A SEPARATION METHOD FOR MAXIMAL COVERING LOCATION PROBLEMS WITH FUZZY PARAMETERS

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Abstract. Our paper discusses a novel computational approach to the extended Maximal Covering Location Problem (MCLP). We consider a fuzzy-type formulation of the generic MCLP and develop the necessary theoretical and numerical aspects of the proposed Separation Method (SM). A specific structure of the originally given MCLP makes it possible to reduce it to two auxiliary Knapsack-type problems. The equivalent separation we propose reduces essentially the complexity of the resulting computational algorithms. This algorithm also incorporates a conventional relaxation technique and the scalarizing method applied to an auxiliary multiobjective optimization problem. The proposed solution methodology is next applied to Supply Chain optimization in the presence of incomplete information. We study two illustrative examples and give a rigorous analysis of the obtained results.
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1. Introduction

Optimization of modern technological processes and the corresponding computer oriented methods are nowadays a usual and efficient approach to the practical de-

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velopment of several engineering applications (see e.g., [1,5-7,9,10,11,15,18,23]). In our contribution we study an extended MCLP model with an incomplete information and propose a relative simple approach to the effective numerical treatment of this problem. The obtained theoretic and computational results are next applied to the resilient Supply Chain Management System Optimization. The requested optimal design of an optimal management operation can be formalized as a specific MCLP [10]. In that case the information incompleteness mentioned above can be adequately described by an eligibility matrix with the fuzzy structure and the systems "resilience" is related to this incomplete modelling framework.

Let us recall that the conventional and extended MCLP formulations constitute a family of challenging optimization problems with numerous practical applications. It has a decisive role in the success of a Supply Chain management, with several applications including location of industrial plants, landfills, hubs, cross-docks, etc (see e.g., [1,3,8-10,12-15,18,20,22,24]). A well-known MCLP and the related decision making involve the delivery of a manufactured product to the end customer or/and to a warehouse. In a classical MCLP, one seeks the location of a number of facilities on a network in such a way that the covered "population" is maximized [14,24]. MCLP was first introduced by Church and ReVelle [14] on a network, and since then, several extensions to the original problem have been made. A variety of numerical approaches have been proposed to the practical treatment of distinct MCLPs. Recently several heuristical methods are actively used in the practical treatment of the MCLP based models. We refer to [8-10,12-15,18,20,22] for some effective heuristic and metaheuristic algorithms and for further references. Note that heuristics and metaheuristics have usually been employed in order to solve large size MCLPs (see e.g., [3,13,18,20]). A recent interest to MCLPs has arisen out the uncertainty of model parameters, such as demands or/and locations of demand nodes [9,10,24]. The solution procedure (Separation Method) we propose is generally based on an exact optimization procedure. However it also can incorporate some heuristic procedures for solving the obtained auxiliary problems.

This paper is devoted to a further theoretic and numerical development of a newly elaborated solution method for the MCLPs, namely, to the so called Separation Method (see [7]). The optimization approach we follow includes an equivalent transformation (separation) of the original MCLP and solution of two auxiliary Knapsack-type problems (see e.g., [16] and references therein). The proposed SM reduces the complexity of the original problem. Moreover, one can apply various methods to the resulting auxiliary problems. In this paper we use a usual relaxation scheme for the purpose of a concrete computation [12,16]. We also apply the standard scalarizing of an intermediate multiobjective optimization problem we obtain. And, it should be noted already at this point that the MCLP based optimization approach we propose can be effectively implemented (at the prototype stage) in a concrete optimal design of a decision or management system. Concretely, this SM involved approach is applied in our paper to the optimal design of a resilient Supply Chain scheme for a typical manufactures - customers delivery. Finally note that SM we propose in fact involves a suitable (equivalent) decomposition of an initially given MCLP. This fact, namely, the consideration of two resulting auxiliary problems makes it also possible to extend this method to some applied large-scale MCLP (see e.g., [3]).

The remainder of our paper is organized as follows: Section 2 contains an abstract problem formulation and some necessary theoretical concepts and facts. In Section 3 we develop a theoretic basis of the SM. This section also includes a necessary characterization of the obtained auxiliary problems. Section 4 discusses the appropriate numerical schemes in the context of the the initially given and auxiliary optimization problems. We use our main theoretic results and finally propose an implementable and well-determined algorithm for an effective numerical treatment of the originally given MCLP. This algorithm also incorporates the conventional relaxation technique. Section 5 contains two computational examples of an optimal resilient Supply Chain design. These practically oriented examples illustrate the implementability of the resulting computational algorithms and usability of the proposed solution procedure. Section 6 summarizes our contribution.

2. Problem formulation and preliminaries

We start by introducing the main optimization problem with a fuzzy structure. The MCLP we study has the following form:

$$\text{(1)} \qquad \begin{aligned} &\text{maximize} \quad J(z(y)) := \sum_{j=1}^{n} w_j z_j \\ &\text{subject to} \quad \begin{cases} \sum_{i=1}^{l} y_i = k \in \mathbb{N}, \ l > k, \\ z_j \leq \sum_{i=1}^{l} a_{ij} y_i, \\ z \in \mathbb{B}^n, \ y \in \mathbb{B}^l \end{cases} \end{aligned}$$

Here $w_j \in \mathbb{R}_+, \ j = 1, ..., n$ are given nonnegative objective "weights" and variables $z_j, \ j = 1, ..., n$ determine the "facilities to be served". By y_i, where $i = 1, ..., l$, we define the generic decision variables of the problem under consideration and $k \in \mathbb{N}$ in (1) describes the total amount of the facilities to be located. Elements a_{ij}, where

$$1 \geq a_{ij} \geq 0, \sum_{i=1,...,l} a_{ij} \geq 1,$$

are components of the so called "eligibility matrix"

$$A := (a_{ij})_{j=1,...,n}^{i=1,...,l}$$

associated with the eligible sites that provide a covering of the demand points indexed by $j = 1, ..., n$. The admissible values of the elements of the matrix A are "distributed" on the interval $[0,1]$. Note that the second index in (1), namely, $i = 1, ..., l$ is related to the given "facilities sites". Finally, the admissible sets $\mathbb{B}^n$ and $\mathbb{B}^l$ in the main problem (1) are defined as follows:

$$\mathbb{B}^n := \{0,1\}^n, \quad \mathbb{B}^l := \{0,1\}^l.$$

Note that the objective functional $J(\cdot)$ from (1) has a linear structure. We use the following vectorial notation

$$z := (z_1, ..., z_n)^T, \quad y := (y_1, ..., y_l)^T.$$

The implicit dependence

$$J(z(y)) = \langle w, z \rangle,$$
$$w := (w_1, ..., w_n)^T$$

of the objective functional J on the vector y is given by the corresponding (componentwise) inequalities constraints

$$z \leq A^T y$$

in (1). By $\langle \cdot, \cdot \rangle$ we denote here the scalar product in the corresponding Euclidean space. A vector pair (z, y) that satisfies all the constraints in (1) is next called an admissible pair for the main problem (1). Note that the objective functional does not depend explicitly on the problem variable y.

The abstract optimization framework (1) provides a constructive and modelling approach for various practically oriented problems (see e.g., [1,9,11,13,18]). Following [14] we next call the main optimization problem (1) a Maximal Covering Location Problem (MCLP). Let us also refer to [24] for a detailed discussion on the applied interpretation of the MCLP (1). The main problem (1) is formulated under the general (non-binary) assumption related to the elements a_{ij} of the eligibility matrix A. This corresponds to a suitable modelling approach under incomplete information (see e.g., [10] and references therein). Roughly speaking every value of an admissible parameter a_{ij} in (1) has a fuzzy nature (similar to [8]). This fuzzy MCLP under consideration provides an adequate formal framework for the resilient Supply Chain Optimization (see Section 5). Let us also observe that the "resilience" concept is understood here as a kind of robustness of the optimization approach we develop. This robustness is considered with respect to a possible incomplete information about the main mathematical model (robustness with respect to uncertainties in the modelling approach). Note that the possible incompleteness of the mathematical model mentioned above and the robustness requirement for a selected optimization approach constitute the common (and adequate) attributes for a realistic Supply Chain optimal design.

The mathematical characterization of (1) can evidently be given in terms of the classic integer programming (see e., g. [11,16,19] for mathematical details). Let us note that (1) possesses an optimal solution (an optimal pair)

$$(z^{opt}, y^{opt}) \in \mathbb{B}^n \otimes \mathbb{B}^l,$$

where

$$z^{opt} := (z_1^{opt}, ..., z_n^{opt})^T,$$
$$y^{opt} := (y_1^{opt}, ..., y_l^{opt})^T.$$

This fact is a direct consequence of the basic results from [11,16,19]. Let us also note that the conventional problem (1) can also be easily extended to the "multi-valued" version, where the admissible sets $\mathbb{B}^n$ and $\mathbb{B}^l$ are replaced by

$$\tilde{\mathbb{B}}^n := \{0, 1, ..., N_n\}^n,$$
$$\tilde{\mathbb{B}}^l := \{0, 1, ..., N_l\}^l,$$

where $N_n,\ N_l \in \mathbb{N}$.

Our aim is to develop a simple and effective numerical approach to the sophisticated MCLP (1). Facility location has a decisive role in success of Supply Chains with applications in many production and service facilities. It has been a focal center of interest in the last century among practitioners and scholars. For a detailed introduction to location models, one may refer to [15,23,24]. In general the literature of covering models is too diverse to be exhaustively studied in this paper. Although some of known publications in the literature of MCLP are included in this paper, one may refer to valuable reviews for further information.

3. Analytical foundations of the separation method

We next separate the originally given MCLP (1) and introduce two auxiliary optimization problems. These formal constructions provide a necessary basis for the future numerical development. The first optimization problem can be formulated as follows

$$\text{maximize} \quad \sum_{j=1}^{n} \mu_j \sum_{i=1}^{l} a_{ij} y_i$$
$$\text{subject to} \quad \begin{cases} \sum_{i=1}^{l} y_i = k, \quad y \in \mathbb{B}^l, \\ \mu_j \in [0,1] \quad \forall j = 1, ..., n \end{cases} \tag{2}$$

The second auxiliary problem has the following specific form:

$$\text{maximize} \quad J(z) := \sum_{j=1}^{n} w_j z_j$$
$$\text{subject to} \quad \begin{cases} z_j \leq \sum_{i=1}^{l} a_{ij} \hat{y}_i \\ z \in \mathbb{B}^n \end{cases} \tag{3}$$

where $\hat{y} \in \mathbb{B}^l$ is optimal solution of problem (2). The components of $\hat{y}$ are denoted as $\hat{y}_i,\ i = 1, ..., l$. The existence of an optimal solution for (2) is a direct consequence of the results from [11,19]. The same is also true with respect to the auxiliary problem (3). Let

$$\hat{z} \in \mathbb{B}^n,\ \hat{z} := (\hat{z}_1, ..., \hat{z}_n)^T$$

be an optimal solution to (3). Evidently, problem (3) coincides with the originally given MCLP (1) in a specific case of a fixed variable $y = \hat{y}$. Let us note that in general $\hat{y} \neq y^{opt}$.

The first auxiliary problem, namely, problem (2) can be interpreted as a usual linear scalarization of the following multiobjective optimization problem (vector optimization):

$$\text{(4)} \quad \begin{array}{ll} \text{maximize} & \{\sum_{i=1}^{l} a_{i1} y_i, ..., \sum_{i=1}^{l} a_{in} y_i\} \\ \text{subject to} & \begin{cases} \sum_{i=1}^{l} y_i = k, \\ y \in \mathbb{B}^l \end{cases} \end{array}$$

Recall that a scalarizing of a multi-objective optimization problem is an adequate numerical approach, which means formulating a single-objective optimization problem such that optimal solutions to the single-objective optimization problem are Pareto optimal solutions to the multi-objective optimization problem. We next assume that the multipliers

$$\mu_j,\ j = 1, ..., n$$

in (2) are chosen by such a way that problems (2) and (4) are equivalent (see e.g., [2,11,19] for necessary details). In this particular case we call (2) an adequate scalarizing of (4). We discuss shortly the adequate scalizing in Section 4.3.

It is easy to see that problems (2) and (3) have a structure of a so-called Knapsack problem (see [16] and references therein). Various efficient numerical algorithms are recently proposed for a generic Knapsack problem. We refer to [16] for a comprehensive overview about the modern implementable numerical approaches to this basic optimization problem.

The relevance and the main motivation of the auxiliary optimization problems (2) and (3) introduced can be stated by the following abstract result.

Theorem 3.1. *Assume* (z^{opt}, y^{opt}) *is an optimal solution of* (1) *and* (2) *is an adequate scalarizing of* (4). *Let* $\hat{y}$ *be an optimal solutions of* (2) *and* $\hat{z}$ *be an optimal solution of the auxiliary problem* (3). *Then* (1) *and* (3) *possess the same optimal values, that is*

$$J(z^{opt}(y^{opt})) = J(\hat{z}). \tag{5}$$

Moreover, in the case problems (1), (2), *and* (3) *possess unique solutions we additionally have*

$$(z^{opt}, y^{opt}) = (\hat{z}, \hat{y}).$$

Proof. Since

$$\sum_{i=1}^{l} \hat{y}_i = k,$$

and

$$\hat{z}_j \leq \sum_{i=1}^{l} a_{ij}\hat{y}_i,$$

we conclude that $(\hat{z}, \hat{y})$ is an admissible pair for the original MCLP (1). Taking into account the definition of an optimal pair for problem (1), we next deduce

$$J(\hat{z}(\hat{y})) \leq J(z^{opt}(y^{opt})). \tag{6}$$

Let

$$\Gamma = \Gamma_z \otimes \Gamma_y \subset \mathbb{B}^n \otimes \mathbb{B}^l$$

be a solutions set (the set of all optimal solutions) for problem (1). We also define the solutions sets

$$\Gamma_{(2)} \subset \mathbb{B}^l, \quad \Gamma_{(3)} \subset \mathbb{B}^n$$

of problems (2) and (3), respectively. From (6) it follows that

$$\Gamma_{(3)} \otimes \Gamma_{(2)} \subset \Gamma. \tag{7}$$

Taking into account the restrictions associated with the variable y in (1) and (2), we next obtain

$$\Gamma_y \equiv \Gamma_{(2)}. \tag{8}$$

Since (2) is an adequate scalarization of the multi-objective maximization problem (4), we deduce

$$z_j \leq \max_{\substack{\sum_{i=1}^{l} y_i = k, \\ y \in \mathbb{B}^l}} \sum_{i=1}^{l} a_{ij} y_i.$$

This fact implies

$$\Gamma_z \subset \Gamma_{(3)}. \tag{9}$$

Inclusions (7), (9) and the basic equivalence (8) now imply the following crucial equivalence

$$\Gamma_{(3)} \otimes \Gamma_{(2)} \equiv \Gamma. \tag{10}$$

Taking into account the same form of the objective functionals in (1) and (2.3), we immediately obtain the basic relation (5). In a specific case of the one point sets Γ, $\Gamma_{(3)}$ and $\Gamma_{(2)}$ the expected relation

$$(z^{opt}, y^{opt}) = (\hat{z}, \hat{y})$$

is a direct consequence of (10). The proof is completed. □

Theorem 3.1 makes it possible to separate (decompose equivalently) the original sophisticated problem (1) into two relative simple optimization problems. It provides a theoretical basis for effective numerical approaches to the abstract MCLPs and to possible applications.

4. Numerical analysis of the auxiliary problems

This section is dedicated to the numerical aspects related to the two optimization problems obtained in Section 3. Our aim is to develop a resulting self-closed algorithm for an effective numerical treatment of the original MCLP (1).

4.1 A combinatorial algorithm for the first auxiliary problem

We first observe that the auxiliary optimization problem (2) has a simple combinatorial structure. It can be easily solved using the following natural scheme:

$$\begin{aligned} &\hat{y}_i = 1 \text{ if } i \in \hat{I}; \\ &\hat{y}_i = 0 \text{ if } i \in \{1, ..., l\} \setminus \hat{I} \end{aligned} \tag{11}$$

where

$$\begin{aligned} &\hat{I} := \{1 \leq i \leq l \mid S_{\mathcal{A}_i} \in \max_k \{S_{\mathcal{A}_1}, ..., S_{\mathcal{A}_l}\}\}, \\ &S_{\mathcal{A}_i} := \sum_{j=1}^{n} \mu_j a_{ij}, \\ &\mathcal{A}_i := (a_{i1}, ..., a_{in})^T. \end{aligned} \tag{12}$$

Here $\mathcal{A}_i$ is a vector of i-row of the eligibility matrix A and operator $\max_k$ determines an array of k-largest numbers from the given array. Evidently, the choice (11)-(12) determines an optimal solution of (2). Roughly speaking the combinatorial algorithm (11)-(12) assigns the maximal value $\hat{y}_i = 1$ for all vectors $\mathcal{A}_i$ which sum of components $S_{\mathcal{A}_i}$ belongs to the array of k-largest sums of components of all vectors

$$\mathcal{A}_i, \ i = 1, ..., l.$$

It is easy to see that for the given eligibility matrix A with the specific elements a_{ij} (determined in Section 2) the sum of components $S_{\mathcal{A}_i}$ constitutes a specific norm of the given vector $\mathcal{A}_i$. The total complexity of the combinatorial algorithm (11)-(12) can be easily calculated and is equal to

$$O(l \times \log k) + O(k).$$

We refer to [16] for the necessary details.

Let us denote

$$c := \sum_{j=1}^{n} \sum_{i=1}^{l} a_{ij} \hat{y}_i.$$

Then the inequality constraints in (3) imply the generic Knapsack-type constraint with uniform weights

$$\sum_{j=1}^{n} z_j \leq c.$$

We now present a fundamental solvability result for the second auxiliary optimization problem, namely, the Knapsack problem (3).

Theorem 4.1. *The* Knapsack *problem* (3) *can be solved in* $O(nc)$ *time and*

$$O(n+c)$$

space.

The formal proof of Theorem 4.1 can be found in [16].

4.2 A relaxation based approach and the resulting computational scheme

The theoretic and numerical results obtained above, namely, Theorem 1 and the combinatorial choice algorithm (11)-(12) provide a theoretic basis for a novel exact solution method for the originally given MCLP (1). We now need to establish an implementable solution procedure for the effective numerical treatment of the second auxiliary problem (3) from the obtained decomposition (2)-(2.3). This optimization problem, which is NP-hard, has been comprehensively studied in the last few decades and several exact algorithms for its solution can be found in the literature (see [16] and the references therein). Constructive algorithms for this Knapsack problems are mainly based on two basic numerical approaches: branch-and-bound and dynamic programming. Let us also mention here the corresponding combined approach.

In this paper we firstly consider the well-known Lagrange relaxation scheme in the context of the second auxiliary problem (problem (3)). "Relaxing a problem" has various meanings in applied mathematics, depending on the areas where it is defined, depending also on what one relaxes (a functional, the underlying space, etc.). We refer to [2,4-7,12, 21] for various relaxation techniques in the modern optimization. Introducing the Lagrange function

$$\mathcal{L}(z,\lambda) := \sum_{j=1}^{n} w_j z_j - \sum_{j=1}^{n} \lambda_j \big(z_j - \sum_{i=1}^{l} a_{ij}\hat{y}_i\big)$$

associated with the Knapsack problem (3), we obtain the following relaxed problem

$$\text{(13)} \qquad \begin{aligned} &\text{maximize} \quad \mathcal{L}(z,\lambda) \\ &\text{subject to} \quad z \in \mathbb{B}^n \end{aligned}$$

The relaxed problem (13) does not contain the originally given unpleasant inequality constraints. These constraints are now included into the objective function $\mathcal{L}(z,\lambda)$ from (13) as a penalty term

$$\sum_{j=1}^{n} \lambda_j \big(z_j - \sum_{i=1}^{l} a_{ij}\hat{y}_i\big).$$

Recall that all feasible solutions to (3) are also feasible solutions to (13). The objective value of feasible solutions to (3) is not larger than the objective value in (13) (see [16] for the necessary proofs). Thus, the optimal solution value to the relaxed problem (13) is an upper bound to the original problem (3) for any vector of nonnegative Lagrange multipliers

$$\lambda := (\lambda_1, ..., \lambda_n)^T,$$
$$\lambda_j \geq 0,\ j = 1, ..., n.$$

For a concrete numerical solution of the relaxed problem (13) we use here the classic branch-and-bound method (see e.g., [11,16]). In a branch-and-bound algorithm we are interested in achieving the tightest upper bound in (13). Hence, we would like to choose a vector of nonnegative multipliers

$$\hat{\lambda}^{\mathcal{L}} := (\hat{\lambda}_1^{\mathcal{L}}, ..., \hat{\lambda}_n^{\mathcal{L}})^T,$$
$$\hat{\lambda}_j^{\mathcal{L}} \geq 0,\ j = 1, ..., n$$

such that (13) is minimized. This evidently leads to the generic Lagrangian dual problem

$$\text{(14)} \qquad \begin{array}{l} \text{minimize } \ \mathcal{L}(z,\lambda) \\ \text{subject to } \ \lambda \geq 0 \end{array}$$

It is well-known that the Lagrangian dual problem (14) yields the least upper bound available from all possible Lagrangian relaxations. The problem of finding an optimal vector of multipliers $\hat{\lambda}^{\mathcal{L}} \geq 0$ in (14) is in fact a linear programming problem [11,19]. In a typical branch-and-bound algorithm one will often be satisfied with a sub-optimal choice of multipliers $\lambda \geq 0$ if only the bound can be derived quickly. In this case sub-gradient optimization techniques can be applied [19]. The following analytic result is an immediate consequence of our main Theorem 1 and of the basic properties of the primal-dual system (13)-(14).

Theorem 4.2. *Let* $(\hat{z}^{\mathcal{L}}, \hat{\lambda}^{\mathcal{L}})$ *be an optimal solution of the primal-dual system* (13)-(14) *associated with the auxiliary problem* (3)*. Assume that all conditions of* Theorem 1 *be satisfied. Then*

$$\text{(15)} \qquad J(z^{opt}(y^{opt})) \leq J(\hat{z}^{\mathcal{L}}).$$

The obtained estimation (15) *constitutes a tightest upper bound for the optimal value* $J(z^{opt}(y^{opt}))$*.*

We are now ready to formulate a complete (conceptual) algorithm for an effective numerical treatment of the basic MCLP (1).

Algorithm 1.

I. *Given an initial* MCLP (1) *separate it into two auxiliary problems* (2) *and* (3);

II. Apply the combinatorial algorithm (11)-(12) and compute $\hat{y}$;

III. Using $\hat{y}$, construct the Lagrange function $\mathcal{L}(z, \lambda)$ and solve the primal-dual system (13)-(14).

The numerical consistency of the proposed Algorithm 1 is an immediate consequence of the obtained main theoretic results, namely, of Theorem 3.1 and Theorem 4.2. Recall that the Lagrange relaxation scheme is usually applied to the original MCLP (1) (see e.g., [12,16]). In that case the resulting (relaxed) problem and the corresponding Lagrangian dual problem possess a higher complexity in comparison with the proposed "partial" Lagrange relaxation (13)-(14) associated with the original MCLP (1). This is a simple consequence of the proposed SM that reduces the initial problem (1) to two (more simple) auxiliary optimization problem (2)-(3). This fact makes it possible to apply the proposed separation methodology to the large-scale MCLPs that are important and realistic mathematical models for many practically oriented (optimal) decision making systems (see e.g., [7,9,10,14,15,18,20,22,23,24]).

4.3 A remark on the adequate scalarizing procedure

Let us now make a short remark related to the scalarizing procedure used above (see Section 3, problems (2)-(4)).

It can be shown analytically that the values $S_{\mathcal{A}_i}$ in (12) depend on the multipliers vector μ. This is a consequence of the inclusion (9). Recall that (9) constitutes a useful relation of the SM and for the resulting optimization strategy we propose. Since the obtained multiobjective maximation problem (5) has a linear structure, an adequate scalarizing makes it possible to determine every "non-dominant" points (see [11,19] for mathematical details).

On the other hand, a possible "non-adequate" selection of μ geometrically implies a significant "cutting" (restriction) of the feasible region for problem (3). This feasible region restriction can finally eliminate a true optimal solution. Recall that a scalarizing implemented in the objective function from (2) evidently determines the resulting geometry associated with the basic problem (3). On the other side the geometrical properties of a non-adequately scalarized problem can violate the conceptual condition (9).

5. Optimization of the resilient supply chain management system

This section is devoted to applications of the proposed SM to an optimal resilient Supply Chain Management for a system of manufacturing plants - warehouses. Note that the "resilience" of a Supply Chain Management System is modelled here by a fuzzy-type eligibility matrix A (see Section 2). We use here the notation from Section 4 and denote by $\mathcal{A}_i$ a vector of i-row of the eligibility matrix A $(i = 1, ..., l)$ such that

$$A = (\mathcal{A}_1^T ... \mathcal{A}_l^T)^T.$$

Let us firstly point the common applied meaning of the variables and parameters from the general MCLP (1) in the context of the resilient Supply Chain Management system. The binary variables

$$(z, y) \in \mathbb{B}^n \otimes \mathbb{B}^l$$

constitute the main "decision variables" of the problem under consideration. The vector of weights w can be interpreted as a rentability of the final product. Therefore, the maximization of the cost functional $J(\cdot)$ in (1) expresses the maximization of the total profit (total income) generated by the designed Supply Chain system. The complete "decision resource" associated with the decision variable (vector) y is restricted in (1) by a constant (parameter) $k \in \mathbb{N}$. The eligibility matrix "A" is in fact a useful linear modelling framework that establishes the natural relation between the "producer" decision and "recipient". This relation is formally given by the corresponding elements a_{ij} of the matrix A. Our aim now is to apply the developed SM to two practically oriented examples of the optimal Supply Chain Management design in a classic manufactures - warehouses system.

Example 5.1. The simple Supply Chain system that include $n = 8$ manufacturing plants and $l = 5$ warehouses is indicated on Fig. 1.

We also assume that

$$a_{ij} + a_{i'j} \geq 1, \quad i = 1, ...5 \; j = 1, ...8.$$

Here i' is an index that corresponds to a resilient cover of a demand point. The last condition means that at least two feasible facilities (warehouse) cover a given demand point (the manufacturing plants). The corresponding eligibility matrix A is given as follows:

$$A^T = \begin{pmatrix} 0.81286 & 0.0 & 0.0 & 0.62968 & 0.0 \\ 0.25123 & 0.58108 & 0.32049 & 0.89444 & 0.79300 \\ 0.0 & 0.0 & 0.64850 & 0.91921 & 0.94740 \\ 0.54893 & 0.90309 & 0.74559 & 0.50869 & 0.99279 \\ 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.77105 & 0.27081 & 0.65883 & 0.60434 & 0.23595 \\ 0.0 & 0.51569 & 0.0 & 0.0 & 0.57810 \\ 0.64741 & 0.91733 & 0.60562 & 0.63874 & 0.71511 \end{pmatrix}$$

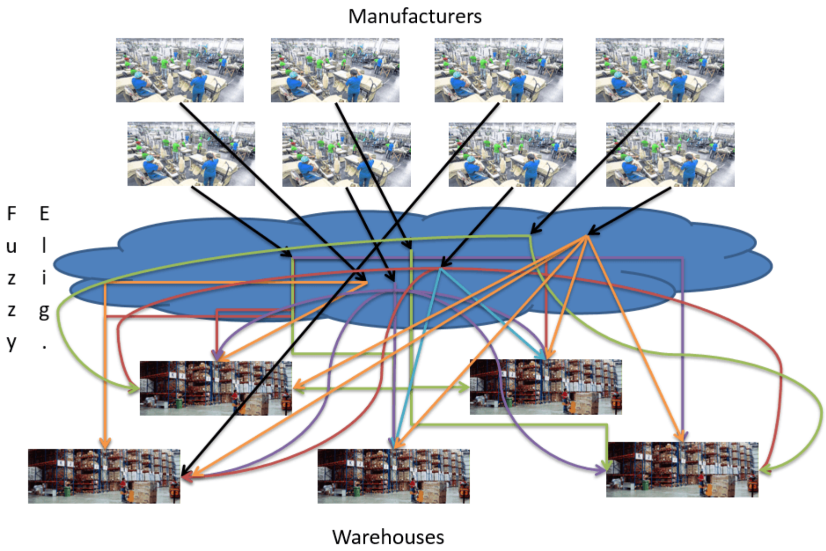

Figure 1: Fuzzy eligibility model

The objective weights

$$w_j \in \mathbb{R}_+, \quad j = 1, ..., 8$$

indicate the service priority and are selected in this example as follows

$$w = (32.0,\ 19.0,\ 41.0,\ 26.0\ 37.0\ 49.0\ 50.0\ 11.0)^T.$$

Note that the fifth demand point in this example has no "resilient" character (only one facility covers this point). We assume that the Supply Chain decision maker is interested opens $k = 2$ facilities. That means

$$\sum_{i=1}^{5} y_i = 2.$$

Moreover, we also define the necessary row vectors (see Section 3) for the combinatorial algorithm (11)-(12):

$$\begin{array}{ll} S_{\mathcal{A}_1} = 8.06295 & S_{\mathcal{A}_2} = 5.86033 \\ S_{\mathcal{A}_3} = 5.30955 & S_{\mathcal{A}_4} = 7.47098 \\ S_{\mathcal{A}_5} = 6.99921 & \end{array}$$

Application of the basic Algorithm 1 leads to the following computational results:

$$\begin{aligned} z^{opt} &= (1,\ 1,\ 0,\ 1,\ 1,\ 1,\ 0,\ 1)^T, \\ y^{opt} &= (1,\ 0,\ 0,\ 1,\ 0)^T, \end{aligned} \tag{16}$$

The corresponding (maximal) value of the objective functional is equal to

$$J(z^{opt}(y^{opt})) = \max_{Problem(1)} J(z(y)) = 174.0$$

Let us also note that the computed scalarizing multiplier μ in the auxiliary problem (2) for the given problem data is equal to

$$\mu = (2.0,\ 2.0,\ 1.0,\ 2.0,\ 2.0,\ 2.0,\ 1.0,\ 2.0)^T.$$

The practical implementation of Algorithm 1 was carried out by using the standard Python package and an author-written program.

For comparison, the given MCLP problem was also solved by a direct application of the standard CPLEX optimization package. We use the concrete problem parameters given above and obtain the same optimal pair as in (16). The CPLEX integer programming solver proceeds with 6 MIP simplex iterations and 0 branch-and-bound nodes for in total 13 binary variables and 9 linear constraints.

Example 5.2. We now consider a formal extension of the previous example (for a double dimension) and put $n = 16,\ l = 10,\ k = 5$. Let

$$w = (29.0,\ 37.0,\ 22.0,\ 42.0,\ 26.0,\ 14.0,\ 27.0,\ 30.0,$$
$$46.0,\ 16.0,\ 10.0,\ 36.0,\ 33.0,\ 39.0,\ 46.0,\ 49.0)^T.$$

The eligibility matrix A is given by rows:

$$\mathcal{A}_1 = (0.846109459436,\ 0.0,\ 0.0,\ 0.582693667799, 0.964574511054,\ 0.798899459366,\ 0.0,\ 0.0,\ 1.0,$$
$$0.300320432977,\ 0.997688107849,\ 0.3335795069, 0.49602683501, 1.0,\ 0.0,\ 0.374671961499),^T$$

$$\mathcal{A}_2 = (0.0,\ 1.0,\ 0.0,\ 0.0,\ 0.741552391071, 0.537788748272, 0.883796533814,\ 0.585368404373,\ 0.0,$$
$$0.860903890172, 0.958028639759, 0.0, 0.186896812387, 0.0, 0.968601622008,\ 0.579580096602)^T,$$

$$\mathcal{A}_3 = (0.407084305512,\ 0.0,\ 0.565187029512, 0.0, 0.420858280659, 0.361836079442,\ 0.472471488805,$$
$$0.0, 0.0,\ 0.696525107652, 0.436819747759,\ 0.0, 0.587300759229,\ 0.0,\ 0.347864951313,\ 0.0)^T,$$

$$\mathcal{A}_4 = (0.208102698902,\ 0.0,\ 0.0,\ 0.0,\ 0.0, 0.346461956794, 0.0,\ 0.0,\ 0.0,\ 0.768124612788,$$
$$0.413970925056, 0.0, 0.97348389961,\ 0.0,\ 0.0,\ 0.0)^T,$$

$$\mathcal{A}_5 = (0.0,\ 0.0,\ 0.965589029405,\ 0.0,\ 0.0, 0.893792904298,\ 0.0,\ 0.723969499937,\ 0.0,$$
$$0.562381237935, 0.78216104002,\ 0.557958082269, 0.671624833192, 0.0,\ 0.601221801206,\ 0.0)^T,$$

$$\mathcal{A}_6 = (0.0,\ 0.0,\ 0.0,\ 0.7732353822,\ 0.0, 0.930557571029, 0.0,\ 0.427721730484,\ 0.0,$$
$$0.818424694417, 0.795450242494, 0.314453291276, 0.645666417485,\ 0.0, 0.0,\ 0.0)^T,$$

$\mathcal{A}_7 = (0.0,\ 0.0,\ 0.0,\ 0.71613857057, 0.0, 0.573866657173, 0.0,\ 0.692538237821,\ 0.0,$
$0.296797567788,\ 0.306871729419, 0.334127066948, 0.0, 0.0,\ 0.0,\ 0.976783604764)^T,$

$\mathcal{A}_8 = (0.448086601628,\ 0.0,\ 0.888380378484, 0.576276602931, 0.939065250623,\ 0.0,\ 0.0,$
$0.773234003255,\ 0.0, 0.414398315721, 0.203669220313,\ 0.35600682894, 0.523619957827,$
$0.0, 0.0,\ 0.527029464076)^T,$

$\mathcal{A}_9 = (0.964964029806,\ 0.0,\ 0.0,\ 0.562565185744, 0.0, 0.0, 0.0,\ 0.0,\ 0.0,\ 0.773260049125,$
$0.468988424786, 0.0, 0.0,\ 0.0,\ 0.0,\ 0.794463270734)^T,$

$\mathcal{A}_{10} = (0.0,\ 0.0,\ 0.0,\ 0.545222010668,\ 0.0, 0.0,\ 0.536645142919, 0.212898303253,\ 0.0,$
$0.197891148706,\ 0.471120100438, 0.0,\ 0.0, 0.0,\ 0.0,\ 0.0)^T.$

The basic Algorithm 1 was applied to this example. We obtain the following optimal solution:

$$z^{opt} = (1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1, 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1)^T,$$
$$y^{opt} = (1,\ 1,\ 1,\ 0,\ 1, 0,\ 0,\ 1,\ 0,\ 0)^T.$$

The obtained scalarizing multiplier μ in the auxiliary problem (2) for the given problem data is equal to

$$\mu = (2.0,\ 2.0,\ 4.0,\ 4.0,\ 2.0,\ 0.0,\ 8.0,\ 2.0,\ 1.0,\ 2.0,\ 6.0,\ 1.0,\ 0.0,$$
$$1.0,\ 1.0,\ 1.0)^T.$$

Finally, the calculated optimal value of the objective functional is equal to

$$J(z^{opt}(y^{opt})) = \max_{Problem(1)} J(z(y)) = 502.$$

Let us note that the successful application of the proposed computationalalgorithm to the above high-dimensional problem indicates a possible usability of this approach in the effective solution procedures of large-scale MCLPs.

Finally let us note that the CPLEX based comparatively analysis and the computational results obtained in Example 5.1 and Example 5.2 illustrate the realisability and effectiveness of the Separation Method developed in our paper.

6. Conclusion

In this contribution, we proposed a conceptually new numerical approach to a wide class of Maximal Covering Location Problems with the fuzzy-type eligibility matrices. This computational algorithm is next applied to the optimal design of a practically motivated Resilient Supply Chain Management System. The developed computational scheme is based on a novel separation approach to the initially given maximization problem. The SM we propose makes it possible to reduce the original sophisticated problem to two Knapsack-type optimization problems. The first one constitutes a generic linear scalarization of a multi-objective optimization problem and the second auxiliary problem is a version of the classic Knapsack formulation. Application of the conventional Lagrange relaxation in combination with a specific combinatorial algorithm leads to an implementable algorithm for the given fuzzy-type Maximal Covering Location Problem.

Theoretical and computational methodologies we present in this contribution can be applied to various generalizations of the basic MCLP. One can combine the elaborated separation scheme with the conventional branch-and-bound method, with the celebrated dynamic programming approach or/and with an alternative exact or heuristic numerical algorithm. Let us finally note that we discussed here only main theoretic aspects of the newly elaborated approach and presented the corresponding conceptual solution procedure. The basic methodology we developed needs further comprehensively numerical examinations that includes solutions of several MCLPs.

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