HYPERMATRIX REPRESENTATIONS OF SINGLE POWER CYCLIC HYPERGROUPS

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Abstract. Cyclic hypergroups are special type of hypergroups that have some importance for their applications in different fields. In this paper, we deal with hypermatrix representations of single power cyclic hypergroups. First, we consider single power cyclic hypergroups with infinite period, define a commutative semihyperring and construct non-trivial hypermatrix representations over our defined semihyperring. Then we do the same for single power cyclic hypergroups with finite period. Many properties of these hypermatrix representations are presented.
Keywords: cyclic hypergroup, representation.

1. Introduction

Hypergroup theory was known for the first time in 1934 at the eighth Congress of Scan- dinavian Mathematicians, when Marty [14] gave the definition of hypergroup as a generalization of the notion of the group, illustrated some applications and showed its utility in the study of groups, algebraic functions and relational fractions. Recently, the hypergroups are studied from the theoretical point of view and for their applications to many subjects of pure and applied mathematics: geometry, topology, cryptography and code theory, graphs and hypergraphs, probability theory, binary relations, theory of fuzzy and rough sets, automata theory, economy, etc. (see [1, 7, 10, 13, 17]). A hypergroup is an algebraic structure similar to a group, but the composition of two elements is a non-empty set. Representation theory is a branch of mathematics, known in 1896 in the work of the German mathematician F. G. Frobenius (see [8]) , that has lots of applications in physics, number theory, etc. It was first known to study representations of algebraic structures (groups, rings, topological spaces, etc) by representing their elements as linear transformations of vector spaces. More

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precisely, a representation makes an abstract algebraic object more concrete by transforming it into matrix and its algebraic operation into matrix addition or multiplication. The concept of representation theory was later generalized to study hyperstructures (see [9, 17, 18, 19]) in which the hypergroup's elements are represented as hypermatrices over some semihyperring. The representation of hypergroups depends on the choice of the semihyperring in which the entries of the hypermatrices belong to. This makes representations of hypergroups much harder than that of representation of groups because of the hyperoperations of the semihyperring which are not always the standard operations.

This paper is a connection between hyperstuctures and representation theory. It is constructed as follows: after an introduction, Section 2 presents some basic definitions related to hyperstructures and hypermatrix representations. Sections 3 and 4 present non-trivial hypermatrix representations of single power cyclic hypergroups with infinite and finite period respectively and study their properties.

2. Basic definitions

In this section, we present some definitions related to hyperstructures and matrix representations that are used throughout the paper.

Let H be a non-empty set. Then, a mapping $\circ : H \times H \to \mathcal{P}^*(H)$ is called a *hyperoperation* on H, where $\mathcal{P}^*(H)$ is the family of all non-empty subsets of H. The couple $(H, \circ)$ is called a *hypergroupoid*. In the above definition, if A and B are two non-empty subsets of H and $x \in H$, then we define:

$$A \circ B = \bigcup_{\substack{a \in A \\ b \in B}} a \circ b, \;\; x \circ A = \{x\} \circ A \text{ and } A \circ x = A \circ \{x\}.$$

An element $e \in H$ is called an *identity* of $(H, \circ)$ if $x \in x \circ e \cap e \circ x$, for all $x \in H$ and it is called a *scalar identity* of $(H, \circ)$ if $x \circ e = e \circ x = \{x\}$, for all $x \in H$. If e is a scalar identity of $(H, \circ)$, then e is the unique identity of $(H, \circ)$. The hypergroupoid $(H, \circ)$ is said to be *commutative* if $x \circ y = y \circ x$, for all $x, y \in H$. A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if it is associative, i.e., for every $x, y, z \in H$, we have $x \circ (y \circ z) = (x \circ y) \circ z$ and is called a *quasihypergroup* if for every $x \in H$, $x \circ H = H = H \circ x$. This condition is called the reproduction axiom. The couple $(H, \circ)$ is called a *hypergroup* if it is a semihypergroup and a quasihypergroup.

Cyclic semihypergroups have been studied by Corsini [4], De Salvo and Freni [11], Vougiouklis [16], Leoreanu [12]. Cyclic semihypergroups are important not only in the sphere of finitely generated semihypergroups but also for interesting combinatorial implications.

A hypergroup $(H, \circ)$ is cyclic if there exist $h \in H$ and $s \in \mathbb{N}$ such that

$$H = h \cup h^2 \cup \cdots \cup h^s \cup \cdots .$$

If $H = h \cup h^2 \cup \cdots \cup h^s$ then H is a cyclic hypergroup with finite period. Otherwise, H is called cyclic hypergroup with infinite period. Here, $h^s = \underbrace{h \circ h \circ \ldots \circ h}_{s \text{ times}}$. It is clear that for all $\alpha \in H$ there exist $r \in \mathbb{N}$ such that $\alpha \in h^r$. It is a *single-power cyclic hypergroup* if there exist $h \in H$ and $s \in \mathbb{N}$ such that

$$H = h \cup h^2 \cup \cdots \cup h^s \cup \cdots \text{ and } h \cup h^2 \cup \cdots \cup h^{s-1} \subset h^s, \text{ for all } s \in \mathbb{N}.$$

A *semihyperring* $(R, +, \cdot)$ is a hyperstructure with two hyperoperations $+$ and $\cdot$ where $+$ and $\cdot$ are associative hyperoperations and $\cdot$ is distributive with respect to $+$, i.e., $x \cdot (y + z) \subseteq x \cdot y + x \cdot z$ and $(x + y) \cdot z \subseteq x \cdot z + y \cdot z$ for every $x, y, z \in R_1$. An element $0 \in R$ is called the *zero element* in R if $0+a = a+0 = a$ and $0 \cdot a = a \cdot 0 = 0$ for all $a \in R$. An element $1 \in R$ is called a *scalar unit* in R if $1 \cdot a = a \cdot 1 = a$ for all $a \in R$.

A *hypermatrix* is a matrix with entries from a semihyperring. The hyperproduct of two $m \times n$ and $n \times r$ hypermatrices (a_{ij}) and (b_{ij}) is the $n \times r$ hypermatrix defined as

$$(a_{ij})(b_{ij}) = \Big\{(c_{ij}) : c_{ij} \in \sum_{k=1}^{n} a_{ik} b_{kj}\Big\}.$$

Let $(R, +, \cdot)$ be a commutative semihyperring and $(H, \circ)$ be a hypergroup then a map $\rho : H \to M_k(R)$ is said to be an *inclusion representation* if for all $\alpha, \beta \in H$, we have

$$\rho(\alpha \circ \beta) = \{\rho(\lambda) : \lambda \in \alpha \circ \beta\} \subseteq \rho(\alpha)\rho(\beta).$$

It is called a *good representation* if $\rho(\alpha \circ \beta) = \rho(\alpha)\rho(\beta)$. And it is a *faithful representation* if it is an injective good representation.

Throughout this paper, we define $exp(\alpha) = min\{i : \alpha \in h^i\} \geq 0$ for all $\alpha \in H$ where H is a cyclic hypergroup with generator h. We call $e \in H$ a trivial element and write $exp(e) = 0$ if for all $\alpha \in H$, we have $e \circ \alpha = \alpha \circ e = \alpha$.

3. Representation of single power cyclic hypergroups with infinite period

The terms "single-power cyclic hypergroup" and "cyclic hypergroup of infinite period" were introduced by Vougiouklis and appeared in 1981 [16]. Moreover, the theory of "representations" on hyperstructures was introduced by Vougiouklis, as one can see in [17]. Also, see [5].

In this section, we construct a commutative semihyperring on P, the set of non negative integers, introduce some hypermatrix representations of single power cyclic hypergroups with infinite period and present some of their interesting properties. Our work is a generalization of a part of a previous work done by the authors (see [2]) on a special single power cyclic hypergroup with infinite period associated to the braid group.

Throughout this section, we define $M_k(R_1)$ as the set of all $k\times k$ hypermatrices with entries from $R_1 = (P, \oplus, \odot)$, $(H, \circ)$ is a single power cyclic hypergroup with infinite period having a generator $h \in H$ (unless it is mentioned differently). It is clear that $h \in h^2 \subset \ldots \subset h^{s-1} \subset h^s \subset \ldots$ for all $s > 2$.

Theorem 3.1 ([2])**.** *Let P be the set of non negative integers and $a, b \in P$. Then $R_1 = (P, \oplus, \odot)$ is a commutative semihyperring with scalar unit and zero element where $\odot$ is the standard multiplication and $a \oplus b = \{0, 1, \ldots, a + b\}$ if $a, b \neq 0$, $a \oplus 0 = 0 \oplus a = a$.*

Proposition 3.2. *$(R_1, \oplus)$ is a single power cyclic hypergroup with infinite period and generator 1.*

Proof. We have that $(R_1, \oplus)$ is commutative and associative by Theorem 3.1. We prove now that the reproduction axiom is satisfied. If $m \in R_1$ then we have two cases; $m = 0$ and $m > 0$. If $m = 0$ then $m \oplus R_1 = \{m \oplus n : n \in R_1\} = \{n : n \in R_1\} = R_1$. If $m > 0$ then $m \oplus R_1 = \{m \oplus r : r \in R_1\} = \bigcup_{r \in R_1} \mathbb{Z}_{m+r+1} = R_1$. Thus, $(R_1, \oplus)$ is a hypergroup.

For every $m \neq 0 \in R_1$ we have that

$$m \in 1^m = \underbrace{1 \oplus 1 \oplus \ldots \oplus 1}_{m \text{ times}} = \{0, 1, \ldots, m\}.$$

It is easy to see that $R_1 = 1 \cup 1^2 \cup \ldots$ and $1 \in 1^2 \subset \ldots \subset 1^{s-1} \subset 1^s \subset \ldots$ for all $s \in \mathbb{N}$. Therefore, $(R_1, \oplus)$ is single power cyclic hypergroup with infinite period and generator 1. □

Definition 3.3. *A representation $\rho : H \to M_k(R_1)$ is said to be reducible if it has a proper non trivial invariant subspace, i.e., there exist*

$$S \neq \{0\} \subset R_1^k = \underbrace{R_1 \times \ldots \times R_1}_{k \ times}$$

such that $\rho(x)u \in S$, for all $x \in H$ and $u \in S$. Otherwise, it is said to be irreducible.

Definition 3.4. *A representation $\rho : H \to M_k(R_1)$ is said to be unitary relative to a non-zero symmetric hypermatrix M if $\rho(x)M\rho(x)^T = \rho(x)^T M\rho(x) = M$, for all $x \in H$. Here, T is the matrix transpose.*

Proposition 3.5. *Let $(H, \circ)$ be a commutative hypergroup (not necessary a single power cyclic hypergroup) and $\rho : H \to M_k(R_1)$ be an inclusion (or good) representation of H. Then $\rho^T : H \to M_k(R_1)$ defined as $\rho^T(\alpha) = (\rho(\alpha))^T$ is a representation of H where T is the matrix transpose.*

Proof. Let $\alpha, \beta \in H$ and ρ be an inclusion representation such that $\rho(\alpha) = A$ and $\rho(\beta) = B$. We have that $\rho(\alpha \circ \beta) = \rho(\beta \circ \alpha)$ as $(H, \circ)$ is commutative. And having ρ an inclusion representation implies that $\rho(\alpha \circ \beta) = \rho(\beta \circ \alpha) \subseteq$

$\rho(\beta)\rho(\alpha) = BA$. We have that $\rho^T(\alpha \circ \beta) = \{\rho^T(\lambda) : \lambda \in \alpha \circ \beta\} = \{(\rho(\lambda))^T : \lambda \in \alpha \circ \beta\}$. And since $\rho(\alpha \circ \beta) = \{\rho(\gamma) : \gamma \in \alpha \circ \beta\} \subseteq BA$, it follows that $\rho^T(\alpha \circ \beta) \subseteq (BA)^T$. Using the definition of hyperproduct and the fact that R_1 is commutative, it is easy to see that $(BA)^T = A^T B^T$. Therefore, $\rho^T(\alpha \circ \beta) \subseteq \rho^T(\alpha)\rho^T(\beta)$.

The case when ρ is a good representation is done in a similar manner. □

Proposition 3.6. *Let $(H, \circ)$ be a commutative hypergroup (not necessary a single power cyclic hypergroup) and $\rho : H \to M_k(R_1)$ is a unitary representation relative to a hypermatrix M then $\rho^T : H \to M_k(R)$ is a unitary representation relative to M.*

Proof. Proposition 3.5 asserts that $\rho^T : H \to M_k(R)$ is a representation of H. Let $\alpha \in H$ such that $\rho(\alpha) = A$. Having ρ unitary relative to M implies that $AMA^T = A^T MA = M$. The latter can be written again as $(A^T)^T MA^T = A^T M(A^T)^T = M$. Thus, ρ^T is unitary relative to M. □

Proposition 3.7. *Let $(H, \circ)$ be a hypergroup (not necessary a single power cyclic hypergroup) and $\rho : H \to M_k(R_1)$ be the trivial map defined by $\rho(\alpha) = I_k$ where I_k is the $k \times k$ identity matrix. Then ρ is a good representation of H.*

Proof. Let $\alpha, \beta \in H$ with $\rho(\alpha) = \rho(\beta) = I_k$. We have that $\rho(\alpha \circ \beta) = \{\rho(\lambda) : \lambda \in \alpha \circ \beta\} = I_k$ and $\rho(\alpha)\rho(\beta) = I_k I_k = I_k$ using the definition of hyperproduct. Thus, $\rho(\alpha \circ \beta) = \rho(\alpha)\rho(\beta)$. □

Proposition 3.8. *Let $(H, \circ)$ be a cyclic hypergroup with infinite period having h as a generator and $\rho : H \to M_k(R_1)$ an inclusion (or good) representation of H satisfying $\rho(h) = I_k$. Then ρ is the trivial representation.*

Proof. Let α be a non trivial element in $H = h \cup h^2 \cup \ldots$. Then there exist $r \in R_1$ such that $\alpha \in h^r$. The latter implies that $\rho(\alpha) \in \rho(h^r) \subseteq (\rho(h))^r = I_k^r = I_k$. Thus, $\rho(\alpha) = I_k$.

If ρ is a good representation then the proof results from having ρ an inclusion representation. □

Proposition 3.9. *Let $\rho : H \to GL_k(R_1)$ be an inclusion diagonal representation of H. Then ρ is the trivial representation. Here, $GL_k(R_1)$ is the set of all matrices with entries from R_1 having non-zero determinant.*

Proof. Let $h \in H$ be a generator of $(H, \circ)$ then there exist $a_1, \ldots, a_k \in R_1 \setminus \{0\}$ such that

$$\rho(h) = \begin{pmatrix} a_1 & 0 & \ldots & 0 \\ 0 & a_2 & \ldots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & \ldots & 0 & a_k \end{pmatrix}.$$

Since $h \in h^2$ and ρ is an inclusion representation, it follows that $\rho(h) \in \rho(h^2) \subseteq (\rho(h))^2$. We get now that

$$\begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & \dots & 0 & a_k \end{pmatrix} \subseteq \begin{pmatrix} a_1^2 & 0 & \dots & 0 \\ 0 & a_2^2 & \dots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & \dots & 0 & a_k^2 \end{pmatrix}.$$

The latter implies that $a_i^2 = a_i$ for $i = 1, \dots, k$ which is equivalent to $a_i = 1$ or $a_i = 0$. Having $\rho : H \to GL_k(R_1)$ implies that $a_i = 1$ for $i = 1, \dots, k$. We get now that $\rho(h) = I_k$. Proposition 3.8 completes the proof. □

Proposition 3.10. *$\rho : H \to M_1(R_1)$ is a non-zero inclusion (or good) representation of H of degree 1 if and only if ρ is the trivial representation.*

Proof. If ρ is the trivial representation then Proposition 3.7 asserts that ρ is a good representation of H.

Let h be a generator of $(H, \circ)$ and $\rho : H \to M_1(R)$ is an inclusion representation of H. Then there exist $a \in R_1$ such that $\rho(h) = a$. Having $h \in h^2$ and ρ an inclusion representation imply that $\rho(h) \in \rho(h^2) \subseteq (\rho(h))^2 = a^2$. The latter implies that $a = a^2$. Since $a > 0$, it follows that $\rho(h) = 1$. Proposition 3.8 asserts that ρ is the trivial representation. □

Theorem 3.11. *Let $(H, \circ)$ be a cyclic hypergroup with infinite period (not necessary single power), $\rho : H \to M_k(R_1)$ with $k \geq 2$, $m \in R_1$ and $\alpha \in H$ with $exp(\alpha) = a$. If $\rho(\alpha) = (a_{ij})$ and*

$$a_{ij} = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{if } i < j \text{ or } j < i < k; \\ ma, & \text{if } 1 < i < k \text{ and } j = k \end{cases}$$

i.e.,

$$\rho(\alpha) = \begin{pmatrix} 1 & 0 & \dots & 0 & ma \\ 0 & 1 & \dots & 0 & ma \\ \vdots & 0 & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & ma \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

then ρ and ρ^T are reducible and unitary inclusion representations of H.

Proof. Let $\alpha, \beta \in H$ satisfying $exp(\alpha) = a$ and $exp(\beta) = b$. It is easy to see that if $a = 0$ or $b = 0$ ($\alpha = e$ or $\beta = e$ respectively) then

$$\rho(\alpha \circ \beta) = \begin{cases} \rho(\alpha), & \text{if } b = 0; \\ \rho(\beta), & \text{if } a = 0. \end{cases} = \begin{cases} \rho(\alpha)\rho(\beta), & \text{if } b = 0; \\ \rho(\alpha)\rho(\beta), & \text{if } a = 0. \end{cases}.$$

If $a, b > 0$ then $\rho(\alpha \circ \beta) = \{\rho(\lambda) : \lambda \in \alpha \circ \beta\}$. Easy computations show that $\rho(\alpha)\rho(\beta) = (c_{ij})$ where

$$(c_{ij}) = \begin{pmatrix} 1 & 0 & \dots & 0 & ma \oplus mb \\ 0 & 1 & \dots & 0 & ma \oplus mb \\ \vdots & 0 & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & ma \oplus mb \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

We have that $\rho(\alpha \circ \beta) = \{\rho(\lambda) : \lambda \in \alpha \circ \beta\}$ and $exp(\lambda) = i \leq exp(\alpha) + exp(\beta)$. We get now that

$$\rho(\lambda) = \begin{pmatrix} 1 & 0 & \dots & 0 & mi \\ 0 & 1 & \dots & 0 & mi \\ \vdots & 0 & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & mi \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

Having $mi \leq ma + mb$ implies that $\rho(\alpha \circ \beta) \subseteq \rho(\alpha)\rho(\beta)$.

Since $\rho(\alpha)(e_1) = e_1$ for all $\alpha \in H$, it follows that ρ is reducible as $< e_1 >$ is an invariant subspace of ρ where $e_1 = (1, \underbrace{0, \dots, 0}_{k-1 \text{ times}})^T$ and T is the transpose. Easy computations show that ρ is unitary relative to

$$M = (m_{ij}) = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \dots & 1 & 0 \\ 0 & \dots & \dots & 0 & 0 \end{pmatrix}$$

where

$$m_{ij} = \begin{cases} 1, & \text{if } i = j < k; \\ 0, & \text{otherwise.} \end{cases}$$

The proof that ρ^T is a reducible and unitary inclusion representation of H is done in a similar manner. □

Proposition 3.12. *The matrix found in the proof of Theorem 3.11 is unique if and only if $k = 2$.*

Proof. The matrix found in the proof of Theorem 3.11 is not unique for all $k > 2$ as ρ is unitary also relative to

$$N = \begin{pmatrix} 1 & 2 & 0 & \dots & \dots & 0 \\ 2 & 1 & 0 & \dots & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \dots & \dots & 1 & 0 \\ 0 & \dots & \dots & \dots & 0 & 0 \end{pmatrix}.$$

For the case $k = 2$, we use the same proof as that of Proposition 4.7 in [2]. $\square$

Theorem 3.13. *Let $(H, \circ)$ be a single power cyclic hypergroup with infinite period. The representation $\rho : H \to M_k(R_1)$ with $k \geq 2$ defined in Theorem 3.11 is a good representation if and only if ρ is the trivial representation or $m = 1$ and for all non trivial elements $\alpha, \beta \in H$ with $r \leq exp(\alpha) + exp(\beta)$ there exist $\lambda \in \alpha \circ \beta$ satisfying $exp(\lambda) = r$.*

Proof. Let ρ be a non trivial good representation and h a generator of H $(m \neq 0)$. We have that

$$\rho(h) = \begin{pmatrix} 1 & 0 & \dots & 0 & m \\ 0 & 1 & \dots & 0 & m \\ \vdots & 0 & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & m \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

and $\rho(h^2) = \{\rho(\lambda) : \lambda \in h^2\} \subseteq \{\rho(h), \rho(a_0), \rho(a_2)\}$ where a_0, a_2 are elements in H (if they exist) with $exp(a_0) = 0, exp(a_2) = 2$.

Since ρ is a good representation of H, it follows that

$$\rho(h^2) = \rho(h)^2 = \begin{pmatrix} 1 & 0 & \dots & 0 & m \oplus m \\ 0 & 1 & \dots & 0 & m \oplus m \\ \vdots & 0 & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & m \oplus m \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

We get now $|\rho(h)^2| = 2m + 1 = |\rho(h^2)| \leq 3$. The latter implies that $m = 1$.

For all non trivial elements $\alpha, \beta \in H$, we have that $\rho(\alpha \circ \beta) = \{\rho(\lambda) : \lambda \in \alpha \circ \beta\} = \rho(\alpha)\rho(\beta)$. The latter implies that

$$\left\{ \begin{pmatrix} 1 & 0 & \dots & 0 & exp(\lambda) \\ 0 & 1 & \dots & 0 & exp(\lambda) \\ \vdots & 0 & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & exp(\lambda) \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} : \lambda \in \alpha \circ \beta \right\}$$

$$= \begin{pmatrix} 1 & 0 & \dots & 0 & exp(\alpha) \oplus exp(\beta) \\ 0 & 1 & \dots & 0 & exp(\alpha) \oplus exp(\beta) \\ \vdots & 0 & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & exp(\alpha) \oplus exp(\beta) \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

Since $exp(\alpha) \oplus exp(\beta) = \{0, 1, \cdots, exp(\alpha) + exp(\beta)\}$, it follows that for all $r \leq exp(\alpha) + exp(\beta)$ there exist $\lambda \in \alpha \circ \beta$ satisfying $exp(\lambda) = r$.

Proposition 3.7 asserts that if ρ is the trivial representation then it is a good representation. Also, it is easy to see that ρ is a good representation when the condition: $m = 1$ and for all non trivial elements $\alpha, \beta \in H$ with $r \leq exp(\alpha) + exp(\beta)$ there exist $\lambda \in \alpha \circ \beta$ satisfying $exp(\lambda) = r$, is satisfied. □

Next, we give an example on a faithful representation of a single power cyclic hypergroup with infinite period.

Example 3.14. Let $n \in R_1$ and $\rho : (R_1, \oplus) \to M_k(R_1)$ be defined as follows:

$$\rho(n) = \begin{pmatrix} 1 & 0 & \dots & 0 & n \\ 0 & 1 & \dots & 0 & n \\ \vdots & 0 & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & n \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

Since $(R_1, \oplus)$ is generated by 1 and $n \in 1^i$ for all $i \geq n$, it follows that $exp(n) = n$. Theorems 3.11 and 3.13 assert that ρ is a good representation of $(R_1, \oplus)$.

To prove that ρ is faithful let $n, n' \in R_1$ satisfying $\rho(n) = \rho(n')$. It is easy to see that $n = n'$.

Theorem 3.15. *Let $(H, \circ)$ be a commutative single power cyclic hypergroup, $\rho : H \to M_2(R_1)$, for all non trivial elements $\alpha, \beta \in H$ and $r \leq exp(\alpha) + exp(\beta) \in R_1$ there exist $\lambda \in \alpha \circ \beta$ satisfying $\rho(\lambda) = r$. Then ρ is a good homomorphism if and only if ρ is the trivial representation, or ρ is given by*

$$\rho(\alpha) = \begin{pmatrix} 1 & exp(\alpha) \\ 0 & 1 \end{pmatrix}$$

or by its transpose.

Proof. Propositions (3.5 and 3.7) and Theorem 3.13 assert that ρ is a good homomorphism of H if either ρ is the trivial representation, given by

$$\rho(\alpha) = \begin{pmatrix} 1 & exp(\alpha) \\ 0 & 1 \end{pmatrix}$$

or by its transpose.. We need to prove the backward direction now.

Since for all non trivial elements $\alpha, \beta \in H$ and $r \leq exp(\alpha) + exp(\beta) \in R_1$ there exist $\lambda \in \alpha \circ \beta$ satisfying $\rho(\lambda) = r$, it follows that $(H, \circ)$ admits a trivial element e. Having ρ a good representation of H and $e \circ e = e$ imply that $\rho(e) = \rho(e \circ e) = \rho(e)\rho(e)$. The latter implies that $\rho(e) = I_2$.

Let

$$\rho(h) = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $a, b, c, d \in R_1$. The proof is the same as that of Theorem 4.12 in [2]. □

Next, we give an example on an inclusion representation that is not of the previous forms.

Proposition 3.16. *Let $(H,\circ)$ be a commutative single power cyclic hypergroup and $\rho : H \to M_2(R_1)$ defined as follows:*

$$\rho(\alpha) = \begin{pmatrix} 1 & exp(\alpha) \\ exp(\alpha) & 2 \end{pmatrix}.$$

Then ρ is an inclusion representation of H. Moreover, ρ is not a good representation of H.

Proof. Theorem 3.15 asserts that ρ is not a good representation of H.

To prove that ρ is an inclusion representation of H, let $\alpha, \beta \in H$ with $exp(\alpha), exp(\beta) > 0$ (If $exp(\alpha) = 0$ or $exp(\beta) = 0$ then $\rho(\alpha \circ \beta) = \rho(\alpha)\rho(\beta)$.). We have that

$$\rho(\alpha \circ \beta) = \{\rho(\lambda) : \lambda \in \alpha \circ \beta\} = \Big\{ \begin{pmatrix} 1 & exp(\lambda) \\ exp(\lambda) & 2 \end{pmatrix} : \lambda \in \alpha \circ \beta \Big\}.$$

Also, we have that

$$\rho(\alpha)\rho(\beta) = \begin{pmatrix} 1 \oplus exp(\alpha)exp(\beta) & 2exp(\alpha) \oplus exp(\beta) \\ exp(\alpha) \oplus 2exp(\beta) & exp(\alpha)exp(\beta) \oplus 4 \end{pmatrix}.$$

Since $exp(\lambda) \leq exp(\alpha) + exp(\beta)$, it follows that $\rho(\alpha \circ \beta) \subseteq \rho(\alpha)\rho(\beta)$. □

4. Representation of single power cyclic hypergroups with finite period

In this section, we construct a commutative semihyperring on P', the set of non negative integers less than or equal to r, introduce some hypermatrix representations of single power cyclic hypergroups with finite period r and present some of their interesting properties. Our work is a generalization of a part of a previous work done by the authors on a special single power cyclic hypergroup with finite period associated to the symmetric group.

Throughout this section, we define $M_k(R_2)$ as the set of all $k \times k$ hypermatrices with entries from $R_2 = (P', \oplus, \odot)$ and $(H, \circ)$ is a single power cyclic hypergroup with finite period r and generator h (unless it is mentioned differently). It is clear that $h \in h^2 \subset \ldots \subset h^{r-1} \subset h^r$.

Theorem 4.1. *Let $P' = \{0, 1, \ldots, r\}$ and $a, b \in P'$. Then $R_2 = (P', \oplus, \odot)$ is a commutative semihyperring with scalar unit and zero element where*

$$a \odot b = \begin{cases} ab, & \text{if } ab \leq r; \\ r, & \text{otherwise} \end{cases}$$

and

$$a \oplus b = \begin{cases} \{0, 1, \ldots, a+b\}, & \textit{if } a, b \neq 0 \quad \textit{and } a+b \leq r; \\ \{0, 1, \ldots, r\}, & \textit{if } a+b > r; \\ a, & \textit{if } b = 0; \\ b, & \textit{if } a = 0. \end{cases}$$

Proof. It is clear that $(P', \oplus)$ and $(P', \odot)$ are commutative. Also, $(P', \oplus)$ is associative as

$$a \oplus (b \oplus c) = (a \oplus b) \oplus c = \begin{cases} \{0, 1, \ldots, a+b+c\}, & \text{if } a, b, c \neq 0 \text{ and } a+b+c \leq r; \\ \{0, 1, \ldots, r\}, & \text{if } a+b+c > r; \\ b \oplus c, & \text{if } a = 0; \\ a \oplus c, & \text{if } b = 0; \\ a \oplus b, & \text{if } c = 0. \end{cases}$$

In a similar manner, we can show that $(P', \odot)$ is associative with 1 as a scalar unit as $1 \odot a = a$ and

$$a \odot (b \odot c) = (a \odot b) \odot c = \begin{cases} abc, & \text{if } abc \leq r; \\ r, & \text{if } abc > r. \end{cases}$$

Since $(P', \oplus)$ and $(P', \odot)$ are commutative, it suffices to show that $a \odot (b \oplus c) \subseteq (a \odot b) \oplus (a \odot c)$. We have that

$$a \odot (b \oplus c) = \begin{cases} 0, & \text{if } a = 0; \\ \{0, a, \ldots, a(b+c)\}, & \text{if } a, b, c \neq 0 \text{ and } a(b+c) \leq r; \\ \{0, a, \ldots, r\}, & \text{if } a, b, c \neq 0 \text{ and } a(b+c) > r; \\ a \odot (0 \oplus c) = a \odot c, & \text{if } b = 0; \\ a \odot (b \oplus 0) = a \odot b, & \text{if } c = 0; \end{cases}$$

and

$$(a \odot b) \oplus (a \odot c) = \begin{cases} 0, & \text{if } a = 0; \\ \{0, 1, \ldots, ab+ac\}, & \text{if } a, b, c \neq 0 \text{ and } ab+ac \leq r; \\ \{0, 1, \ldots, r\}, & \text{if } a, b, c \neq 0 \text{ and } ab+ac > r; \\ a \odot c, & \text{if } b = 0; \\ a \odot b, & \text{if } c = 0. \end{cases}$$

Thus, $a \odot (b \oplus c) \subseteq (a \odot b) \oplus (a \odot c)$. It is easy to see that 1 is the scalar unit and 0 is the zero element of R. □

Proposition 4.2. $(R_2, \oplus)$ *is single power cyclic hypergroup with finite period* r *and generator* 1.

Proof. We have that $(R_2, \oplus)$ is commutative and associative by Theorem 4.1. We prove now that the reproduction axiom is satisfied. If $m \in R_2$ then we have two cases: $m = 0$ and $m > 0$. If $m = 0$ then $m \oplus R_2 = \{m \oplus n : n \in R_2\} = \{n : n \in R_2\} = R_2$. If $m > 0$ then $R_2 = m \oplus r \subseteq m \oplus R_2$. Thus, $(R_2, \oplus)$ is a hypergroup.

For every $m \neq 0 \in R_2$ we have that $m \in 1^m = \{0, 1, \dots, m\}$ and $R_2 = 1^r$. It is easy to see that $1 \in 1^2 \subset \dots \subset 1^{r-1} \subset 1^r$. Therefore, $(R_2, \oplus)$ is single power cyclic hypergroup with finite period r and generator 1. □

Proposition 4.3. *Let $(H, \circ)$ be a commutative hypergroup (not necessary single power cyclic hypergroup) and $\rho : H \to M_k(R_2)$ be an inclusion (or good) representation of H. Then $\rho^T : H \to M_k(R_2)$ defined as $\rho^T(\alpha) = (\rho(\alpha))^T$ is a representation of H where T is the matrix transpose.*

Proof. The proof is similar to that of Proposition 3.5. □

Proposition 4.4. *Let $(H, \circ)$ be a commutative hypergroup (not necessary single power cyclic hypergroup) and $\rho : H \to M_k(R_2)$ is a unitary representation relative to a hypermatrix M. Then $\rho^T : H \to M_k(R_2)$ is a unitary representation relative to M.*

Proof. The proof is similar to that of Proposition 3.6. □

Proposition 4.5. *Let $(H, \circ)$ be a hypergroup (not necessary single power cyclic hypergroup) and $\rho : H \to M_k(R_2)$ be the trivial map defined by $\rho(\alpha) = I_k$ where I_k is the $k \times k$ identity matrix. Then ρ is a good representation of H.*

Proof. Let $\alpha, \beta \in H$ with $\rho(\alpha) = \rho(\beta) = I_k$. We have that $\rho(\alpha \circ \beta) = \{\rho(\lambda) : \lambda \in \alpha \circ \beta\} = I_k$ and $\rho(\alpha)\rho(\beta) = I_k I_k = I_k$ using the definition of hyperproduct. Thus, $\rho(\alpha \circ \beta) = \rho(\alpha)\rho(\beta)$. □

Proposition 4.6. *Let $(H, \circ)$ be a cyclic hypergroup with finite period r having h as a generator and $\rho : H \to M_k(R_2)$ an inclusion (or good) representation of H satisfying $\rho(h) = I_k$. Then ρ is the trivial representation.*

Proof. Let $\alpha \in H$. Then there exist $s \in R_2$ such that $\alpha \in h^s$. The latter implies that $\rho(\alpha) \in \rho(h^s) \subseteq (\rho(h))^s = I_k^s = I_k$. Thus, $\rho(\alpha) = I_k$.

If ρ is a good representation then the proof follows from having ρ an inclusion representation. □

Proposition 4.7. *$\rho : H \to M_1(R_2)$ is a non-zero inclusion (or good) representation of H of degree 1 if and only if one of the following is satisfied:*

1. *ρ is the trivial representation; or*

2. *$\rho(\alpha) = r$ for all $\alpha \in H$ and H does not have a trivial element.*

Proof. If ρ is the trivial representation then Proposition 4.5 asserts that ρ is a representation of H.

Let h be a generator of $(H, \circ)$ and $\rho : H \to M_1(R_2)$ a representation of H. Then there exist $a \in R_2$ such that $\rho(h) = a \leq r$. Having $h \in h^2$ and ρ an inclusion representation imply that

$$\rho(h) \in \rho(h^2) \subseteq (\rho(h))^2 = a \odot a = \begin{cases} a^2, & \text{if } a^2 \leq r; \\ r, & \text{otherwise.} \end{cases}$$

The latter implies that $a = a \odot a$. Since $a > 0$, it follows that $\rho(h) = 1$ or $\rho(h) = r$. If $\rho(h) = 1$ then Proposition 4.6 asserts that ρ is the trivial representation. If $\rho(h) = r$ then $\rho(h) \in \rho(h^n) \subseteq (\rho(h))^n = \underbrace{r \odot \ldots \odot r}_{n \text{ times}} = r$ for all $n \in \mathbb{N}$. It is easy to see that $\rho(\alpha) = r$ for all $\alpha \in H$.

Let $\alpha, \beta \in H$. we have that $\rho(\alpha \circ \beta) = \{\rho(\lambda) : \lambda \in \alpha \circ \beta\} = r = r \odot r = \rho(\alpha)\rho(\beta)$. Then ρ is a good representation of H. □

Theorem 4.8. *Let $(H, \circ)$ be a cyclic hypergroup with finite period r (not necessary single power), $\rho : H \to M_k(R_2)$ with $k \geq 2$, $m \in R_2$ and $\alpha \in H$ with $exp(\alpha) = a$. If $\rho(\alpha) = (a_{ij})$ and*

$$a_{ij} = \begin{cases} 1, & \textit{if } i = j; \\ 0, & \textit{if } i < j \textit{ or } j < i < k; \\ m \odot a, & \textit{if } 1 < i < k \textit{ and } j = k \end{cases}$$

i.e., $\rho(\alpha) = \begin{pmatrix} 1 & 0 & \ldots & 0 & m \odot a \\ 0 & 1 & \ldots & 0 & m \odot a \\ \vdots & 0 & \ddots & \vdots & \vdots \\ 0 & \ldots & 0 & 1 & m \odot a \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix}$. *Then ρ and ρ^T are reducible and unitary inclusion representations of H.*

Proof. Let ρ be a non trivial representation of H, i.e., $m \neq 0$ and let $\alpha, \beta \in H$ satisfying $exp(\alpha) = a \leq r$ and $exp(\beta) = b \leq r$. It is easy to see that if $a = 0$ or $b = 0$ then $\rho(\alpha \circ \beta) = \rho(\alpha)\rho(\beta)$. If $a, b > 0$ then $\rho(\alpha \circ \beta) = \{\rho(\lambda) : \lambda \in \alpha \circ \beta\}$. Easy computations show that $\rho(\alpha)\rho(\beta) = (c_{ij})$ where

$$(c_{ij}) = \begin{pmatrix} 1 & 0 & \ldots & 0 & m \odot a \oplus m \odot b \\ 0 & 1 & \ldots & 0 & m \odot a \oplus m \odot b \\ \vdots & 0 & \ddots & \vdots & \vdots \\ 0 & \ldots & 0 & 1 & m \odot a \oplus m \odot b \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix}.$$

We have that $\rho(\alpha \circ \beta) = \{\rho(\lambda) : \lambda \in \alpha \circ \beta\}$ and $exp(\lambda) = i \leq exp(\alpha) + exp(\beta)$. We get now that

$$\rho(\lambda) = \begin{pmatrix} 1 & 0 & \dots & 0 & m \odot i \\ 0 & 1 & \dots & 0 & m \odot i \\ \vdots & 0 & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & m \odot i \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

Since

$$m \odot i = \begin{cases} mi \leq ma + mb, & \text{if } mi \leq r; \\ r, & \text{otherwise} \end{cases}$$

and

$$m \odot a \oplus m \odot b = \begin{cases} \mathbb{Z}_{ma+mb+1}, & \text{if } ma + mb \leq r; \\ \mathbb{Z}_{r+1}, & \text{otherwise}; \end{cases}$$

it follows that $\rho(\alpha \circ \beta) \subseteq \rho(\alpha)\rho(\beta)$.

Having $\rho(\alpha)(e_1) = e_1$ for all $\alpha \in H$ implies that ρ is reducible as $< e_1 >$ is an invariant subspace of ρ where $e_1 = (1, \underbrace{0, \dots, 0}_{k-1 \text{ times}})^T$ and T is the transpose.

Easy computations show that ρ is unitary relative to

$$M = (m_{ij}) = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \dots & 1 & 0 \\ 0 & \dots & \dots & 0 & 0 \end{pmatrix}$$

where

$$m_{ij} = \begin{cases} 1, & \text{if } i = j < k; \\ 0, & \text{otherwise.} \end{cases}$$

The proof that ρ^T is a reducible and unitary inclusion representation of H is done in a similar manner. □

Theorem 4.9. *Let $(H, \circ)$ be a single power cyclic hypergroup with finite period r. The representation $\rho : H \to M_k(R_2)$ with $k \geq 2$ defined in Theorem 4.8 is a good representation if and only if one of the following us satisfied:*

1. *ρ is the trivial representation; or*

2. *$r = 2$ and H admits a trivial element; or*

3. *$m = 1$ and for all $\alpha, \beta \in H$ with $s \in exp(\alpha) \oplus exp(\beta)$ there exist $\lambda \in \alpha \circ \beta$ satisfying $exp(\lambda) = s$.*

Proof. Let ρ be a good non trivial representation of H and h a generator of H. Then there exist $m \neq 0$ such that

$$\rho(h) = \begin{pmatrix} 1 & 0 & \dots & 0 & m \\ 0 & 1 & \dots & 0 & m \\ \vdots & 0 & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & m \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

And $\rho(h^2) = \{\rho(\lambda) : \lambda \in h^2\} \subseteq \{\rho(h), \rho(a_0), \rho(a_2)\}$ where a_0, a_2 are elements in H (if they exist) with $exp(a_0) = 0, exp(a_2) = 2$.

Since ρ is a good representation of H, it follows that

$$\rho(h^2) = \rho(h)^2 = \begin{pmatrix} 1 & 0 & \dots & 0 & m \oplus m \\ 0 & 1 & \dots & 0 & m \oplus m \\ \vdots & 0 & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & m \oplus m \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

Having

$$m \oplus m = \begin{cases} \{0, 1, \dots, 2m\}, & \text{if } 2m < r; \\ \{0, 1, \dots, r\}, & \text{otherwise;} \end{cases}$$

implies that

$$\begin{aligned} |\rho(h)^2| &= \begin{cases} 2m + 1, & \text{if } 2m < r; \\ r + 1, & \text{otherwise} \end{cases} \\ &= |\rho(h^2)| \leq 3. \end{aligned}$$

The latter implies that $m = 1$ or $r = 2$.

We consider first the case $m = 1$. For all non trivial elements $\alpha, \beta \in H$, we have that $\rho(\alpha \circ \beta) = \{\rho(\lambda) : \lambda \in \alpha \circ \beta\} = \rho(\alpha)\rho(\beta)$. The latter implies that

$$\begin{aligned} &\left\{ \begin{pmatrix} 1 & 0 & \dots & 0 & exp(\lambda) \\ 0 & 1 & \dots & 0 & exp(\lambda) \\ \vdots & 0 & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & exp(\lambda) \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} : \lambda \in \alpha \circ \beta \right\} \\ &= \begin{pmatrix} 1 & 0 & \dots & 0 & exp(\alpha) \oplus exp(\beta) \\ 0 & 1 & \dots & 0 & exp(\alpha) \oplus exp(\beta) \\ \vdots & 0 & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & exp(\alpha) \oplus exp(\beta) \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}. \end{aligned}$$

Since

$$exp(\alpha) \oplus exp(\beta) = \begin{cases} \{0, 1, \ldots, exp(\alpha) + exp(\beta)\}, & \text{if } \exp(\alpha) + \exp(\beta) < r; \\ \{0, 1, \ldots, r\}, & \text{otherwise}; \end{cases}$$

it follows that for all $s \in exp(\alpha) \oplus exp(\beta)$ there exist $\lambda \in \alpha \circ \beta$ satisfying $exp(\lambda) = s$.

We consider now the case when $r = 2$. If $r = 2$ then $H = h \cup h^2$ and $h \in h^2$. $|(\rho(h))^2| = 3 = |\rho(h^2)| = |\{\rho(\lambda) : \lambda \in h^2\}|$. The latter implies existence of elements of $exp : 0, 1, 2$ in H.

It is easy to see that ρ is a good representation if $m = 1$ and for all $\alpha, \beta \in H$ with $s \in exp(\alpha) \oplus exp(\beta)$ there exist $\lambda \in \alpha \circ \beta$ satisfying $exp(\lambda) = s$ or if $r = 2$ and H admits a trivial element. □

Next, we give an example on a faithful representation of a single power cyclic hypergroup with finite period.

Example 4.10. Let $n \in R_2$ and $\rho : (R_2, \oplus) \to M_k(R_2)$ defined as follows:

$$\rho(n) = \begin{pmatrix} 1 & 0 & \ldots & 0 & n \\ 0 & 1 & \ldots & 0 & n \\ \vdots & 0 & \ddots & \vdots & \vdots \\ 0 & \ldots & 0 & 1 & n \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix}.$$

Since $(R_2, \oplus)$ is generated by 1 and $n \in 1^i$ for all $i \geq n$, it follows that $exp(n) = n$. Theorems 4.8 and 4.9 assert that ρ is a good representation of $(R_2, \oplus)$.

To prove that ρ is faithful let $n, n' \in R_2$ satisfying $\rho(n) = \rho(n')$. It is easy to see that $n = n'$.

5. Conclusion

After the introduction of hyperstructures by Marty, there have been a number of generalizations of this fundamental concept. One of these generalizations was the concept of hypermatrix representations. This paper studied the notion of hypermatrix representations of single power cyclic hypergroups and proved some of their interesting properties. Several results were obtained for both cases; single power cyclic hypergroups with infinite period as well as with finite period.

For a future work, it will be interesting to generalize our work to cyclic hypergroups by studying their non trivial hypermatrix representations.

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