1.1]).

Proposition 2.2. *Let X be a connected heptavalent (G, s)-transitive graph for some $G \leq \mathrm{Aut}(X)$ and $s \geq 1$. Let $v \in V(X)$. Then $s \leq 3$ and one of the following holds:*

(1) *For $s = 1$, $G_v \cong \mathbb{Z}_7$, D_{14}, F_{21}, D_{28}, $F_{21} \times \mathbb{Z}_3$;*
(2) *For $s = 2$, $G_v \cong F_{42}$, $F_{42} \times \mathbb{Z}_2$, $F_{42} \times \mathbb{Z}_3$, $\mathrm{PSL}(3,2)$, A_7, S_7, $\mathbb{Z}_2^3 \rtimes \mathrm{SL}(3,2)$ or $\mathbb{Z}_2^4 \rtimes \mathrm{SL}(3,2)$;*
(3) *For $s = 3$, $G_v \cong F_{42} \times \mathbb{Z}_6$, $\mathrm{PSL}(3,2) \times \mathrm{S}_4$, $\mathrm{A}_7 \times \mathrm{A}_6$, $\mathrm{S}_7 \times \mathrm{S}_6$, $(\mathrm{A}_7 \times \mathrm{A}_6) \rtimes \mathbb{Z}_2$, $\mathbb{Z}_2^6 \rtimes (\mathrm{SL}(2,2) \times \mathrm{SL}(3,2))$ or $[2^{20}] \rtimes (\mathrm{SL}(2,2) \times \mathrm{SL}(3,2))$.*

To extract a classification of connected heptavalent symmetric graphs of order $2p$ for a prime p from Cheng and Oxley [4], we introduce the graphs $G(2p, r)$. Let V and V' be two disjoint copies of $\mathbb{Z}_p$, say $V = \{0, 1, \cdots, p-1\}$ and $V' = \{0', 1', \cdots, (p-1)'\}$. Let r be a positive integer dividing $p-1$ and $H(p, r)$ the unique subgroup of Z_p^* of order r. Define the graph $G(2p, r)$ to have vertex set $V \cup V'$ and edge set $\{xy' \mid x - y \in H(p, r)\}$.

Proposition 2.3. *Let X be a connected heptavalent symmetric graph of order $2p$ with p a prime. Then X is isomorphic to $K_{7,7}$ or $G(2p, 7)$ with $7 \mid (p-1)$. Furthermore, $\mathrm{Aut}(G(2p, 7)) = (\mathbb{Z}_p \rtimes \mathbb{Z}_7) \rtimes \mathbb{Z}_2$.*

From [9, pp.12-14], [24, Theorem 2] and [14, Theorem A], we may obtain the following proposition by checking the orders of non-abelian simple groups:

Proposition 2.4. *Let p be a prime, and let G be a non-abelian simple group of order $|G| \,\big|\, (2^{25}{\cdot}3^5{\cdot}5^2{\cdot}7{\cdot}p)$. Then G has 3-prime factor, 4-prime factor or 5-prime factor, and is one of the following groups:*

Table 1: **Non-abelian simple $\{2, 3, 5, 7, p\}$−groups**

3-prime factor					
G	Order	G	Order	G	Order
A_5	$2^2{\cdot}3{\cdot}5$	PSL(2, 8)	$2^3{\cdot}3^2{\cdot}7$	PSU(4, 2)	$2^6{\cdot}3^4{\cdot}5$
A_6	$2^3{\cdot}3^2{\cdot}5$	PSL(2, 17)	$2^4{\cdot}3^2{\cdot}17$	PSU(3, 3)	$2^5{\cdot}3^3{\cdot}7$
PSL(2, 7)	$2^3{\cdot}3{\cdot}7$	PSL(3, 3)	$2^4{\cdot}3^3{\cdot}13$		
4-prime factor					
G	Order	G	Order	G	Order
A_7	$2^3{\cdot}3^2{\cdot}5{\cdot}7$	PSL(2, 27)	$2^2{\cdot}3^3{\cdot}7{\cdot}13$	PSU(5, 2)	$2^{10}{\cdot}3^5{\cdot}5{\cdot}11$
A_8	$2^6{\cdot}3^2{\cdot}5{\cdot}7$	PSL(2, 31)	$2^5{\cdot}3{\cdot}5{\cdot}31$	PSp(4, 4)	$2^8{\cdot}3^2{\cdot}5^2{\cdot}17$
A_9	$2^6{\cdot}3^4{\cdot}5{\cdot}7$	PSL(2, 49)	$2^4{\cdot}3{\cdot}5^2{\cdot}7^2$	PSp(6, 2)	$2^9{\cdot}3^4{\cdot}5{\cdot}7$
A_{10}	$2^7{\cdot}3^4{\cdot}5^2{\cdot}7$	PSL(2, 81)	$2^4{\cdot}3^4{\cdot}5{\cdot}41$	M_{11}	$2^4{\cdot}3^2{\cdot}5{\cdot}11$
PSL(2, 11)	$2^2{\cdot}3{\cdot}5{\cdot}11$	PSL(2, 127)	$2^7{\cdot}3^2{\cdot}7{\cdot}127$	M_{12}	$2^6{\cdot}3^3{\cdot}5{\cdot}11$
PSL(2, 13)	$2^2{\cdot}3{\cdot}7{\cdot}13$	PSL(3, 4)	$2^6{\cdot}3^2{\cdot}5{\cdot}7$	J_2	$2^7{\cdot}3^3{\cdot}5^2{\cdot}7$
PSL(2, 16)	$2^4{\cdot}3{\cdot}5{\cdot}17$	PSU(3, 4)	$2^6{\cdot}3{\cdot}5^2{\cdot}13$	$P\Omega^+(8, 2)$	$2^{12}{\cdot}3^5{\cdot}5^2{\cdot}7$
PSL(2, 19)	$2^2{\cdot}3^2{\cdot}5{\cdot}19$	PSU(3, 5)	$2^4{\cdot}3^2{\cdot}5^3{\cdot}13$	Sz(8)	$2^6{\cdot}5{\cdot}7{\cdot}13$
PSL(2, 25)	$2^3{\cdot}3{\cdot}5^2{\cdot}13$	PSU(3, 8)	$2^9{\cdot}3^4{\cdot}7{\cdot}19$	${}^2F_4(2)'$	$2^{11}{\cdot}3^3{\cdot}5^2{\cdot}13$
5-prime factor					
G	Order	G	Order	G	Order
A_{11}	$2^7{\cdot}3^4{\cdot}5^2{\cdot}7{\cdot}11$	PSL(2, 449)	$2^6{\cdot}3^2{\cdot}5^2{\cdot}7{\cdot}449$	PSp(8, 2)	$2^{16}{\cdot}3^5{\cdot}5^2{\cdot}7{\cdot}17$
A_{12}	$2^9{\cdot}3^5{\cdot}5^2{\cdot}7{\cdot}11$	PSL(2, 2^6)	$2^6{\cdot}3^2{\cdot}5{\cdot}7{\cdot}13$	M_{22}	$2^7{\cdot}3^2{\cdot}5{\cdot}7{\cdot}11$
PSL(2, 29)	$2^2{\cdot}3{\cdot}5{\cdot}7{\cdot}29$	PSL(4, 4)	$2^{12}{\cdot}3^4{\cdot}5^2{\cdot}7{\cdot}17$	$P\Omega^-(8, 2)$	$2^{12}{\cdot}3^4{\cdot}5{\cdot}7{\cdot}17$
PSL(2, 41)	$2^3{\cdot}3{\cdot}5{\cdot}7{\cdot}41$	PSL(5, 2)	$2^{10}{\cdot}3^2{\cdot}5{\cdot}7{\cdot}31$	$G_2(4)$	$2^{12}{\cdot}3^3{\cdot}5^2{\cdot}7{\cdot}13$
PSL(2, 71)	$2^3{\cdot}3^2{\cdot}5{\cdot}7{\cdot}71$				

Next we construct some heptavalent symmetric graphs of order $6p$ with p a prime. To do this, we need to introduce the so called coset graph (see [18, 23]) constructed from a finite group G relative to a subgroup H of G and a union D of some double cosets of H in G such that $D^{-1} = D$. The *coset graph* $\mathrm{Cos}(G, H, D)$ of G with respect to H and D is defined to have vertex set $[G : H]$, the set of right cosets of H in G, and edge set $\{\{Hg, Hdg\} \mid g \in G, d \in D\}$. The graph $\mathrm{Cos}(G, H, D)$ has valency $|D|/|H|$ and is connected if and only if D generates the group G. The action of G on $V(\mathrm{Cos}(G, H, D))$ by right multiplication induces a vertex-transitive automorphism group, which is arc-transitive if and only if D is a single double coset. Moreover, this action is faithful if and only if $H_G = 1$, where H_G is the largest normal subgroup of G in H. Clearly, $\mathrm{Cos}(G, H, D) \cong \mathrm{Cos}(G, H^\alpha, D^\alpha)$ for every $\alpha \in \mathrm{Aut}(G)$. For more details regarding coset graphs, see, for example, [7, 16, 23].

Construction 2.5. Let $G \cong S_8$ and $M \cong A_7$. Then by Atlas [6], G has a maximal subgroup $K \cong \mathbb{Z}_2^4 \rtimes S_4$ and M has a maximal subgroup $H \cong \mathbb{Z}_2^3 \rtimes \mathrm{PSL}(3, 2)$ such that $K \cap H \cong \mathbb{Z}_2^3 \rtimes S_4$. Take an element g of order 2 in $K \backslash (K \cap H)$. Then $H \cap H^g \cong \mathbb{Z}_2^3 \rtimes S_4$, $\langle H, g\rangle = G$ and hence the coset graph:

$$\mathcal{C}_{30} = \mathrm{Cos}(G, H, HgH)$$

is a connected heptavalent symmetric graph of order 30. By Magma [3], $\mathrm{Aut}(\mathcal{C}_{30}) \cong \mathrm{S}_8$, and any connected heptavalent symmetric graph of order 30 admitting S_8 as an arc-transitive automorphism group is isomorphic to $\mathcal{C}_{30}$.

The following two graphs are coset graphs of order 78 constructed from the simple group $\mathrm{PSL}(2, 13)$.

Construction 2.6. Let $G \cong \mathrm{PSL}(2, 13)$, and take the following four elements:

$a = (1, 13, 2, 12, 9, 5, 14)(3, 4, 7, 10, 11, 6, 8)$,
$b = (2, 5)(3, 7)(6, 11)(8, 10)(9, 12)(13, 14)$,
$x = (1, 4)(3, 10)(6, 14)(7, 8)(9, 12)(11, 13)$,
$y = (1, 4)(2, 6)(5, 11)(8, 9)(10, 12)(13, 14)$.

Then $G = \langle a, b, x\rangle \cong \mathrm{PSL}(2, 13)$ and $H = \langle a, b\rangle \cong D_{14}$. Define the following two coset graphs:

$$\mathcal{C}^1_{78} = \mathrm{Cos}(G, H, HxH), \quad \mathcal{C}^2_{78} = \mathrm{Cos}(G, H, HyH).$$

By Magma [3], $\mathrm{Aut}(\mathcal{C}^1_{78}) \cong \mathrm{PSL}(2, 13)$, $\mathrm{Aut}(\mathcal{C}^2_{78}) \cong \mathrm{PGL}(2, 13)$, and any connected heptavalent symmetric graph admitting $\mathrm{PSL}(2, 13)$ as an arc-transitive group is isomorphic to $\mathcal{C}^1_{78}$ or $\mathcal{C}^2_{78}$.

3. Main result

This section is devoted to classify connected heptavalent symmetric graphs of order $6p$ for each prime p.

Theorem 3.1. *Let X be a connected heptavalent symmetric graph of order $6p$ with p a prime. Then X is isomorphic to one of the following graphs:*

Table 2: **Heptavalent symmetric graphs of order** $6p$

X	s-transitivity	$\mathrm{Aut}(X)$	Comments
$\mathcal{C}_{30}$	2-transitive	S_8	Construction 2.5, $p = 5$
$\mathcal{C}^1_{78}$	1-transitive	$\mathrm{PSL}(2, 13)$	Construction 2.6, $p = 13$
$\mathcal{C}^2_{78}$	1-transitive	$\mathrm{PGL}(2, 13)$	Construction 2.6, $p = 13$

Proof. Let $A = \mathrm{Aut}(X)$. If $p = 2$ or 3, then $|V(X)| = 12$ or 18. By [17] and [5], there is no connected heptavalent symmetric graph of order 12 or 18. If $p = 5$, then $|V(X)| = 30$. By [5] and Construction 2.5, $X \cong \mathcal{C}_{30}$ and $A \cong \mathrm{S}_8$. Thus, we may assume that $p \geq 7$.

Let $v \in V(X)$. Then by Proposition 2.2, $|A_v| \bigm| 2^{24}{\cdot}3^4{\cdot}5^2{\cdot}7$ and hence $|A| \bigm| 2^{25}{\cdot}3^5{\cdot}5^2{\cdot}7{\cdot}p$. We separate the proof into two cases: A has a solvable minimal subgroup; A has no solvable minimal normal subgroup.

Case 1. A has a solvable minimal normal subgroup.

Let N be a solvable minimal normal subgroup of A. Then $|N| \bigm| 2^{25}{\cdot}3^5{\cdot}5^2{\cdot}7{\cdot}p$, and N is elementary abelian. Thus, $N \cong \mathbb{Z}_q^k$ with $q = 2, 3, 5, 7$ or p and k a positive integer. By Proposition 2.1, N is semiregular and X_N is also a connected heptavalent A/N-symmetric graphs. It follows that $|N| \bigm| 6p$ and $N \cong \mathbb{Z}_2$, $\mathbb{Z}_3$ or $\mathbb{Z}_p$. Note that there is no connected heptavalent graph of odd order. Thus, $N \ncong \mathbb{Z}_2$. Since there is no connected heptavalent regular graph of order 6, we have that $N \ncong \mathbb{Z}_p$. This forces that $N \cong \mathbb{Z}_3$ and X_N is a heptavalent symmetric graph of order $2p$. By Proposition 2.3, $X_N \cong K_{7,7}$ or $G(2p, 7)$.

Assume that $X_N \cong K_{7,7}$. Then $p = 7$ and $A/N \lesssim \mathrm{S}_7 \,\mathrm{wr}\, \mathrm{S}_2$. By Magma [3], $(\mathrm{S}_7 \times \mathrm{S}_7) \rtimes \mathbb{Z}_2$ has minimal arc-transitive subgroups $\mathbb{Z}_7^2 \rtimes \mathbb{Z}_2$ or $\mathbb{Z}_7^2 \rtimes \mathbb{Z}_4$. Thus, A/N has an arc-transitive subgroup $M/N \cong \mathbb{Z}_7^2 \rtimes \mathbb{Z}_2$ or $\mathbb{Z}_7^2 \rtimes \mathbb{Z}_4$. By "N/C-Theorem" (see [13, Chapter I, Theorem 4.5]), $M/C_M(N) \lesssim \mathrm{Aut}(N) \cong \mathbb{Z}_2$. Thus, $7^2 \bigm| |C_M(N)|$. Let H be a Sylow 7-subgroup of $C_M(N)$. Then $\mathbb{Z}_7^2 \cong H$ is normal in M. Note that $p = 7$. Thus, the quotient graph X_H has order 6. By Propositions 2.1, H is semiregular and hence $H \cong \mathbb{Z}_7$, a contradiction.

Assume that $X_N \cong G(2p, 7)$. Then $A/N \lesssim (\mathbb{Z}_p \rtimes \mathbb{Z}_7) \rtimes \mathbb{Z}_2$. Since A/N is arc-transitive on X_N, we have that $A/N \cong (\mathbb{Z}_p \rtimes \mathbb{Z}_7) \rtimes \mathbb{Z}_2$. Clearly, A/N has a normal subgroup $M/N \cong \mathbb{Z}_p$. By "N/C-theorem", $M/C_M(N) \lesssim \mathrm{Aut}(N) \cong \mathbb{Z}_2$. It follows that $M = C_M(N)$ and $M = \mathbb{Z}_p \times \mathbb{Z}_3$. This implies that M has a characteristic subgroup $H \cong \mathbb{Z}_p$. Since $M \trianglelefteq A$, we can deduce that $H \trianglelefteq A$. Thus, X_H is a connected heptavalent graph of order 6, a contradiction.

Case 2. A has no solvable minimal normal subgroup.

For convenience, we still use N to denote a minimal normal subgroup of A. Then N is non-solvable. Let $N = T^k$ with T a non-abelian simple group and k a positive integer. Then T has at least 3-prime factors. Since $|T| \bigm| 2^{25}{\cdot}3^5{\cdot}5^2{\cdot}7{\cdot}p$, we have that T is one of the simple groups listed in Proposition 2.4. By Proposition 2.1, $|N| = 6p|N_v|$ or $3p|N_v|$.

Assume that $k \geq 2$. Since T is a non-abelian simple group, we have that $2^2 \bigm| |T|$ and $T_v \neq 1$. If $p > 7$, then $p \nmid |T|$ because $p^2 \nmid |N| = |T^k|$. It follows that p divides the order of X_N. By Proposition 2.1, $N = T^k$ is semiregular and hence $T_v = 1$, a contradiction. Thus, $p = 7$ and $N = T^k$ has at most two orbits on $V(X)$. It follows that $3{\cdot}7 \bigm| |T^k|$. Note that $|T^k| \bigm| 2^{25}{\cdot}3^5{\cdot}5^2{\cdot}7^2$. Thus, T is a simple $\{2, 3, 7\}$-group or $\{2, 3, 5, 7\}$-group and $k = 2$. By Proposition 2.4, $T \cong \mathrm{PSL}(2, 7)$, $\mathrm{PSL}(2, 8)$, A_7, A_8 or $\mathrm{PSL}(3, 4)$. If $T \cong \mathrm{PSL}(2, 8)$ or A_8, then $N \cong \mathrm{PSL}(2, 8)^2$ or A_8^2 and $|N_v| = |N|/(3{\cdot}7)$ or $|N|/(2{\cdot}3{\cdot}7)$. However, by Magma [3], $\mathrm{PSL}(2, 8)^2$ or A_8^2 has no subgroups of such orders, a contradiction. If $T \cong \mathrm{PSL}(2, 7)$, then $|N_v| = |N|/(3{\cdot}7) = 2^6{\cdot}3{\cdot}7$ or $|N|/(2{\cdot}3{\cdot}7) = 2^5{\cdot}3{\cdot}7$. By Magma [3], $N_v \cong D_8 \times \mathrm{PSL}(2, 7)$, $\mathbb{Z}_4 \times \mathrm{PSL}(2, 7)$ or $\mathbb{Z}_2^2 \times \mathrm{PSL}(2, 7)$. Note that $N_v \trianglelefteq A_v$. By Proposition 2.2, $N_v \cong \mathbb{Z}_2^2 \times \mathrm{PSL}(2, 7)$ and $A_v \cong \mathrm{S}_4 \times \mathrm{PSL}(2, 7)$.

It follows that N is transitive on $V(X)$ and hence N is arc-transitive. By Magma [3], there is no connected heptavalent symmetric graph of order 6·7 admitting $\mathrm{PSL}(2,7)^2$ as an arc-transitive group. If $T \cong \mathrm{A}_7$, then $|N_v| = 2^6{\cdot}3^3{\cdot}5^2{\cdot}7$ or $2^5{\cdot}3^3{\cdot}5^2{\cdot}7$. By Magma [3], $N_v \cong \mathrm{A}_7 \times \mathrm{S}_5$ or $\mathrm{A}_7 \times \mathrm{A}_5$. However, by Proposition 2.2, A_v has no normal subgroups isomorphic to $\mathrm{A}_7 \times \mathrm{S}_5$ or $\mathrm{A}_7 \times \mathrm{A}_5$, a contradiction. If $T \cong \mathrm{PSL}(3,4)$, then by Magma [3], $N_v \cong (\mathbb{Z}_2^4 \rtimes \mathrm{A}_5) \times \mathrm{PSL}(3,4)$. Similarly, A_v has no such normal subgroup by Proposition 2.2, a contradiction.

Thus, $k = 1$ and $N = T$ is a non-abelian simple group. It follows that $2^2 \,\big|\, |N|$ and $N_v \neq 1$. By Proposition 2.1, N has at most two orbits on $V(X)$ and $|N| = 6p|N_v|$ or $3p|N_v|$.

Subcase 2.1. Suppose that $p = 7$. Then $|N| \,\big|\, 2^{25}{\cdot}3^5{\cdot}5^2{\cdot}7^2$.

Note that $3{\cdot}7 \,\big|\, |N|$. By Proposition 2.4, N is isomorphic to one of the following simple groups:

$$\mathrm{PSL}(2,7),\ \mathrm{PSL}(2,8),\ \mathrm{PSU}(3,3),\ \mathrm{A}_7,\ \mathrm{A}_8,\ \mathrm{A}_9,\ \mathrm{A}_{10},$$
$$\mathrm{PSL}(2,49),\ \mathrm{PSL}(3,4),\ \mathrm{PSp}(6,2),\ \mathrm{J}_2,\ \mathrm{P}\Omega^+(8,2).$$

Since N is transitive or has two orbits on $V(X)$, we have that $|N_v| = |N|/42$ or $|N| = |N|/21$. It follows that N has a subgroup of order $|N|/42$ or $|N|/21$. By Magma [3], $\mathrm{PSL}(2,8)$, $\mathrm{PSU}(3,3)$, A_8, A_9, A_{10}, $\mathrm{PSL}(2,49)$, $\mathrm{PSL}(3,4)$, $\mathrm{PSp}(6,2)$, J_2 and $\mathrm{P}\Omega^+(8,2)$ have no subgroups of such orders. Thus, $N \cong \mathrm{PSL}(2,7)$ or A_7.

Let $N \cong \mathrm{PSL}(2,7)$. Then $|N_v| = 2^2$ or 2^3, and by Atlas [6], $N_v \cong D_8$, $\mathbb{Z}_4$ or $\mathbb{Z}_2^2$. The normality of N in A implies that $N_v \trianglelefteq A_v$. By Proposition 2.2 and Remark of [11, Theorem 1.1], the only possible is $A_v \cong \mathrm{PSL}(3,2) \times \mathrm{S}_4$ and $N_v \cong \mathbb{Z}_2^2$. It follows that N is transitive on $V(X)$ and hence $A = A_vN$. Set $C = C_A(N)$. Then $C \cap N = 1$ because N is a non-abelian simple group. If $C = 1$, then by "N/C-Theorem" (see [13, Chapter I, Theorem 4.5]), $A \cong A/C \lesssim \mathrm{Aut}(N) \cong \mathrm{PGL}(2,7)$. Since $|A_v| = |A|/42$, we have that $7 \not|\, |A_v|$, a contradiction. Note that A has no solvable minimal normal subgroup. Thus, C is non-solvable. Clearly, $C \cong CN/N \trianglelefteq A/N = A_vN/N \cong A_v/N_v \cong \mathrm{PSL}(3,2) \times \mathrm{S}_3$. It forces that C has a normal subgroup $M \cong \mathrm{PSL}(3,2)$. Since N is transitive on $V(X)$ and $7 \,\big|\, |M|$, we have that $H = M \times N \cong \mathrm{PSL}(2,7)^2$ is arc-transitive and $H_v \cong \mathrm{PSL}(2,7) \times \mathbb{Z}_2^2$. However, $\mathrm{PSL}(2,7) \times \mathbb{Z}_2^2$ can not be as a vertex stabilizer of a heptavalent symmetric graph by Proposition 2.2, a contradiction.

Let $N \cong \mathrm{A}_7$. Then $|N_v| = 2^2{\cdot}3{\cdot}5$ or $2^3{\cdot}3{\cdot}5$. By Atlas [6], $N_v \cong \mathrm{A}_5$ or S_5. Since $N \trianglelefteq A$, we have that $N_v \trianglelefteq A_v$. Thus, A_v has a normal subgroup isomorphic to A_5 or S_5. However, by Proposition 2.2, A_v has no such normal subgroup, a contradiction.

Subcase 2.2. Suppose that $p > 7$. Then $|N| \,\big|\, 2^{25}{\cdot}3^5{\cdot}5^2{\cdot}7{\cdot}p$.

Since $3p \,\big|\, |N|$, we have that N is simple $\{2,3,p\}$, $\{2,3,5,p\}$, $\{2,3,7,p\}$ or $\{2,3,5,7,p\}$-group, and by Proposition 2.4, we can get the information of the group N, the prime p and $|N_v|$ as the following table:

By Atlas [6] and Magma [3], except for the following groups:

Table 3: **The group N, the prime p and the order $|N_v|$**

3-prime factor					
N	p	$\|N_v\|$	N	p	$\|N_v\|$
$\mathrm{PSL}(2,17)$	17	$2^3{\cdot}3$ or $2^4{\cdot}3$	$\mathrm{PSL}(3,3)$	13	$2^3{\cdot}3^2$ or $2^4{\cdot}3^2$
4-prime factor					
N	p	$\|N_v\|$	N	p	$\|N_v\|$
$\mathrm{PSL}(2,11)$	11	$2{\cdot}5$ or $2^2{\cdot}5$	$\mathrm{PSL}(2,127)$	127	$2^6{\cdot}3{\cdot}7$ or $2^7{\cdot}3{\cdot}7$
$\mathrm{PSL}(2,13)$	13	$2{\cdot}7$ or $2^2{\cdot}7$	$\mathrm{PSU}(3,4)$	13	$2^5{\cdot}5^2$ or $2^6{\cdot}5^2$
$\mathrm{PSL}(2,16)$	17	$2^3{\cdot}5$ or $2^4{\cdot}5$	$\mathrm{PSU}(3,8)$	19	$2^8{\cdot}3^3{\cdot}7$ or $2^9{\cdot}3^3{\cdot}7$
$\mathrm{PSL}(2,19)$	19	$2{\cdot}3{\cdot}5$ or $2^2{\cdot}3{\cdot}5$	$\mathrm{PSU}(5,2)$	11	$2^9{\cdot}3^4{\cdot}5$ or $2^{10}{\cdot}3^4{\cdot}5$
$\mathrm{PSL}(2,25)$	13	$2^2{\cdot}5^2$ or $2^3{\cdot}5^2$	$\mathrm{PSp}(4,4)$	17	$2^7{\cdot}3{\cdot}5^2$ or $2^8{\cdot}3{\cdot}5^2$
$\mathrm{PSL}(2,27)$	13	$2{\cdot}3^2{\cdot}7$ or $2^2{\cdot}3^2{\cdot}7$	M_{11}	11	$2^3{\cdot}3{\cdot}5$ or $2^4{\cdot}3{\cdot}5$
$\mathrm{PSL}(2,31)$	31	$2^4{\cdot}5$ or $2^5{\cdot}5$	M_{12}	11	$2^5{\cdot}3^2{\cdot}5$ or $2^6{\cdot}3^2{\cdot}5$
$\mathrm{PSL}(2,81)$	41	$2^3{\cdot}3^3{\cdot}5$ or $2^4{\cdot}3^3{\cdot}5$	${}^2F_4(2)'$	13	$2^{10}{\cdot}3^2{\cdot}5^2$ or $2^{11}{\cdot}3^2{\cdot}5^2$
5-prime factor					
N	p	$\|N_v\|$	N	p	$\|N_v\|$
A_{11}	11	$2^6{\cdot}3^3{\cdot}5^2{\cdot}7$ or $2^7{\cdot}3^3{\cdot}5^2{\cdot}7$	$\mathrm{PSL}(4,4)$	11	$2^{11}{\cdot}3^3{\cdot}5^2{\cdot}7$ or $2^{12}{\cdot}3^3{\cdot}5^2{\cdot}7$
A_{12}	11	$2^8{\cdot}3^4{\cdot}5^2{\cdot}7$ or $2^9{\cdot}3^4{\cdot}5^2{\cdot}7$	$\mathrm{PSL}(5,2)$	31	$2^9{\cdot}3{\cdot}5{\cdot}7$ or $2^{10}{\cdot}3{\cdot}5{\cdot}7$
$\mathrm{PSL}(2,29)$	29	$2{\cdot}5{\cdot}7$ or $2^2{\cdot}5{\cdot}7$	$\mathrm{PSp}(8,2)$	17	$2^{15}{\cdot}3^4{\cdot}5^2{\cdot}7$ or $2^{16}{\cdot}3^4{\cdot}5^2{\cdot}7$
$\mathrm{PSL}(2,41)$	41	$2^2{\cdot}5{\cdot}7$ or $2^3{\cdot}5{\cdot}7$	M_{22}	11	$2^6{\cdot}3{\cdot}5{\cdot}7$ or $2^7{\cdot}3{\cdot}5{\cdot}7$
$\mathrm{PSL}(2,71)$	71	$2^2{\cdot}3{\cdot}5{\cdot}7$ or $2^3{\cdot}3{\cdot}5{\cdot}7$	$\mathrm{P}\Omega^-(8,2)$	17	$2^{11}{\cdot}3^3{\cdot}5{\cdot}7$ or $2^{12}{\cdot}3^3{\cdot}5{\cdot}7$
$\mathrm{PSL}(2,449)$	449	$2^5{\cdot}3{\cdot}5^2{\cdot}7$ or $2^6{\cdot}3{\cdot}5^2{\cdot}7$	$G_2(4)$	13	$2^{11}{\cdot}3^2{\cdot}5^2{\cdot}7$ or $2^{12}{\cdot}3^2{\cdot}5^2{\cdot}7$
$\mathrm{PSL}(2,2^6)$	13	$2^5{\cdot}3{\cdot}5{\cdot}7$ or $2^6{\cdot}3{\cdot}5{\cdot}7$			

$$\mathrm{PSL}(2,17),\ \mathrm{PSL}(3,3),\ \mathrm{PSL}(2,11),\ \mathrm{PSL}(2,13),$$
$$\mathrm{PSL}(2,16),\ \mathrm{PSL}(2,19),\ \mathrm{PSL}(2,25),\ \mathrm{M}_{11},\ \mathrm{M}_{12},\ \mathrm{A}_{12},$$

the remaining simple groups listed in Table 3 do not have subgroups of order $|N_v| = |N|/3p$ or $|N|/6p$. Thus, next we deal with these nine groups.

Set $C = C_A(N)$. Then $C \cap N = 1$ and $C \trianglelefteq A$. If $C \neq 1$, then C is non-solvable because A has no solvable minimal normal subgroup. Note that $p > 7$ and $p \,\big|\, |N|$. Thus, $p \nmid |C|$. It follows that C has at least p orbits on $V(X)$. By Proposition 2.1, C is semiregular and hence $|C| \,\big|\, 6p$. This implies that C is solvable, a contradiction. Thus, $C = 1$. By "N/C-Theorem", $A \cong A/C \lesssim \mathrm{Aut}(N)$, that is, A is almost simple with socle N.

Let $N \cong \mathrm{PSL}(2,17)$. Then $|N_v| = 2^3{\cdot}3$ or $2^4{\cdot}3$. By Atlas [6], $\mathrm{PSL}(2,17)$ has no subgroups of order $2^4{\cdot}3$ and $N_v \cong \mathrm{S}_4$. Clearly, N is transitive on $V(X)$. Recall that $N_v \trianglelefteq A_v$. By Proposition 2.2, $A_v \cong \mathrm{PSL}(3,2) \times \mathrm{S}_4$. It follows that $\mathrm{PSL}(3,2) \cong A_v/N_v \cong A/N \lesssim \mathrm{Out}(N) \cong \mathbb{Z}_2$, a contradiction.

Let $N \cong \mathrm{PSL}(3,3)$. Then $|N_v| = 2^3{\cdot}3^2$ or $2^4{\cdot}3^2$. By Magma [3], N_v has a normal Sylow 3-subgroup $P \cong \mathbb{Z}_3^2$. Thus, P is characteristic in N_v. The normality of N_v in A_v implies that $P \trianglelefteq A_v$. However, by Proposition 2.2, A_v has no normal subgroup isomorphic to $\mathbb{Z}_3^2$, a contradiction.

Let $N \cong \mathrm{PSL}(2,11)$. Then $|N_v| = 2{\cdot}5$ or $2^2{\cdot}5$. By Atlas [6], $\mathrm{PSL}(2,11)$ has no subgroups of order $2^2{\cdot}5$ and $N_v \cong D_{10}$. However, by Proposition 2.2, A_v has no normal subgroups isomorphic to D_{10}, a contradiction.

Let $N \cong \mathrm{PSL}(2,16)$, $\mathrm{PSL}(2,19)$, $\mathrm{PSL}(2,25)$, M_{11}, M_{12} or A_{12}. Then by Atlas [6] and Magma [3], $N_v \cong \mathbb{Z}_2^4 \rtimes \mathbb{Z}_5$, A_5, $\mathbb{Z}_5^2 \rtimes \mathbb{Z}_4$, S_5, $\mathrm{A}_6 \rtimes \mathbb{Z}_2^2$ or S_{10}. Similarly, A_v has no such normal subgroups, a contradiction.

Finally, let $N \cong \mathrm{PSL}(2,13)$. Then $|N_v| = 2{\cdot}7$ or $2^2{\cdot}7$. By Atlas [6], $\mathrm{PSL}(2,13)$ has no subgroups of order $2^2{\cdot}7$ and $N_v \cong D_{14}$. Since $|N|/|N_v| = 6{\cdot}13$, we have that N is transitive on $V(X)$, and since $7 \,\big|\, |N_v|$, we have that N is arc-transitive. By Construction 2.6, $X \cong \mathcal{C}_{78}^1$ or $\mathcal{C}_{78}^2$, and $A \cong \mathrm{PSL}(2,13)$ or $\mathrm{PGL}(2,13)$. □

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