A CHARACTERIZATION OF MATHIEU GROUPS BY THEIR ORDERS AND CHARACTER DEGREE GRAPHS

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Abstract. Let G be a finite group. The character degree graph $\Gamma(G)$ of G is the graph whose vertices are the prime divisors of character degrees of G and two vertices p and q are joined by an edge if pq divides some character degree of G. Let $L_n(q)$ be the projective special linear group of degree n over finite field of order q. Xu et al. proved that the Mathieu groups are characterized by the order and one irreducible character degree. Recently Khosravi et al. have proven that the simple groups $L_2(p^2)$, and $L_2(p)$ where $p \in \{7, 8, 11, 13, 17, 19\}$ are characterizable by the degree graphs and their orders. In this paper, we give a new characterization of Mathieu groups by using the character degree graphs and their orders.

Keywords: Character degree graph, Mathieu group, simple group, character degree.

1. Introduction

All groups in this note are finite. Let G be a finite group and let $\mathrm{Irr}(G)$ be the set of irreducible characters of G. Denote by $\mathrm{cd}(G) = \{\chi(1) : \chi \in \mathrm{Irr}(G)\}$, the set of character degrees of G. Some author have studied the Mathieu groups by considering the properties of element orders [1, 11]. Some authors studied the properties of groups by investigating the character degrees [13]. In this paper, we will study the groups by considering the character degree graph. Recall that the graph $\Gamma(G)$ is called *character degree graph* whose vertices are the prime divisors of character degrees of the group G and two vertices p and q are joined by an edge if pq divides some character degree of G [10]. Xu et al. in [13] have shown that Mathieu groups are determined by some character degree and their orders. Khosravi et. al. in [6, 15, 9] proved that the groups $L_2(p^2)$, where p is a

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prime, and $L_3(q)$ where $q \in \{4, 5, 7, 8, 9\}$, are characterizable by their character degree graphs and orders. Khosravi et. al. in [5] investigated the influence of the character degree graph and order of the simple groups of order less than 6000, on the structure of group. As the development of this topic, we give a new characterization of the Mathieu groups by their character degree graphs and orders. The following theorem is proved.

Main Theorem 1.1. *The following statements hold*

(1) *Let* $L \in \{M_{11}, M_{23}, M_{24}\}$. *If* G *is a finite group such that* $\Gamma(G) = \Gamma(L)$ *and* $|G| = |L|$, *then* $G \cong L$.

(2) *Let* $L := M_{12}$. *If* G *is a finite group such that* $\Gamma(G) = \Gamma(L)$ *and* $|G| = |L|$, *then* $G \cong L$ *or* $G \cong A_4 \times M_{11}$.

We introduce some notation here. Let S_n be the symmetric group of degree n. Let $L_n(q)$ be the projective special linear group of degree n over finite field of order q. Let G be a group and let r be a prime, then denote the set of Sylow r-subgroups G_r of G by $\mathrm{Syl}_r(G)$. If H is a characteristic subgroup of G, we write $H \operatorname{ch} G$. All other symbols are standard (see [2]).

2. Some preliminary results

In this section, we give some lemmas to prove the main theorem.

Lemma 2.1. *Let* $A \trianglelefteq G$ *be abelian. Then* $\chi(1)$ *divides* $|G : A|$ *for all* $\chi \in \mathrm{Irr}(G)$.

Proof. See Theorem 6.5 of [4]. □

Lemma 2.2. *Let* $N \trianglelefteq G$ *and let* $\chi \in \mathrm{Irr}(G)$. *Let* θ *be an irreducible constituent of* χ_N *and suppose that* $\theta_1, \cdots, \theta_t$ *are distinct conjugates of* θ *in* G. *Then* $\chi_N = e\sum_{i=1}^{t} \theta_i$, *where* $e = [\chi_N, \theta]$ *and* $t = |G : I_G(\theta)|$. *Also* $\theta(1) \mid \chi(1)$ *and* $\frac{\chi(1)}{\theta(1)} \mid \frac{|G|}{|N|}$.

Proof. Theorems 6.2, 6.8 and 11.29 of [4]. □

Lemma 2.3. *Let* G *be a non-solvable group. Then* G *has a normal series* $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, *such that* K/H *is a direct product of isomorphic non-abelian simple groups and* $|G/K| \,\big|\, |\mathrm{Out}(K/H)|$.

Proof. See Lemma 1 of [12]. □

Lemma 2.4. *Let* G *be a finite solvable group of order* $p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n}$, *where* $p_1, p_2, \cdots, p_n$ *are distinct primes. If* $kp_n + 1 \nmid p_i^{a_i}$ *for each* $i \le n-1$ *and* $k > 0$, *then the Sylow* p_n*-subgroup is normal in* G.

Proof. See Lemma 2 of [13]. □

We also need the structure of non-abelian simple groups whose largest prime divisor is 11 or 23.

Lemma 2.5. *If S is a finite non-abelian simple group such that $\{11\} \subseteq \pi(S) \subseteq \{2,3,5,11\}$, then S is isomorphic to one of the following simple groups listed as in Table 1.*

Proof. See [14]. $\square$

Table 1. Simple groups S with $\{11\} \subseteq \pi(S) \subseteq \{2,3,5,7,11\}$

S	Order of S	$Out(S)$	S	Order of S	$\|Out(S)\|$
$U_5(2)$	$2^{10} \cdot 3^5 \cdot 5 \cdot 11$	2	$U_6(2)$	$2^{15} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$	6
$L_2(11)$	$2^2 \cdot 3 \cdot 5 \cdot 11$	2	M_{11}	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	1
M_{12}	$2^6 \cdot 3^3 \cdot 5 \cdot 11$	2	M_{22}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	2
HS	$2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$	2	M^cL	$2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$	2
A_{11}	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$	2	A_{12}	$2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$	2

Lemma 2.6. *If S is a finite non-abelian simple group except for alternating group such that $\{23\} \subseteq \pi(S) \subseteq \{2,3,5,7,11,13,17,19,23\}$, then S is isomorphic to one of the following simple groups listed as in Table 2.*

Proof. See [14]. $\square$

Table 2. Simple group S with $\{23\} \subseteq \pi(S) \subseteq \{2,3,5,7,11,13,17,19,23\}$

S	Order of S	$\|Out(S)\|$
$L_2(23)$	$2^3 \cdot 3 \cdot 11 \cdot 23$	2
$U_3(23)$	$2^7 \cdot 3^2 \cdot 11 \cdot 13^2 \cdot 23^2$	4
M_{23}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	1
M_{24}	$2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	1
Co_3	$2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$	1
Co_2	$2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$	1
Co_1	$2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$	1
Fi_{23}	$2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23$	1

3. The proof of Main Theorem

In this section, we will prove the main theorem.

Proof of Main Theorem

Proof. We prove the results by the following cases.

Case 1. $L = M_{11}$.

Then $|L| = 2^4 \cdot 3^2 \cdot 5 \cdot 11$. It is easy to get from [2], that $\mathrm{cd}(L) = \{1, 10, 11, 16, 44, 45, 55\}$. So the graph $\Gamma(L)$ has the vertices $\{2,3,5,11\}$, the prime 5 is adjacent to the primes 2, 3, and 11, but the prime 3 is not adjacent to the primes 2 and 11. By in $\Gamma(G)$, there is a character χ such that $\chi(1)$ is divisible by 55.

We can conclude that $O_{11}(G) = 1 = O_5(G)$. In fact, if $O_{11}(G) \neq 1$, then since $|G_{11}| = 11$, then $O_{11}(G)$ is a Sylow 11-subgroup. Then by Lemma 2.1, there is a character $\chi \in \mathrm{Irr}(G)$ such that $\chi(1)\big||G : O_{11}(G)|$, a contradiction. Hence we have $O_{11}(G) = 1$. Similarly, $O_5(G) = 1$.

Assumed that G is a solvable group. Let M be a minimal normal subgroup of G. Then M is an elementary abelian p-group where $p = 2$ or $p = 3$. Since in $\Gamma(G)$ the prime 5 is adjacent to the primes 2 and 3, then we can assume that $|M|\big|2^3$ or $|M| = 3$.

Case 1.1. Let M be a 3-group. Let H/M be a Hall subgroup of order $2^4 \cdot 5 \cdot 11$. Then $|G/M : H/M| = 3$ and so $(G/M)/(L/M) \hookrightarrow S_3$, where $L/M = Core_{G/M}(H/M)$. Therefore $11\big||L/M|$. By Lemma 2.4, Q/M is normal in L/M, where Q/M is a Sylow 11-subgroup of L/M. Hence $Q \trianglelefteq G$ and $|Q| = 33$. Therefore $O_{11}(G) \neq 1$, a contradiction.

Case 1.2. Let M be a 2-group. Then $|M| = 2^k$ with $1 \leq k \leq 3$. Let H/M be a Hall subgroup of order $3^3 \cdot 5 \cdot 11$. Then $|G/M : H/M| = 2^{4-k}$.

Let $k = 3, 2$ or 1. Then $G/H_G \hookrightarrow S_2$, $G/H_G \hookrightarrow S_4$ or $G/H_G \hookrightarrow S_8$ respectively. Then in these three cases, $11\big||H_G|$. By Lemma 2.4, Q/M, the Sylow 11-subgroup of H_G/M, is also normal in H_G/M. It follows that $Q \trianglelefteq G$. Since $|Q| = 2^k{\cdot}11$, then $G_{11} \trianglelefteq H_G \, ch \, G$ and so G_{11} is normal in G, a contradiction.

Therefore G is non-solvable and so by Lemma 2.3, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that K/H is a direct product of isomorphic non-abelian simple groups and $|G/K|\big||\mathrm{Out}(K/H)|$.

We will prove that $11 \in \pi(K/H)$. Assume the contrary, then obviously by Lemma 6(d) of [7] and Lemma 2.13 of [8], $|\mathrm{Out}(K/H)|$ is not divisible by 11. If $11\big||H|$, then there is a Hall $\{p, 7\}$-subgroup D of H, where p is a prime and $p \in \{2, 3, 5\}$, then by considering group order and Lemma 2.4, D is cyclic and so D is abelian. By Lemma 2.1, $\chi(1)\big||G : D|$, a contradiction. Therefore $11\big||K/H|$.

In $\Gamma(G)$, the prime 3 is not adjacent to the primes 2 and 11 and so, by Lemma 2.5 and order consideration, K/H is isomorphic to one of the simple groups: $L_2(11)$ or L.

Let $K/M \cong L_2(11)$. By [2], $cd(L_2(11)) = \{1, 5, 10, 11, 12\}$ and so in $\Gamma(L_2(11))$, the prime 11 is adjacent to the prime 3. It follows that the prime 2 is adjacent to the prime 3 in $\Gamma(G)$, a contradiction.

Let $K/M \cong L$. Then $M = 1$ and $G \cong L$ by order consideration.

Case 2. $L = M_{12}$.

Then $|L| = 2^6 \cdot 3^3 \cdot 5 \cdot 11$. By [2], $\mathrm{cd}(L) = \{1, 11, 16, 45, 54, 55, 66, 99, 120, 144, 176\}$ and so the graph $\Gamma(G)$ is complete with vertex set $\{2, 3, 5, 11\}$.

Similarly as Case 1, we can prove that $O_{11}(G) = 1$.

Assumed that G is a solvable group. Let M be a minimal normal subgroup of G. Then M is an elementary abelian p-group where $p = 2$ or $p = 3$(in fact, if $p = 5$, then since $|G_5| = 5 = |M|$, there is a character $\chi \in \mathrm{Irr}(G)$ such that $\chi(1)\big||G : M|$, contradicting Lemma 2.1).

Let M be a 3-group. Then $|M| = 3^k$ with $1 \leq k \leq 2$ since $\Gamma(G)$ is complete. Let H/M be a Hall subgroup of order $2^6 \cdot 5 \cdot 11$. Then $|G/M : H/M| = 3^{3-k}$ and so $\frac{G}{H_G} \hookrightarrow S_9$ when $k = 1$ or $\frac{G}{H_G} \hookrightarrow S_3$ when $k = 2$. It follows that $11\big||H_G|$.

Let Q/M be a Sylow 11-subgroup of H_G/M. Since $|H_G/M|\big||H/M| = 2^6 \cdot 5 \cdot 11$, then Q/M is normal in H_G and so $Q \trianglelefteq G$. Since $|Q| = 3^k \cdot 11$, then $O_{11}(G)$ is normal in G, a contradiction.

Let M be a 2-group. Then $|M| = 2^k$ with $1 \leq k \leq 5$ since $\Gamma(G)$ is complete. Let H/M be a Hall subgroup of order $3^3 \cdot 5 \cdot 11$ of G/M. Then $|G/M : H/M| = 2^{6-k}$.

Let $3 \leq k \leq 5$. Then $G/H_G \cong S_8$ when $k = 3$, $G/H_G \cong S_4$ when $k = 4$, or $G/H_G \cong S_2$ when $k = 5$. In these three cases, $11\big||H_G|$. Let Q/M be a Sylow 11-subgroup of H_G/M. Since $|H_G/M|\big||H/M| = 3^3 \cdot 5 \cdot 11$, then Q/M is normal in H_G/M and so $Q \trianglelefteq G$. Since $|Q| = 2^k \cdot 11$, then $O_{11}(G)$ is normal in G, a contradiction.

Let $1 \leq k \leq 2$. Let Q/M be a Sylow 11-subgroup of H/M. Then by Lemma 2.4, Q/M is normal in H/M, in particularly, $Q \trianglelefteq H$. Since $|Q| = 2^k \cdot 11$, then G_{11} is normal in H and so $N/C := \frac{N_G(G_{11})}{C_G(G_{11})} \lesssim Z_{10}$. If $N/C \cong Z_{10}$ or $N/C \cong Z_5$, then $C_G(G_{11})$ is a $\{2, 3, 11\}$-group. It is easy to see that $G_{11}\, ch\, C$ and so $G_{11} \trianglelefteq G$. If $N/C \cong Z_2$, then N/C is a $\{2, 3, 5, 11\}$-group and so $G_{11}\, ch\, C$ and so $G_{11} \trianglelefteq G$. If $N = C$, then also we have $G_{11} \trianglelefteq G$. So in these cases, we rule out since $O_{11}(G) = 1$.

Therefore, G is non-solvable and so by Lemma 2.3, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that K/H is a direct product of isomorphic non-abelian simple groups and $|G/K|\big||Out(K/H)|$. Similarly as Case 1, we can show that $11\big||K/H|$. Therefore by Lemma 2.5, K/H is isomorphic to $L_2(11)$, M_{11} or L.

Let $K/H \cong L_2(11)$. Then $L_2(11) \leq G/H \leq Aut(L_2(11))$ and $|G/K| \mid |Out(L_2(11))| = 2$. If $G/H \cong L_2(11)$, then $|H| = 2^2 \cdot 3$. By [2], $cd(L_2(11)) = \{1, 5, 10, 11, 12\}$ and so in $\Gamma(L_2(11))$, the primes $2, 3$ are not adjacent to the prime 11. Since $5 \nmid |H|$, then this case can't occur. Similarly we can rule out the two cases $G/H \cong Z_2.L_2(11)$ and $G/H \cong SL_2(11)$.

Let $K/H \cong M_{11}$. Since $|Out(M_{11})| = 1$ and $cd(M_{11}) = \{1, 10, 11, 16, 44, 45, 55\}$, then $G/H \cong M_{11}$ and $|H| = 12$. On the other hand, in $\Gamma(M_{11})$, the prime 3 is not adjacent to the primes 2 and 11. Therefore $G = A_4 \times M_{11}$.

Let $K/H \cong L$. Then $H = 1$ and so order consideration implies that $G \cong L$.

Case 3. $L = M_{23}$.

Then $|L| = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$ and $|Out(L)| = 1$.

By [2], cd$(L) = \{1, 22, 45, 230, 231, 253, 770, 896, 990, 1035, 2024\}$ and so in $\Gamma(L)$, the prime 7 is not adjacent to the prime 23.

We will prove $O_{23}(G) = 1$. Assume the contrary, then $O_{23}(G)$ is normal in G. But $|G|_{23} = 23$ and so $O_{23}(G)$ is an abelian normal Sylow 23-subgroup of G. It follows that for all $\chi \in \mathrm{Irr}(G)$, $\chi(1)\big||G : O_{23}(G)|$, contradicting Lemma 2.1. Similarly, we can conclude that $O_{11}(G) = O_7(G) = O_5(G) = 1$.

Assumed that G is a solvable group. Let M be a minimal normal subgroup of G. Then M is an elementary abelian p-group where $p = 2$ or $p = 3$.

Let M be a 3-group. Then $|M| = 3$ since in $\Gamma(G)$ the prime 3 is adjacent to the prime 23. Let H/M be a Hall subgroup of order $2^7 \cdot 5 \cdot 7 \cdot 11 \cdot 23$. Then

$|G/M : H/M| = 3$ and so $\frac{G}{H_G} \hookrightarrow S_3$. It follows that $23 \big| |H_G|$. Let Q/M be a Sylow 23-subgroup of H_G/M. Since $|H_G/M| \big| |H/M| = 2^7 \cdot 5 \cdot 7 \cdot 11 \cdot 23$, then Q/M is normal in H_G/M and so $Q \trianglelefteq G$. Since $|Q| = 3 \cdot 23$, then $O_{23}(G)$ is normal in G, a contradiction.

Let M be a 2-group. Then $|M| = 2^k$ with $1 \leq k \leq 6$ since in $\Gamma(G)$, the prime 2 is adjacent to the prime 23. Let H/M be a Hall subgroup of order $3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$ of G/M. Then $|G/M : H/M| = 2^{7-k}$.

Let $3 \leq k \leq 6$. Then $G/H_G \cong S_{16}$ when $k = 3$, $G/H_G \cong S_8$ when $k = 4$, $G/H_G \cong S_4$ when $k = 5$, or $G/H_G \cong S_2$ when $k = 6$. In these four cases, $23 \big| |H_G|$. Let Q/M be a Sylow 23-subgroup of H_G/M. Since $|H_G/M| \big| |H/M| = 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$, then Q/M is normal in H_G/M and so $Q \trianglelefteq H_G$. Since $|Q| = 2^k \cdot 23$, then G_{23} is normal in G.

Let $1 \leq k \leq 2$. Let Q/M be a Sylow 23-subgroup of H/M. Then by Lemma 2.4, Q/M is normal in H/M, in particularly, $Q \trianglelefteq H$. Since $|Q| = 2^k \cdot 23$, then G_{23} is normal in H and so $N/C := \frac{N_G(G_{23})}{C_G(G_{23})} \lesssim Z_{22}$. If $N/C \cong Z_{11}$ or Z_{22}, then C is a $\{2, 3, 5, 7, 23\}$-group and by Lemma 2.4 G_{23} is normal in C. If $N/C \cong Z_2$, then C is a $\{2, 3, 5, 7, 11, 23\}$-group and also G_{23} is normal in C. In these cases, we have $O_{23}(G) \neq 1$, a contradiction.

Therefore, G is non-solvable and so by Lemma 2.3, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that K/H is a direct product of isomorphic non-abelian simple groups and $|G/K| \big| |Out(K/H)|$. Similarly as Case 1, we can show that $7, 11, 23 \big| |K/H|$. Therefore by Lemma 2.6, K/H is isomorphic to L.

It is easy to get that G is isomorphic to L by group order.

Case 4. $L = M_{24}$.

In this case, $|L| = 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$ and $Mult(L) = 1$.

Similarly as Case 1, we can prove that $O_{23}(G) = O_{11}(G) = O_7(G) = O_5(G) = 1$.

By [2], $cd(L) = \{1, 23, 45, 231, 252, 253, 483, 770, 990, 1035, 1265, 1771, 2024, 2277, 3312, 3520, 5313, 5544, 5796, 10395\}$. It means that the graph $\Gamma(G)$ is complete.

Assumed that G is a solvable group. Let M be a minimal normal subgroup of G. Then M is an elementary abelian p-group with $p = 2$ or 3.

Let M be a 3-group. Let H/M be a Hall subgroup of G/M of order $2^{10} \cdot 5 \cdot 7 \cdot 11 \cdot 23$. Then similarly as Case 1, we can get that $G/H_G \hookrightarrow S_9$ or $G/H_G \hookrightarrow S_3$ since $\Gamma(G)$ is complete. In both cases, $23 \mid |H_G|$. By Lemma 2.4, we know that $G_{23}M/M$ is normal in H_G/M(note that the Sylow 23-subgroup of H_G is also a Sylow 23-subgroup of G) and so $G_{23}M \trianglelefteq G$. We know that $G_{23} \trianglelefteq G_{23}M$. But $H_G \, ch \, G$. Hence $G_{23} \trianglelefteq G$, a contradiction.

Let M be a 2-group. Let H/M be a Hall subgroup of G/M with order $3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$. Let Q/M be a Sylow 23-subgroup of H/M. Then by Lemma 2.4, Q/M is normal in H/M and so $Q \trianglelefteq H$. We have $|Q| = 2^k \cdot 23$ and $G_{23} \, ch \, Q$. So G_{23} is normal in H. Similarly as Case 2, G_{23} is normal in G, a contradiction.

Therefore, G is non-solvable and so by Lemma 2.3, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that K/H is a direct product of isomorphic non-abelian simple groups and $|G/K|\big||Out(K/H)|$. Similarly as Case 1, we can show that $5, 7, 11, 23\big||K/H|$. Therefore by Lemma 2.6, K/H is isomorphic to M_{23} or L.

If $K/H \cong M_{23}$, then $G/H \cong M_{23}$ since $Mult(M_{23}) = 1$ and $\mathrm{Out}(M_{23}) = 1$. It follows that $|H|$ is a $\{2, 3\}$. Since in $\Gamma(M_{23})$, the prime 7 is not adjacent to the prime 23 and $\Gamma(G)$ is complete, then we can rule out this case.

If $K/H \cong L$, then $H = 1$ and so order consideration forces $G \cong L$.

This completes the proof of Main Theorem. □

4. Some applications

Huppert in [3] gave the following conjecture related to character degrees of finite simple groups.

Conjecture[3] Let H be any simple nonabelian group and G a group such that $cd(G) = cd(H)$. Then $G \cong H \times A$, where A is abelian.

Then we have the following theorem.

Corollary 4.1. *Let $L \in \{M_{11}, M_{12}, M_{23}, M_{24}\}$ and G a group such that $cd(G) = cd(L)$. Then $G \cong L \times A$, where A is abelian.*

We first show the following easy result.

Lemma 4.2. *Let G be a finite group. If $p^a\big|\chi(1)$ for some $\chi \in Irr(G)$ and $p^{a+1} \nmid \eta(1)$ for all $\eta \in Irr(G)$. Then $p^a\big||G|$. In particular, if $a = 1$ and G is simple group, then $|G|_p = p$.*

Proof. It is easy to get from lemma 2.1. If $a = 1$ and G is simple, then by Problem 3.4 of [4], have $|G|_p = p$. □

Proof of Corollary 4.1.

Proof. By Lemma 4.2, and Main Theorem 1.1, $G/H \cong L$. Order consideration implies the result.

This completes the proof. □

References

[1] G. Chen, *A new characterization of sporadic simple groups*, Algebra Colloq., 3 (1996), 49–58.

[2] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, R. A. Wilson, *Atlas of finite groups*, Oxford University Press, Eynsham, 1985, Maximal subgroups and ordinary characters for simple groups, With computational assistance from J. G. Thackray.

[3] B. Huppert, *Some simple groups which are determined by the set of their character degrees. I*, Illinois J. Math., 44 (2000), 828–842.

[4] I. M. Isaacs, *Character theory of finite groups*, Dover Publications, Inc., New York, 1994, Corrected reprint of the 1976 original [Academic Press, New York;

[5] B. Khosravi, B. Khosravi, B. Khosravi, Z. Momen, *Recognition by character degree graph and order of simple groups of order less than 6000*, Miskolc Math. Notes, 15 (2014), 537–544.

[6] B. Khosravi, B. Khosravi, B. Khosravi, Z. Momen, *Recognition of the simple group* $PSL(2, p^2)$ *by character degree graph and order*, Monatsh. Math., 178 (2015), 251–257.

[7] A. S. Kondrat'ev, V. D. Mazurov, *Recognition of alternating groups of prime degree from the orders of their elements*, Sibirsk. Mat. Zh., 41 (2000), 359–369, iii.

[8] S. Liu, *OD-characterization of some alternating groups*, Turkish J. Math., 39 (2015), 395–407.

[9] S. Liu, Y. Xie, *A characterization of* $L_3(4)$ *by its character degree graph and order*, SpringerPlus., 5 (2016), 242(6 pages).

[10] O. Manz, R. Staszewski, W. Willems, *On the number of components of a graph related to character degrees*, Proc. Amer. Math. Soc., 103 (1988), 31–37.

[11] C. Shao, Q. Jiang, Y.Y. Shao, *A new characterization of Mathieu groups*, Southeast Asian Bull. Math., 38 (2014), 283–288.

[12] H. Xu, G. Chen, Y. Yan, *A new characterization of simple* K_3*-groups by their orders and large degrees of their irreducible characters*, Comm. Algebra, 42 (2014), 5374–5380.

[13] H. Xu, Y. Yan, G. Chen, *A new characterization of Mathieu-groups by the order and one irreducible character degree*, J. Inequal. Appl., (2013), 2013:209, 6.

[14] A.V. Zavarnitsine, *Finite simple groups with narrow prime spectrum*, Sib. Èlektron. Mat. Izv., 6 (2009), 1–12.

[15] R. Zhang, S. Liu, *A characterization of linear groups* $L_3(q)$ *by their character degree graphs and orders*, Bol. Soc. Mat. Mex. (3), (2016), 1–9.

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