A NOTE ON Z-ALGEBRA

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Abstract. This paper introduces a new notion of algebra called Z-algebra and some of its properties are discussed in detail. It reveals also that Z-algebra is completely different from some of other abstract algebras.
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1. Introduction

Algebraic structures play an important role in mathematics with wide range of applications in many disciplines such as theoretical physics, computer sciences,control engineering, information sciences, coding theory etc. Several algebraic structure have been introduced in the recent past. K. Iseki and S. Tanaka [5], introduced a class of abstract algebra: BCK-algebra. Also Y. Imai and K. Iseki [6], dealt about BCI-algebras. Then, Hu Q.P. and Li. X [4], have given the

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notion of BCH-algebra which is the generalization of BCI and BCK-algebras. Neggers et.al [13, 14, 15] introduced the notions of B-algebras, Q-algebras and d-algebras. In 2010, K. Megalai and A. Tamilarasi [18], introduced TM-algebra. During 2011, Keawrahun and Leerawat [8] introduced new structured algebra called SU-Algebra. Prabayak and Leerawat [16], discussed ideals and congruences in KU-algebras. With all these ideas, in this paper, a new notion of algebra called Z-algebra is defined and some substructures are also established in this algebra .

2. Preliminares

This section recalls the definitions of various algebras defined by different authors in the algebraic formulation of the propositional calculus and implicational calculus.

Definition 2.1 ([5]). *A BCK-algebra* $(X, *, 0)$ *is a nonempty set* X *with constant* 0 *and a binary operation * by satisfying the following conditions:*

1. $((x * y) * (x * z)) * (z * y) = 0$
2. $0 * x = 0$
3. $x * x = 0$
4. $(x(x * y)) * y = 0$
5. $x * y = 0$ *and* $y * x = 0$ *imply* $x = y$, $\forall x, y \in X$.

Definition 2.2 ([14]). *A Q-algebra* $(X, *, 0)$ *is a nonempty set* X *with constant* 0 *and a binary operation * by satisfying the following conditions:*

1. $x * 0 = x$
2. $x * x = 0$
3. $(x * y) * z = (x * z) * y$, $\forall\ x, y \in X$.

Definition 2.3 ([7]). *A BH-algebra* $(X, *, 0)$ *is a nonempty set* X *with constant* 0 *and a binary operation * satisfying the following conditions:*

1. $x * 0 = x$
2. $x * x = 0$
3. $x * y = 0$ *and* $y * x = 0$ *implies* $x = y$, $\forall\ x, y \in X$.

Definition 2.4 ([18]). *A TM-algebra* $(X, *, 0)$ *is a nonempty set* X *with constant* 0 *and a binary operation * satisfying the following conditions:*

1. $x * 0 = x$

2. $(x * y) * (x * z) = z * y$, $\forall x, y \in X$.

Definition 2.5 ([8]). *A SU-algebra $(X, *, 0)$ is a nonempty set X with constant 0 and a binary operation * satisfying the following conditions:*

1. $((x * y) * (x * z)) * (y * z) = 0$
2. $x * 0 = x$
3. *if* $x * y = 0 \Rightarrow x = y$, $\forall x, y, z \in X$. $\forall x, y \in X$.

Definition 2.6. *Let X be any non-empty set. A filter of X, is a subset $S(\neq \phi)$ of X, if $0 \notin S$ and $\forall x, y \in S \implies x \triangle y \in S$, where $x \triangle y = x * (x * y)$.*

3. The structure of Z-algebra

In this section, a new structure of algebra, namely Z-algebra which is an algebra based on propositional calculus is intoduced and the relation between Z-algebra and other algebras is investigated. A substructue Z-Subalgebra is also defined.

Definition 3.1. *e A Z-algebra $(X, *, 0)$ is a nonempty set X with constant 0 and a binary operation * satisfying the following conditions:*

(3.1) $x * 0 = 0$

(3.2) $0 * x = x$

(3.3) $x * x = x$

(3.4) $x * y = y * x, when\ x \neq 0\ and\ y \neq 0, \forall x, y \in X.$

Example 3.2. Let $X = \{0, 1, 2, 3\}$ be a set with constant 0 and a binary operation * is defined on X by the Cayley's table as follows.

*	0	1	2	3
0	0	1	2	3
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

Then, $(X, *, 0)$ is a Z-algebra.

Example 3.3. Let $(X = \mathbb{R}, *, 0)$ where $x, y \in \mathbb{R}$. Define a binary operation * on X by,

$$x * y = \begin{cases} y, & if\ (x = 0\ and\ y \neq 0)\ or\ (x \neq 0\ and\ y = 0) \\ x, & x = y. \\ xy, & x \neq y. \end{cases}$$

Then, $(X, *, 0)$ be Z-algebra.

Example 3.4. The following table shows $(X = \{0, a, b, c\}, *, 0)$ is a Z-algebra.

$*$	0	a	b	c
0	0	a	b	c
a	0	a	c	b
b	0	c	b	a
c	0	b	a	c

Definition 3.5. *Let S be a non empty subset of a Z-algebra of X. Then, S is called a Z-Subalgebra of X, if $x * y \in S$, $\forall x, y \in S$.*

Example 3.6. Consider the Z-algebra in example 3.2. Then, the subsets $A = \{1, 3\} \subset X$ and $B = \{2, 3\} \subset X$ are Z-Subalgebras of X, but the subset $C = \{1, 2, 3\} \subset X$ is not a Z-Subalgebra of X.

Proposition 3.7. *Let $(X, *, 0)$ be a Z-algebra. Then, it is not a BCK-algebra* [5]*, Q-algebra* [14]*, BH-algebra* [7]*, SU- algebra* [8]*, BM-algebra* [10]*, B-algebra* [13]*, BF-algebra* [19]*, BRK-algebra* [3]*, RG-algebra* [12] *and TM-algebra* [18]*.*

Proof. In all the above algebras cited here except a Z-algebra, we have, $x * 0 = x$. But by definition 3.1, for a Z-algebra we have, $x * 0 = 0 \neq x$.
This completes the proof. □

Proposition 3.8. *Let $(X, *, 0)$ be a Z-algebra. Then, it is not a d-algebra* [15]*, KU-algebra* [16]*, BE-algebra* [9]*, PS-algebra* [17]*, CI-algebra* [2]*, BCI-algebra* [6]*, BCH-algebra* [4]*, BP-algebra* [1]*, BO-algebra* [11] *and BCL-algebra* [20]*.*

Proof. In all the above algebras cited here except a Z-algebra, we have, $x * x = 0$. But by definition 3.1, for a Z-algebra we have, $x * x = x \neq 0$.
This completes the proof. □

Proposition 3.9. *Let X be a Z-algebra. Then, the following results hold for all $x, y, z \in X$.*

1. $(x * (x * (y * x))) = x, \ if \ y = 0$
2. $x * y = (y * 0) * (x * 0)$
3. $(x * y) * [(y * x) * (x * y)] = y * x$
4. $0 * (x * y) = (0 * x) * (0 * y)$
5. $(X, *, 0), \quad x * (0 * y) = y * x, \forall x \neq y.$

Proof. Let (X,*,0) be Z-algebra, $x, y \in X$.

1. Suppose $y = 0$, then $(x * (x * (y * x))) = (x * (x * (0 * x))) = (x * (x * x))$ by 3.1 $= (x * x)$ by 3.3 $= x$ by 3.3

2. $x * y = (x * y) * 0 = 0 = 0 * 0$ by 3.3 $= (y * 0) * (x * 0)$ by 3.4

3. $(x * y) * [(y * x) * (x * y)] = (x * y) * [(y * x) * (y * x)]$ by 3.4 $= (x * y) * (y * x)$ 3.3 $= (y * x) * (y * x)$ by 3.1 $= y * x$ by 3.3

4. $0 * (x * y) = (x * y)$ by 3.2 $= (0 * x) * (0 * y)$ by 3.2

5. $x * (0 * y) = (0 * x) * (0 * y) = x * y = y * x$. □

Definition 3.10. *Let $(X, *, 0)$ and $(Y, \Delta, 0')$ be two Z-algebras. A mapping $f : X \to Y$ of a Z-algebra is called a homomorphism, if $f(x*y) = f(x)\Delta f(y), \forall\, x, y \in X$.*

Definition 3.11. *Let $(X, *, 0)$ and $(Y, \Delta, 0')$ be two Z-algebras and $f : X \to Y$ of is a homomorphism. Then, Kernal of f is the subset of X is defined by $Ker(f) = \{x \in X : f(x) = 0'\}$.*

Lemma 3.12. *If $f : X \to Y$ be an homomorphism of Z-algebras, then $f(0) = 0'$, $0 \in X$.*

Proof. Let $f : X \to Y$ be an homomorphism of Z-algebras. Then, $f(0) = f(0 * 0) = f(0)\Delta f(0) = 0'$ □

Theorem 3.13. *Let $(X, *, 0)$ and $(Y, \Delta, 0')$ be two Z-algebras. Let $f : X \to Y$ be a surjective Z- homomorphism. If A is Z-subalgebra of X, then f(A) is Z-subalgebra of Y.*

Proof. Let X and Y be two Z-algebras.

Let $f : X \to Y$ be a homomorphism and A be a Z-subalgebra of X.

Now, $a, b \in A \Rightarrow a * b \in A$, $\therefore f(a), f(b) \in f(A)$.

$\Rightarrow f(a)\Delta f(b) = f(a * b) \in f(A)$.

Hence, f(A) is a Z-algebra of Y. □

Theorem 3.14. *Let $(X, *, 0)$ and $(Y, \Delta, 0')$ be two Z-algebras and B be Z-subalgebra of Y. Let $f : X \to Y$ be a homomorphism. Then $f^{-1}(B)$ is a Z-algebra of X.*

Proof. It is known that, $f^{-1}(B) = \{x \in X | f(x) = y \; for \; some \; y \in B\}$.

Assume that $x and y \in f^{-1}(B)$. Then $f(x) and f(y) \in B$.

Since B is a Z-subalgera of Y, $f(x)\Delta f(y) \in B$. Also, since f is homomorphism, $f(x * y) = f(x)\Delta f(y) \in B$.

$\therefore x * y \in f^{-1}(B)$.

Hence, $f^{-1}(B)$ is a Z-algebra of X. □

4. Z-Ideals

In this section, the notions of Z-ideal and Z-closed ideal of a Z-algebra are discussed.

Definition 4.1. *Let X be a Z-algebra and I be a subset of X. Then, I is called a Z-ideal of X, if it satisfies the following conditions:*

1. $0 \in I$
2. $x * y \in I$ *and* $y \in I \Rightarrow x \in I$

Example 4.2. Consider the Z-algebra in example 3.2. Then, $I = \{0, 1, 2\} \subset X$ is a Z-ideal of X.

Definition 4.3. *An ideal I of a Z-algebra $(X, *, 0)$ is said to be Z-closed ideal, if $0 * x \in A$, $\forall x \in I$.*

Proposition 4.4. *Let I be a Z-ideal of a Z-algebra X. If $x \in I$ and y*x =0, then $y \in I$.*

Proof. Let $x \in I$ and $y * x = 0$.
$\Rightarrow x \in I$ *and* $y * x \in I$.
Since I is a Z-ideal , $y \in I$ is obtained. □

Theorem 4.5. *Let $(X, *, 0)$ and $(Y, \Delta, 0')$ be two Z-algebras and $f : X \to Y$ be a surjective homomorphism. Then, we have,*

1. *if A is Z-Ideal of X , then f(A) is Z-Ideal of Y.*
2. *if B is Z-Ideal of Y, then $f^{-1}(B)$ is a Z-Ideal of X.*

Theorem 4.6. *Let $f : X \to Y$ be a homomorphism of Z-algebras. Then, Ker(f) is an ideal of X.*

Proof. Obviously, $0 \in Ker(f)$, $\because f(0) = 0'$.
Let $x * y \in Ker(f)$ and $y \in Ker(f)$, so that $f(x * y) = 0', f(y) = 0'$.
That is $f(x)\Delta f(y) = 0' \Rightarrow f(x)\Delta 0' = 0' \Rightarrow f(x) = 0'$.
Hence $x \in Ker(f)$.
So, $Ker(f)$ is an ideal of X. □

5. Z-Filters

In this section, the notion of Z-filter on a Z-algebra is studied.

Definition 5.1. *Let X be a Z-algebra and $A \subset X$ with $0 \notin A$. Then, A is said to be a Z-filter on X, if $\forall x, y\ (x \neq y) \in A \Rightarrow x \triangle y = x * (x * y) \in A$.*

Example 5.2. Consider the Z-algebra in example 3.4. Then, $A = \{a, b\} \subset X$ and $B = \{a, b, c\} \subset X$ are Z-filters on X.

Remark 5.3. The substructures of Z-filter and Z-subalgebra of X are totally different in general. i.e a Z-subalgebra is not a Z-filter and conversely Z-filter is not a Z-subalgebra.

Example 5.4. Consider the Z-algebra in example 3.4.

Then, $A = \{0, a\} \subset X$ is a Z-subalgebra, but it is not Z-filter, since $0 \in A$.

And, $B = \{a, b\} \subset X$ is a Z-filter, but it is not Z-subalgebra, since $a * b = c \notin B$.

Also, $C = \{0, a, b\} \subset X$ is neither a Z-subalgebra nor a Z-filter.

Remark 5.5. By the definitions 4.1 and 5.1, a Z-filter and Z-ideal are also different substructures in a Z-algebra.

Theorem 5.6. *Let X and Y be two Z- algebras. Let $f : X \to Y$ be a surjective Z- homomorphism. If A is Z- filter of X, then f(A) is Z- filter of Y.*

Proof. Let $a, b \in f(A)$. Then, $a = f(x), b = f(y)$ *for some* $x,\ y \in X$.

Since A is a Z-filter of X, $x \triangle y = x * (x * y) \in A$.

Also $f(x \triangle y) \in f(A)$.

Now

$$\begin{aligned} a \triangle b &= a * (a * b) \\ &= f(x) * (f(x) * f(y)) \\ &= f(x * (x * y)) \\ &= f(x \triangle y) \end{aligned}$$

Hence, $a \triangle b \in f(A)$, and f(A)is a Z-filter of Y. □

Remark 5.7. Let $f : X \to Y$ be a homomorphism of Z-algebras. Then, Ker(f) is not a filter of X.

6. Conclusion and future work

This article introduces the new class of algebra, Z-algebra by taking the theory of sets and propositional calculus as the backend. It has been observed that Z-algebra is not like other algebras. So, it is important that the Z-algebras play an independent role in the axiomatic algebraic system. Interestingly, this concept can further be extended to fuzzification of Z-algebras, Rough set and Soft set application to Z-algebras and derivation on Z-algebras in future.

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