ON THE SOLVABILITY OF TWO-POINT IN TIME PROBLEM FOR PDE

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Abstract. We prove that the solution of problem for homogeneous partial differential equation of the second order in time variable in which nonhomogeneous local two-point conditions are given, and infinite order in spatial variables, may not exist in the class of entire functions in the case when the characteristic determinant of the problem equals zero identically. In the case of existence of the solution of the problem, we propose the formula of finding its particular solutions.

Keywords: Multipoint conditions, differential-symbol method, quasipolynomial solutions.

1. Introduction

It is known [1] that multipoint (n-point) problems for PDE are ill-posed problems. The investigations of such problems originate from problems for ODE with multipoint conditions which are called Vallee-Poussin problems [2, 3, 4]. Well-posed solvability of multipoint in time problems for linear PDE, basing on the metric approach, has been investigated in article [5] for the first time. This paper points out the problem of small denominators, which is typical for multipoint problems. Moreover, it was proved that the classes of uniqueness of solution of the multipoint in time problem for PDE are significantly different

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from the classes of uniqueness of the solution of the corresponding Cauchy problem for the same equations. Applying a special technique of below estimation of small denominators to investigation of the n-point problems and the problems with integral conditions for equations and systems of PDEs in the bounded domains, in recent years has been obtained the series of new results (see works [6, 7, 8] and bibliography). For higher order nonlinear hyperbolic equations nonclassical boundary value problems of Vallee-Poussin type are studied in [9].

Papers [10, 11, 12] are devoted to establishing the classes of unique solvability of problems with local multipoint conditions in time for PDEs in unbounded domains $\{(t,x)\colon t \in (0,T),\ x \in \mathbb{R}^s\}$, where $T > 0$ and $s \in \mathbb{N}$.

In all above-mentioned researches, for correct solvability of multipoint problems for PDEs the authors assumed the characteristic determinant of the problem to be different from identical zero.

This article deals with investigation of solvability of problem with nonhomogeneous local two-point in time conditions for homogeneous differential equations of second order in time and arbitrary order in spatial variables in the class of entire functions in the case when the characteristic determinant of the problem equals zero identically. Note that the set of nontrivial solutions of corresponding homogeneous problem was studied in papers [13, 14].

2. Problem statement

Let's investigate in domain $\mathbb{R}^{s+1}$, $s \in \mathbb{N}$, the solvability of the problem:

$$(2.1) \qquad \left[\frac{\partial^2}{\partial t^2} + 2a\left(\frac{\partial}{\partial x}\right)\frac{\partial}{\partial t} + b\left(\frac{\partial}{\partial x}\right)\right]U(t,x) = 0, \quad t \in \mathbb{R},\ x \in \mathbb{R}^s,$$

$$(2.2) \qquad \begin{aligned} l_{0\partial}U(t,x) &\equiv A_1\left(\frac{\partial}{\partial x}\right)U(0,x) + A_2\left(\frac{\partial}{\partial x}\right)\frac{\partial U}{\partial t}(0,x) = \varphi_0(x), \\ l_{1\partial}U(t,x) &\equiv B_1\left(\frac{\partial}{\partial x}\right)U(h,x) + B_2\left(\frac{\partial}{\partial x}\right)\frac{\partial U}{\partial t}(h,x) = \varphi_1(x), \quad x \in \mathbb{R}^s, \end{aligned}$$

where $\varphi_0(x)$, $\varphi_1(x)$ are given entire functions and at least one of them is nonzero, $h > 0$. In equation (2.1), $a\left(\frac{\partial}{\partial x}\right)$, $b\left(\frac{\partial}{\partial x}\right)$ are differential expressions of the finite or infinite order of the following form:

$$(2.3) \qquad a\left(\frac{\partial}{\partial x}\right) = \sum_{|k|=0}^{\infty} a_k\left(\frac{\partial}{\partial x}\right)^k, \quad b\left(\frac{\partial}{\partial x}\right) = \sum_{|k|=0}^{\infty} b_k\left(\frac{\partial}{\partial x}\right)^k,$$

where $k = (k_1, \ldots, k_s) \in \mathbb{Z}_+^s$, $|k| = k_1 + \ldots + k_s$, $a_k, b_k \in \mathbb{C}$, $\left(\frac{\partial}{\partial x}\right)^k = \frac{\partial^{|k|}}{\partial x_1^{k_1} \ldots \partial x_s^{k_s}}$. The symbols $a(\nu)$ and $b(\nu)$ of differential expressions (2.3) are entire functions for $\nu \in \mathbb{C}^s$.

Differential expressions $A_1\left(\frac{\partial}{\partial x}\right)$, $A_2\left(\frac{\partial}{\partial x}\right)$, $B_1\left(\frac{\partial}{\partial x}\right)$, $B_2\left(\frac{\partial}{\partial x}\right)$ in two-point conditions (2.2) are differential polynomials with complex coefficients, such that the

corresponding symbols $A_1(\nu)$, $A_2(\nu)$, $B_1(\nu)$, $B_2(\nu)$ for each $\nu \in \mathbb{C}^s$ satisfy the conditions:

$$|A_1(\nu)|^2 + |A_2(\nu)|^2 \neq 0, \quad |B_1(\nu)|^2 + |B_2(\nu)|^2 \neq 0.$$

The solution of problem (2.1), (2.2) is understood as the following entire function:

$$U(t,x) = \sum_{\widehat{k} \in \mathbb{Z}_+^{s+1}} u_{\widehat{k}} \, t^{k_0} \, x^k, \quad \widehat{k} = (k_0, k) = (k_0, k_1, \ldots, k_s), \quad u_{\widehat{k}} \in \mathbb{C},$$

of variables t and $x = (x_1, \ldots, x_s)$, where $x^k = x_1^{k_1} \ldots x_s^{k_s}$, that satisfies equation (2.1) and two-point conditions (2.2). The action of differential expression $b\left(\frac{\partial}{\partial x}\right)$ onto function U is defined as:

$$b\left(\frac{\partial}{\partial x}\right) U \equiv \sum_{k \in \mathbb{Z}_+^s} b_k \frac{\partial^{|k|} U}{\partial x_1^{k_1} \ldots \partial x_s^{k_s}}.$$

The action of differential expression $a\left(\frac{\partial}{\partial x}\right)$ onto $\frac{\partial U}{\partial t}$ is defined in a similar way. Note that papers [15, 16] deal with well-posedness of actions of infinite order differential expressions in the classes of entire functions.

Let's find the conditions under which the solution of problem (2.1), (2.2) exists, and also does not exist in the space of entire functions. By means of the differential-symbol method [17, 18], we shall construct some partial solutions of problem (2.1), (2.2) in the case when the solution of the problem is nonunique.

3. Main results

Consider the ordinary differential equation

(3.1) $$\left[\frac{d^2}{dt^2} + 2a(\nu)\frac{d}{dt} + b(\nu)\right] T(t, \nu) = 0$$

(assume $\nu \in \mathbb{C}^s$) with its normal fundamental system of solutions at the point $t = 0$:

$$T_0(t, \nu) = e^{-a(\nu)t}\left\{a(\nu)\frac{\sinh\left[t\sqrt{a^2(\nu) - b(\nu)}\right]}{\sqrt{a^2(\nu) - b(\nu)}} + \cosh\left[t\sqrt{a^2(\nu) - b(\nu)}\right]\right\},$$

$$T_1(t, \nu) = e^{-a(\nu)t}\frac{\sinh\left[t\sqrt{a^2(\nu) - b(\nu)}\right]}{\sqrt{a^2(\nu) - b(\nu)}},$$

if $a^2(\nu) \neq b(\nu)$, and $T_0(t, \nu) = e^{-a(\nu)t}\{a(\nu)t + 1\}$, $T_1(t, \nu) = t\,e^{-a(\nu)t}$, if $a^2(\nu) = b(\nu)$.

We write down the characteristic determinant of problem (2.1), (2.2) of the form

$$\Delta(\nu) =$$

$$= \begin{vmatrix} A_1(\nu) & A_2(\nu) \\ B_1(\nu)\,T_0(h,\nu) + B_2(\nu)\dfrac{dT_0}{dt}(h,\nu) & B_1(\nu)\,T_1(h,\nu) + B_2(\nu)\dfrac{dT_1}{dt}(h,\nu) \end{vmatrix}.$$

By Poincare Theorem ([19], p. 59), the functions $T_0(t,\nu)$ and $T_1(t,\nu)$, as solutions of Cauchy problem, are entire functions of vector-parameter ν, since the coefficients $a(\nu)$, $b(\nu)$ of equation (3.1) are entire functions by the assumption. Apart from this, the function $\Delta(\nu)$, as a superposition of entire functions, is also entire function.

The order of entire functions $T_0(t,\nu)\,e^{\nu \cdot x}$, $T_1(t,\nu)\,e^{\nu \cdot x}$ by the set of variables $\nu_1, \nu_2, \ldots, \nu_s$ is denoted by p, where $\nu \cdot x = \nu_1 x_1 + \ldots + \nu_s x_s$. Note that $p \in [1; +\infty]$.

Let's denote the class of entire functions $\varphi(x)$, whose the order is less than p', as $A_{p'}$, where $\frac{1}{p} + \frac{1}{p'} = 1$, when $1 < p < \infty$. Also we assume that if $p = 1$ then $A_{p'}$ is a class of entire functions ($p' = \infty$), and if $p = \infty$ then $A_{p'}$ is a class of entire function of exponential type ($p' = 1$).

Denote by $\mathbb{A}_{p'}$ the class of entire functions $U(t, \cdot)$, which for each fixed $t \in \mathbb{R}$ belong to $A_{p'}$.

Theorem 3.1. *Let $\Delta(\nu) \equiv 0$ in $\mathbb{C}^s$ and for certain $x \in \mathbb{R}^s$ at least one of the following two conditions is satisfied:*

$$(3.2) \qquad l_{1\partial} T_1\left(t, \frac{\partial}{\partial x}\right)\varphi_0(x) \neq A_2\left(\frac{\partial}{\partial x}\right)\varphi_1(x)$$

or

$$(3.3) \qquad l_{1\partial} T_0\left(t, \frac{\partial}{\partial x}\right)\varphi_0(x) \neq A_1\left(\frac{\partial}{\partial x}\right)\varphi_1(x)$$

for $\varphi_0, \varphi_1 \in A_{p'}$. Then the solution of problem (2.1), (2.2) does not exist in the class of entire functions $\mathbb{A}_{p'}$.

Proof. Contrary, we assume that in the class $\mathbb{A}_{p'}$ there exists entire solution $U(t, x)$ of equation (2.1), which satisfies conditions (2.2). Let's denote $U(0, x) = \varphi(x)$, $\frac{\partial U}{\partial t}(0, x) = \psi(x)$. Then functions $\varphi(x)$ and $\psi(x)$ are also entire functions and belong to the class $A_{p'}$.

Let's write down the solution of problem (2.1), (2.2) according to the differential-symbol method [17, 18] as the solution of the Cauchy problem for equation (2.1) with initial data φ and ψ in the form

$$(3.4) \qquad U(t, x) = \varphi\left(\frac{\partial}{\partial \nu}\right)\left\{T_0(t, \nu)e^{\nu \cdot x}\right\}\Big|_{\nu=O} + \psi\left(\frac{\partial}{\partial \nu}\right)\left\{T_1(t, \nu)e^{\nu \cdot x}\right\}\Big|_{\nu=O},$$

where $O = (0, \dots, 0)$.

Since the conditions (2.2) are satisfied for $U(t, x)$, we obtain the matrix equation as follows:

$$\begin{pmatrix} l_{0\partial} T_0\left(t, \dfrac{\partial}{\partial x}\right) & l_{0\partial} T_1\left(t, \dfrac{\partial}{\partial x}\right) \\ l_{1\partial} T_0\left(t, \dfrac{\partial}{\partial x}\right) & l_{1\partial} T_1\left(t, \dfrac{\partial}{\partial x}\right) \end{pmatrix} \begin{pmatrix} \varphi(x) \\ \psi(x) \end{pmatrix} = \begin{pmatrix} \varphi_0(x) \\ \varphi_1(x) \end{pmatrix}.$$

Let's act onto the last equation by the matrix differential expression

$$\begin{pmatrix} l_{1\partial} T_1\left(t, \dfrac{\partial}{\partial x}\right) & -l_{0\partial} T_1\left(t, \dfrac{\partial}{\partial x}\right) \\ -l_{1\partial} T_0\left(t, \dfrac{\partial}{\partial x}\right) & l_{0\partial} T_0\left(t, \dfrac{\partial}{\partial x}\right) \end{pmatrix}.$$

From the equalities

$$l_{0\partial} T_1\left(t, \frac{\partial}{\partial x}\right) = A_2\left(\frac{\partial}{\partial x}\right), \quad l_{0\partial} T_0\left(t, \frac{\partial}{\partial x}\right) = A_1\left(\frac{\partial}{\partial x}\right)$$

for arbitrary $x \in \mathbb{R}^s$, it follows that

$$\begin{pmatrix} \Delta\left(\frac{\partial}{\partial x}\right) \varphi(x) \\ \Delta\left(\frac{\partial}{\partial x}\right) \psi(x) \end{pmatrix} = \begin{pmatrix} l_{1\partial} T_1\left(t, \dfrac{\partial}{\partial x}\right) & -l_{0\partial} T_1\left(t, \dfrac{\partial}{\partial x}\right) \\ -l_{1\partial} T_0\left(t, \dfrac{\partial}{\partial x}\right) & l_{0\partial} T_0\left(t, \dfrac{\partial}{\partial x}\right) \end{pmatrix} \begin{pmatrix} \varphi_0(x) \\ \varphi_1(x) \end{pmatrix} =$$

$$= \begin{pmatrix} l_{1\partial} T_1\left(t, \frac{\partial}{\partial x}\right) \varphi_0(x) - A_2\left(\frac{\partial}{\partial x}\right) \varphi_1(x) \\ -l_{1\partial} T_0\left(t, \frac{\partial}{\partial x}\right) \varphi_0(x) + A_1\left(\frac{\partial}{\partial x}\right) \varphi_1(x) \end{pmatrix}.$$

Since $\Delta(\nu) \equiv 0$ in $\mathbb{C}^s$, then $\Delta\left(\frac{\partial}{\partial x}\right)$ is null operator and $\Delta\left(\frac{\partial}{\partial x}\right)\varphi(x) = \Delta\left(\frac{\partial}{\partial x}\right)\psi(x) = 0$, therefore, we have a contradiction with conditions (3.2) and (3.3). The proof is complete. $\qquad\square$

Example 3.1. In the domain $(t, x_1, x_2) \in \mathbb{R}^3$, investigate the solvability of the problem for the equation

$$(3.5) \qquad \left[\frac{\partial^2}{\partial t^2} + 2\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)\frac{\partial}{\partial t} + \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)^2 + 1 \right] U(t, x) = 0$$

with local two-point conditions

$$(3.6) \qquad \begin{aligned} \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right) U(0, x) + \frac{\partial U}{\partial t}(0, x) &= \varphi_0(x), \\ \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right) U(\pi, x) + \frac{\partial U}{\partial t}(\pi, x) &= \varphi_1(x). \end{aligned}$$

For this problem, we have $a(\nu) = \nu_1 + \nu_2$, $b(\nu) = (\nu_1 + \nu_2)^2 + 1$, $A_1(\nu) = B_1(\nu) = \nu_1 + \nu_2$, $A_2(\nu) = B_2(\nu) = 1$, $s = 2$, $h = \pi$.

The normal fundamental system of solutions at the point $t = 0$ of the ODE, corresponding to (3.5)

$$\left[\frac{d^2}{dt^2} + 2(\nu_1 + \nu_2) \frac{d}{dt} + (\nu_1 + \nu_2)^2 + 1 \right] T(t, \nu) = 0$$

has the form

$$T_0(t, \nu) = e^{-(\nu_1 + \nu_2)t} \left\{ (\nu_1 + \nu_2) \sin t + \cos t \right\}, \quad T_1(t, \nu) = e^{-(\nu_1 + \nu_2)t} \sin t.$$

The functions $T_0(t, \nu), T_1(t, \nu)$ are entire functions of the order $p = 1$ by the set of variables ν_1, ν_2.

Write down the characteristic determinant of problem (3.5), (3.6):

$$\Delta(\nu) = \begin{vmatrix} \nu_1 + \nu_2 & 1 \\ (\nu_1 + \nu_2) T_0(\pi, \nu) + \dfrac{dT_0}{dt}(\pi, \nu) & (\nu_1 + \nu_2) T_1(\pi, \nu) + \dfrac{dT_1}{dt}(\pi, \nu) \end{vmatrix} =$$

$$= e^{-(\nu_1 + \nu_2)\pi} \sin \pi \equiv 0.$$

Condition (3.2) for problem (3.5), (3.6) follows from (3.3) and have such a form:

for a certain point $(x_1, x_2) \in \mathbb{R}^2$ inequality

$$(3.7) \qquad -\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right) \varphi_0(x_1 - \pi, x_2 - \pi) \neq \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right) \varphi_1(x_1, x_2)$$

is hold for entire ($p' = \infty$) functions φ_0, φ_1.

So, in the case of fulfillment of condition (3.7), by Theorem 3.1, the solution of nonhomogeneous problem (3.5), (3.6) in the class of entire functions does not exist.

Example 3.2. Investigate the solvability of the two-point problem in the domain of variables $t \in \mathbb{R}$, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ for differential-functional equation

$$(3.8) \quad \frac{\partial^2}{\partial t^2} U(t, x_1, x_2, x_3) + 2 \frac{\partial}{\partial t} U(t, x_1 + 1, x_2 + 1, x_3 - 1) +$$

$$+ 2U(t, x_1 + 1, x_2 + 1, x_3 - 1) - U(t, x_1, x_2, x_3) = 0$$

with nonhomogeneous local conditions

$$(3.9) \qquad U(0, x) + \frac{\partial U}{\partial t}(0, x) = \varphi_0(x), \quad U(1, x) + \frac{\partial U}{\partial t}(1, x) = \varphi_1(x).$$

Differential-functional equation (3.8) can be represented as the differential equation of infinite order

$$\left[\frac{\partial^2}{\partial t^2} + 2e^{\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3}}\frac{\partial}{\partial t} + \left(2e^{\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3}} - 1\right)\right] U(t, x) = 0.$$

For this problem, we have $a(\nu) = e^{\nu_1 + \nu_2 - \nu_3}$, $b(\nu) = 2e^{\nu_1 + \nu_2 - \nu_3} - 1$, $A_1(\nu) = A_2(\nu) = 1$, $B_1(\nu) = B_2(\nu) = 1$, $s = 3$, $h = 1$.

The normal fundamental system of solutions of ODE

$$\left[\frac{d^2}{dt^2} + 2e^{\nu_1 + \nu_2 - \nu_3}\frac{d}{dt} + \left(2e^{\nu_1 + \nu_2 - \nu_3} - 1\right)\right] T(t, \nu) = 0$$

has the form

(3.10)
$$T_0(t, \nu) = e^{-a(\nu)t}\left\{a(\nu)\frac{\sinh\left[t\{a(\nu) - 1\}\right]}{a(\nu) - 1} + \cosh\left[t\{a(\nu) - 1\}\right]\right\},$$

$$T_1(t, \nu) = e^{-a(\nu)t}\frac{\sinh\left[t\{a(\nu) - 1\}\right]}{a(\nu) - 1},$$

where $a(\nu) = e^{\nu_1 + \nu_2 - \nu_3}$. In the case $a(\nu) = 1$, we have $T_0 = e^{-t}(t+1)$, $T_1 = te^{-t}$.

In our case $p = \infty$. Let's calculate the characteristic determinant $\Delta(\nu)$ of problem (3.8), (3.9):

$$\Delta(\nu) =$$

$$= e^{-e^{\nu_1 + \nu_2 - \nu_3}}\frac{\sinh\left[e^{\nu_1 + \nu_2 - \nu_3} - 1\right]}{e^{\nu_1 + \nu_2 - \nu_3} - 1}\left[\left(e^{\nu_1 + \nu_2 - \nu_3} - 1\right)^2 - \left(e^{\nu_1 + \nu_2 - \nu_3} - 1\right)^2\right] \equiv 0.$$

Conditions (3.2) and (3.3) for problem (3.8), (3.9) can be written down as one condition:

the following inequality

(3.11)
$$\varphi_1(x_1, x_2, x_3) \neq e^{1 - 2e^{\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3}}}\varphi_0(x_1, x_2, x_3)$$

holds for entire functions φ_0, φ_1 of exponential type and for certain point $(x_1, x_2, x_3) \in \mathbb{R}^3$.

Therefore, in the case of fulfillment of condition (3.11), according to Theorem 3.1, the solution of nonhomogeneous problem (3.8), (3.9) in the class of entire functions $\mathbb{A}_1$ does not exist.

4. Constructing partial solutions of the two-point problem in the class of existence of nonunique solution of the problem

Let's consider the case when conditions of Theorem 3.1 are not satisfied.

If for problem (2.1), (2.2) $\Delta(\nu) \equiv 0$ in $\mathbb{C}^s$ and for arbitrary $x \in \mathbb{R}^s$ there hold the equalities

$$l_{1\partial}T_1\left(t, \frac{\partial}{\partial x}\right)\varphi_0(x) = A_2\left(\frac{\partial}{\partial x}\right)\varphi_1(x),$$

$$l_{1\partial}T_0\left(t,\frac{\partial}{\partial x}\right)\varphi_0(x) = A_1\left(\frac{\partial}{\partial x}\right)\varphi_1(x)$$

for $\varphi_0, \varphi_1 \in A_{p'}$, then the solution of problem (2.1), (2.2) in class of entire functions $\mathbb{A}_{p'}$ exists but is not unique.

Let's show in examples constructing partial solutions of the problem.

Example 4.1. Consider the problem

$$(4.1)\qquad\left[\frac{\partial^2}{\partial t^2}+2\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}\right)\frac{\partial}{\partial t}+\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}\right)^2+1\right]U(t,x)=0, (t,x)\in\mathbb{R}^3,$$

$$(4.2)\qquad\begin{aligned}\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}\right)U(0,x)+\frac{\partial U}{\partial t}(0,x)&=\varphi_0(x),\\\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}\right)U(\pi,x)+\frac{\partial U}{\partial t}(\pi,x)&=-\varphi_0(x_1-\pi,x_2-\pi),\quad x\in\mathbb{R}^2.\end{aligned}$$

Let's consider the problem as problem (3.5), (3.6), in which for entire functions φ_0, φ_1 and for arbitrary $x \in \mathbb{R}^2$ there holds the condition

$$-\varphi_0(x_1-\pi,x_2-\pi) = \varphi_1(x_1,x_2).$$

Therefore, the conditions of Theorem 3.1 are not satisfied. We show that the solution of problem (4.1), (4.2) exists in the class of entire functions.

Note that the null space of problem (4.1), (4.2) is infinite-dimensional. It contains not only entire solutions, but also classical solutions of the form

$$U(t,x) = \varphi(x_1-t, x_2-t)\cos t,$$

where φ is arbitrary twice continuously differentiable function on $\mathbb{R}^2$.

The entire (partial) solution of problem (4.1), (4.2) can be found, for example, by formula

$$U(t,x) = \varphi_0\left(\frac{\partial}{\partial\nu}\right)\left\{T_1(t,\nu)e^{\nu\cdot x}\right\}\Big|_{\nu=(0,0)} =$$

$$= \varphi_0\left(\frac{\partial}{\partial\nu}\right)\left\{e^{-(\nu_1+\nu_2)t+\nu\cdot x}\sin t\right\}\Big|_{\nu=(0,0)} = \varphi_0(x_1-t, x_2-t)\sin t.$$

If in the formula

$$(4.3)\qquad\qquad U(t,x) = \varphi_0(x_1-t, x_2-t)\sin t$$

we take $\varphi_0(x_1, x_2)$ as an arbitrary twice continuously differentiable function on $\mathbb{R}^2$, then function (4.3) is a classical solution of problem (4.1), (4.2).

Example 4.2. Let's consider the problem

$$(4.4) \quad \frac{\partial^2}{\partial t^2} U\left(t, x_1, x_2, x_3\right) + 2 \frac{\partial}{\partial t} U\left(t, x_1 + 1, x_2 + 1, x_3 - 1\right) +$$
$$+ 2U\left(t,\, x_1 + 1, x_2 + 1, x_3 - 1\right) - U\left(t, x_1, x_2, x_3\right) = 0, \quad (t, x) \in \mathbb{R}^4,$$

$$(4.5) \quad \begin{aligned} U(0, x) + \frac{\partial U}{\partial t}(0, x) &= \varphi_0(x), \\ U(1, x) + \frac{\partial U}{\partial t}(1, x) &= e^{1 - 2e^{\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3}}} \varphi_0(x), \quad x \in \mathbb{R}^3. \end{aligned}$$

Problem (4.4), (4.5) is problem (3.8), (3.9), in which for entire functions $\varphi_0(x)$, $\varphi_1(x)$ of exponential type and for arbitrary $x \in \mathbb{R}^3$ the following condition

$$(4.6) \qquad \varphi_1(x_1, x_2, x_3) = e^{1 - 2e^{\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3}}} \varphi_0(x_1, x_2, x_3).$$

is satisfied.

Hence, the conditions of Theorem 3.1 are not satisfied. The solution of problem (4.4), (4.5) exists in the class of entire functions $\mathbb{A}_1$. Note that the null space of this problem is infinite-dimensional. It contains the functions

$$(4.7) \qquad U\left(t, x\right) = e^{-t} \varphi\left(x\right),$$

where φ is arbitrary continuous on $\mathbb{R}^3$ function.

The entire (partial) solution of problem (4.4), (4.5) for function $\varphi_0(x)$ of the exponential type can be found, for example, using one of the following formulas:

$$U\left(t, x\right) = \varphi_0\left(\frac{\partial}{\partial \nu}\right) \left\{ T_0(t, \nu) e^{\nu \cdot x} \right\} \Big|_{\nu = (0,0,0)}$$

or

$$(4.8) \qquad U\left(t, x\right) = \varphi_0\left(\frac{\partial}{\partial \nu}\right) \left\{ T_1(t, \nu) e^{\nu \cdot x} \right\} \Big|_{\nu = (0,0,0)},$$

where $T_0(t, \nu)$, $T_1(t, \nu)$ are functions (3.10).

Let's consider some cases of different functions $\varphi_0(x)$.

Case 1. Let $\varphi_0(x_1, x_2, x_3) = e^{x_1 + x_2 + 2x_3}$. Then condition (4.6) has the form

$$e^{1 - 2e^{\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3}}} \varphi_0(x_1, x_2, x_3) = e^{-1 + x_1 + x_2 + 2x_3}.$$

By formula (4.8), we find the partial solution of problem (4.4), (4.5):

$$U\left(t, x\right) = e^{\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + 2\frac{\partial}{\partial x_3}} \left\{ T_1(t, \nu) e^{\nu \cdot x} \right\} \Big|_{\nu = (0,0,0)} =$$
$$= T_1(t, 1, 1, 2) e^{x_1 + x_2 + 2x_3} = t\, e^{-t + x_1 + x_2 + 2x_3}.$$

Case 2. Let $\varphi_0(x_1, x_2, x_3) = 1 + 2x_1 - 3x_2 + 2x_3$. Let's calculate

$$e^{1-2e^{\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3}}} \varphi_0(x_1, x_2, x_3) = \left\{ \left(e^{1-2e^{\nu_1+\nu_2-\nu_3}} \right)\Big|_{\nu=(0,0,0)} + \right.$$

$$+ \frac{\partial}{\partial \nu_1}\left(e^{1-2e^{\nu_1+\nu_2-\nu_3}} \right)\Big|_{\nu=(0,0,0)} \frac{\partial}{\partial x_1} + \frac{\partial}{\partial \nu_2}\left(e^{1-2e^{\nu_1+\nu_2-\nu_3}} \right)\Big|_{\nu=(0,0,0)} \frac{\partial}{\partial x_2} +$$

$$\left. + \frac{\partial}{\partial \nu_3}\left(e^{1-2e^{\nu_1+\nu_2-\nu_3}} \right)\Big|_{\nu=(0,0,0)} \frac{\partial}{\partial x_3} + ... \right\} \varphi_0(x_1, x_2, x_3) =$$

$$= e^{-1}\left\{ 1 - 2\frac{\partial}{\partial x_1} - 2\frac{\partial}{\partial x_2} + 2\frac{\partial}{\partial x_3} + ... \right\}\left(1 + 2x_1 - 3x_2 + 2x_3 \right) =$$

$$= e^{-1}\left(7 + 2x_1 - 3x_2 + 2x_3 \right).$$

Therefore, according to equality (4.6), the existence of the solution of problem (4.4), (4.5) is provided by the condition:

$$\varphi_1(x_1, x_2, x_3) = e^{-1}\left(7 + 2x_1 - 3x_2 + 2x_3 \right).$$

Let's find the partial solution of problem (4.4), (4.5) by formula (4.8):

$$U(t, x) = \left\{ 1 + 2\frac{\partial}{\partial \nu_1} - 3\frac{\partial}{\partial \nu_2} + 2\frac{\partial}{\partial \nu_3} \right\}\left\{ T_1(t, \nu)e^{\nu \cdot x} \right\}\Big|_{\nu=(0,0,0)} =$$

$$= T_1(t, 0, 0, 0) + 2\frac{\partial T_1}{\partial \nu_1}(t, 0, 0, 0) - 3\frac{\partial T_1}{\partial \nu_2}(t, 0, 0, 0) +$$

$$+ 2\frac{\partial T_1}{\partial \nu_3}(t, 0, 0, 0) + 2x_1 T_1(t, 0, 0, 0) - 3x_2 T_1(t, 0, 0, 0) + 2x_3 T_1(t, 0, 0, 0).$$

Since

$$T_1(t, 0, 0, 0) = t\, e^{-t},$$

$$\frac{\partial T_1}{\partial \nu_1}(t, 0, 0, 0) = \frac{\partial T_1}{\partial \nu_2}(t, 0, 0, 0) = -t^2 e^{-t}, \quad \frac{\partial T_1}{\partial \nu_3}(t, 0, 0, 0) = t^2 e^{-t},$$

then the solution of problem (4.4), (4.5) has the form

$$U(t, x) = t\, e^{-t}\left\{ 1 + 2x_1 - 3x_2 + 2x_3 + 3t \right\}.$$

Note that the solution above is only partial solution: summed up with elements of null space of problem (4.7) it is also a solution of problem (4.4), (4.5).

5. Conclusions

In this work, we prove that the problem for homogeneous PDE of second order in time variable (in which the nonhomogeneous local two-point conditions are given) and generally infinite order in spatial variables in the class of entire functions, is ill-posed problem in the case when the characteristic determinant identically equals to zero. In the case of existence of nonunique solution of the problem, we propose a method of finding partial solutions. This way is presented on examples.

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