n-FOLD (POSITIVE) IMPLICATIVE FILTERS OF HOOPS

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Abstract. The aim of this paper is to develop the filter theory of hoops. First, the concept of n-fold (positive) implicative (bounded) hoop and n-fold (positive) implicative filter are introduced. Also, the relationship between these filters and other filters on hoops are discussed. Moreover, conditions for a filter becomes a n-fold (positive) implicative filter are given. Finally, we consider some relations between these filters and quotient algebras.

Keywords: Hoop, filter, n-fold (positive) implicative filter, n-fold (positive) implicative (bounded) hoop.

1. Introduction

Hoops are naturally ordered commutative residuated integral monoids, which introduced by B.Bosbach in [6,7], and studied by J.R. Buchi and T.M. Owens in [5]. In the last few years, hoops theory was enriched with deep structure theorems [1,2,6,7]. Many of these results have a strong impact with fuzzy logic. Particularly, from the structure theorem of finite basic hoops [1] one obtains an elegant short proof of the completeness theorem for propositional basic logic, introduced by Hájek in [11]. The filter theory of the logical algebras plays an important role in studying algebras and the completeness of the corresponding non-classical logics. From a logical point of view, various filters correspond to various sets of provable formulas. Also, a filter is called a "deductive stystems" [18]. Jun and Ko gave some characterizations of a deductive system and discussed how to generate a deductive system by a set on BL-algebras (cf. [13]).

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In [14] Jun and Ko introduced the notion of n-fold grisly deductive systems, and gave some conditions for a deductive system becomes a n-fold grisly deductive system on BL-algebras. Moreover, they constructed an extension property for n-fold grisly deductive system on BL-algebras. However, n-fold implicative filter and n-fold grisly deductive system are the same on BL-algebras [12]. In [12,17,19] the authors defined the notion of n-fold implicative filters, n-fold positive implicative filters, n-fold boolean filters, n-fold fantastic filters, n-fold normal filters in BL-algebras and studied the relation among many types of n-fold filters in BL-algebra. At the present, the filter theory of hoops has been widely studied and some important results have been obtained. In particular, in [16], some types of filters in hoops were introduced, and some of their characterizations and relations were presented. These filters include Boolean(implicative, positive implicative, prime and ultra, ect.)filters. M.Kondo proved that for any filter of a hoop, it is a positive implicative filter if and only if it is an implicative and fantastic filter.

The aim of this paper is to extend this research to hoops with the connection of the results obtaining in [10,15,16]. First, we give some basic results, which are needed in the rest papers. Next, we define n-fold(positive) implicative filter and n-fold (positive) implicative hoop and derive some of their characterizations. We obtain that if F is a n-fold (positive) implicative filter then it is also a filter. Finally, we discuss the relations between n-fold implicative filter and n-fold positive implicative filter.

2. Preliminaries

In this section, we recollect some definitions and results which will be used in the following:

Definition 2.1 ([3, 4, 6]). *An algebra $(H, \odot, \to, 1)$ of type $(2, 2, 0)$ is called a hoop if it satisfies the following conditions, for all $x, y, z \in H$:*

$(H1)$ $(H, \odot, 1)$ *is a commutative monoid;*

$(H2)$ $x \to x = 1$;

$(H3)$ $x \odot (x \to y) = y \odot (y \to x)$;

$(H4)$ $x \to (y \to z) = (x \odot y) \to z$.

In $(H, \odot, \to, 1)$, we define a binary relation $'' \leq''$, where $x \leq y$ iff $x \to y = 1$. One can easily prove the binary relation is partial order.

A hoop H is called bounded if there is a least element 0 in H, such that for all $x \in H$, $0 \leq x$. For all $x \in H$, $x^n = \underbrace{x \odot \ldots x}_{n\ times}$ if $n > 0$ and $x^0 = 1$.

Proposition 2.2 ([6, 7]). *Let H be a hoop, then the following properties hold for all $x, y, z \in H$:*

(1) $(H, \leq)$ *is a $\wedge$-semilattice with* $x \wedge y = x \odot (x \to y)$;

(2) $x \odot y \leq z$ *if and only if* $x \leq y \to z$;

(3) $x \leq y$ *if and only if* $x \to y = 1$;
(4) $x \odot y \leq x, y$;
(5) *If* $x \leq y$, *then* $x \odot z \leq y \odot z$;
(6) $x \leq y \to x$;
(7) $x \leq (x \to y) \to y$;
(8) $x \to 1 = 1$;
(9) $1 \to x = x$;
(10) $x \to (y \to z) = y \to (x \to z) = (x \odot y) \to z$;
(11) *If* $x \leq y$, *then* $z \to x \leq z \to y$, $y \to z \leq x \to z$;
(12) $x \to y \leq (z \to x) \to (z \to y)$;
(13) $x \to y \leq (y \to z) \to (x \to z)$.

Definition 2.3 ([9]). *Let H be a hoop and F be a non-empty subset of H. F is called a filter of H if it satisfies the following conditions for every $x, y \in H$:*
$(F1)$ $x, y \in F$ *implies* $x \odot y \in F$;
$(F2)$ $x \in F$ *and* $x \leq y$ *imply* $y \in F$.

Proposition 2.4 ([9]). *Let H be a hoop and F be a subset of H. Then F is a filter if and only if the following conditions hold:*
(1) $1 \in F$;
(2) $x, y \in H$, *if* $x, x \to y \in F$ *then* $y \in F$.

Theorem 2.5 ([1, 8]). *Let F be a filter of H. Define $x\Theta_F y$ if and only if $x \to y \in F$ and $y \to x \in F$. Then Θ_F is a congruence relation on H.*

The set of all congruence classes is denoted by $H/F \doteq \{x/F | x \in H\}$, where $x/F = \{y \in H | x\Theta_F y\}$. In the H/F we define the binary relation $'' \leq ''$, where $x/F \leq y/F$ iff $x \to y \in F$. Then We can easily prove the binary relation is partial order and $(H/F, \otimes, \rightsquigarrow, 1_{H/F})$ is a hoop, where $1_{H/F} = 1/F$, $x/F \otimes y/F = (x \odot y)/F$, $x/F \rightsquigarrow y/F = (x \to y)/F$.

3. n-fold implicative filter

From now on, we denote the $(H, \odot, \to, 1)$ by H.

Definition 3.1. *Let F be a subset of H and $n \in N$. F is called a n-fold implicative filter of H if it satisfies:*
(1) $1 \in F$;
(2) $x^n \to (y \to z) \in F$, $x^n \to y \in F$, *imply* $x^n \to z \in F$, *for all* $x, y, z \in H$.

Example 3.2. Let $H = \{0, a, b, 1\}$, define $\odot$ and $\to$ as follows,

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	a	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	b	1	1
1	0	a	b	1

Then $(H, \odot, \to, 1)$ is a hoop, and it is clear that $\{1\}$ is a 3-fold implicative filter.

Theorem 3.3. *Any n-fold implicative filter is a filter of H.*

Proof. Let F be a n-fold implicative filter of H and x, $x \to y \in F$. Hence $1 \to x \in F$, $1 \to (x \to y) \in F$. But $1 = 1^n$, thus $y = 1 \to y \in F$, that is, F is a filter of H. □

The following example shows that the converse of Theorem 3.3 is not true.

Example 3.4. Let $H = \{0, a, b, 1\}$, with $0 \le a \le b \le 1$, define $\odot$ and $\to$ as follows:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

Then $(H, \odot, \to, 1)$ is a hoop, and it is clear that $F = \{b, 1\}$ is a filter of H, while it is not a $1-$fold implicative filter. Since $a \to (a \to 0) = 1 \in F$ and $a \to a = 1 \in F$ but $a \to 0 = a \notin F$.

Theorem 3.5. *For any $a \in H$, $H(a) \doteq \{x \in H \mid a \le x\}$ the following conditions are equivalent:*

(1) *$H(a)$ is a filter of H;*
(2) *$a \le y$ whenever $a \le x \to y$ and $a \le x$ for all $x, y \in H$.*

Proof. (1) $\Rightarrow$ (2) Let $H(a)$ be a filter of H and $a \le x \to y$ and $a \le y$, then $x \to y \in H(a)$ and $x \in H(a)$. Since $H(a)$ is a filter, $y \in H(a)$, that is, $a \le y$.

(2) $\Rightarrow$ (1) Since $a \le 1$, $1 \in H(a)$. If $a \le x$, $a \le (x \to y)$, then $x \in H(a)$, $x \to y \in H(a)$. By assumption $a \le y$ and hence $y \in H(a)$. Hence $H(a)$ is a filter. □

Theorem 3.6. *Let a be an element of H. If $H(a)$ is a n-fold implicative filter of H, then $a^{n+1} \to (x \to y) = 1$, $a^{n+1} \to x = 1$ imply $a^{n+1} \to y = 1$, for all $x, y \in H$.*

Proof. Let $H(a)$ be a n-fold implicative filter and $a^{n+1} \to (x \to y) = 1$, $a^{n+1} \to x = 1$. Since $a \to (a^n \to (x \to y)) = a^{n+1} \to (x \to y) = 1$, $a^n \to (x \to y) \in H(a)$. Similarly $a^n \to x \in H(a)$. Since $H(a)$ is a n-fold implicative filter, $a^n \to y \in H(a)$, that is, $a \le a^n \to y$. Thus $a^{n+1} \to y = 1$. □

Theorem 3.7. *Let F be a filter of H. Then for all $x, y, z \in H$, the following conditions are equivalent:*

(1) *F is a n-fold implicative filter of H;*
(2) *$x^n \to x^{2n} \in F$ for all $x \in H$;*
(3) *$x^{n+1} \to y \in F$ implies $x^n \to y \in F$;*
(4) *$x^n \to (y \to z) \in F$ implies $(x^n \to y) \to (x^n \to z) \in F$.*

Proof. (1) $\Rightarrow$ (2) Let $x \in H$, hence $x^n \to (x^n \to x^{2n}) = 1 \in F$ and $x^n \to x^n = 1 \in F$. Since F is a n-fold implicative filter of H, we get $x^n \to x^{2n} \in F$.

(2) $\Rightarrow$ (1) Let $x, y, z \in H$ be such that $x^n \to (y \to z) \in F$, $x^n \to y \in F$. Since $(x^n \to (y \to z)) \odot (x^n \to y) \odot x^n \odot x^n = (x^n \wedge (y \to z)) \odot (x^n \wedge y) \leq (y \to z) \odot y = y \wedge z \leq z$ then $(x^n \to (y \to z)) \odot (x^n \to y) \leq x^{2n} \to z$. Since $x^n \to (y \to z) \in F$, $x^n \to y \in F$, we get $(x^n \to (y \to z)) \odot (x^n \to y) \in F$, and so $x^{2n} \to z \in F$. By Proposition 2.2, $x^n \to x^{2n} \leq (x^{2n} \to z) \to (x^n \to z)$. On the other hand, $x^{2n} \to z \in F$ and $x^n \to x^{2n} \in F$ then $x^n \to z \in F$. Hence F is a n-fold implicative filter of H.

(2) $\Rightarrow$ (3) Since (2) holds, F is a n-fold implicative filter of H. On the other hand, $x^{n+1} \to y = x^n \to (x \to y) \in F$ and $x^n \to x = 1 \in F$, hence $x^n \to y \in F$.

(3) $\Rightarrow$ (2) We have $x^{n+1} \to (x^{n-1} \to x^{2n}) = x^{2n} \to x^{2n} = 1 \in F$, hence by (3) $x^n \to (x^{n-1} \to x^{2n}) \in F$. But $x^{n+1} \to (x^{n-2} \to x^{2n}) = x^{2n-1} \to x^{2n} = x^n \to (x^{n-1} \to x^{2n}) \in F$, that is, $x^{n+1} \to (x^{n-2} \to x^{2n}) \in F$, and so $x^n \to (x^{n-2} \to x^{2n}) \in F$. By repeating the process n times we get $x^n \to (x^0 \to x^{2n}) = x^n \to (1 \to x^{2n}) = x^n \to x^{2n} \in F$.

(2) $\Rightarrow$ (4) Let $x^n \to (y \to z) \in F$, then by Proposition 2.2, $x^n \to (y \to z) \leq x^n \to ((x^n \to y) \to (x^n \to z)) = x^n \to (x^n \to ((x^n \to y) \to z)) = x^{2n} \to ((x^n \to y) \to z)$. Hence $x^{2n} \to ((x^n \to y) \to z) \in F$. By (2), we have $x^n \to x^{2n} \in F$, now by Proposition 2.2, $x^{2n} \to ((x^n \to y) \to z) \leq (x^n \to x^{2n}) \to (x^n \to ((x^n \to y) \to z))$. Then we get $(x^n \to y) \to (x^n \to z) = x^n \to ((x^n \to y) \to z) \in F$.

(4) $\Rightarrow$ (2) Since $x^n \to (x^n \to x^{2n}) = x^{2n} \to x^{2n} = 1 \in F$, by (4) we have $x^n \to x^{2n} = (x^n \to x^n) \to (x^n \to x^{2n}) \in F$. $\square$

Theorem 3.8. *If F is a n-fold implicative filter of H, then it is a $(n+1)$-fold implicative filter.*

Proof. Let $x, y \in H$ be such that $x^{n+2} \to y \in F$. By Proposition 2.2, $x^{n+1} \to (x \to y) = x^{n+2} \to y$ and since F is a n-fold implicative filter, by Theorem 3.7 $x^n \to (x \to y) \in F$. Hence $x^{n+1} \to y \in F$, that is, F is a $(n+1)$-fold implicative filter. $\square$

The following example shows that the converse of Theorem 3.8 is not true.

Example 3.9. In Example 3.2, $\{1\}$ is a 3-fold implicative filter, but $\{1\}$ is not a 2-fold implicative filter. Since $b^3 \to 0 = 1 \in \{1\}$ and $b^2 \to 0 = b \neq 1$.

Theorem 3.10. *Let F and G be filters of H such that $F \subseteq G$. If F is a n-fold implicative filter, then G is also a n-fold implicative filter.*

Proof. Let F be a n-fold implicative filter of H. Then by Theorem 3.7 $x^n \to x^{2n} \in F$ for all $x \in H$, and so $x^n \to x^{2n} \in G$ for all $x \in H$. Hence G is a n-fold implicative filter. $\square$

Definition 3.11. *A hoop H is called a n-fold implicative hoop if it satisfies $x^{n+1} = x^n$, for all $x \in H$.*

Theorem 3.12. *If H is a n-fold implicative hoop, then it is a $(n+k)$-fold implicative hoop, where $k \in Z$, $k \geq 0$.*

Proof. Since H is a n-fold implicative hoop, $x^{n+1} = x^n$ for all $x \in H$. But $x^{n+2} = x^{n+1} \odot x = x^n \odot x = x^{n+1}$ and hence H is a $(n+1)$-fold implicative hoop. By repeating the process k times, we get H is a $(n+k)$-fold implicative hoop. □

The following example shows that the converse of Theorem 3.12 is not true.

Example 3.13. In Example 3.2, it is clear that H is a 3-fold implicative hoop, while it is not a 2-fold implicative hoop, since $b^3 \neq b^2$.

Corollary 3.14. *In a n-fold implicative hoop, the concepts of filter and n-fold implicative filter coincide.*

Proof. It follows from Theorem 3.3 and the definition of n-fold implicative hoop. □

Theorem 3.15. *The following conditions are equivalent:*

(1) *H is a n-fold implicative hoop;*
(2) *Every filter of H is a n-fold implicative filter;*
(3) *$\{1\}$ is a n-fold implicative filter;*
(4) *$x^n = x^{2n}$ for all $x \in H$;*
(5) *H/F is a n-fold implicative hoop.*

Proof. (1) ⇒ (2) It is clear by the Corollary 3.14.

(2) ⇒ (3) This is clear.

(3) ⇒ (1) Let $\{1\}$ be a n-fold implicative filter. Since $x^n \to (x^n \to x^{n+1}) = x^{2n} \to x^{n+1} = 1 \in \{1\}$ and $x^n \to x^n = 1 \in \{1\}$, we get $x^n \to x^{n+1} \in \{1\}$, that is, $x^n = x^{n+1}$. Hence H is a n-fold implicative hoop.

(1) ⇒ (4) Let H be a n-fold implicative hoop, hence $x^{n+1} = x^n$ for all $x \in H$.

We have $x^{n+2} = x^{n+1} \odot x = x^n \odot x = x^{n+1} = x^n$. By repeating the process n times, we get $x^n = x^{2n}$ for all $x \in H$.

(4) ⇒ (1) Since $x^n = x^{2n}$ for all $x \in H$, we have $x = x^2$, $x^3 = x^2 \odot x = x^2 \odot x^2 = (x^2)^2 = x^2 = x$, thus $x^3 = x^2 = x$. By the principle of mathematical induction, we get $x^{n+1} = x^n$. Hence H is a n-fold implicative hoop.

(1) ⇒ (5) By Theorem 2.5 and Definition 3.11, it is clear.

(5) ⇒ (1) Let H/F be a n-fold implicative hoop. For all $x \in H$, we have $x^{n+1}/F = x^n/F$, then $x^n/F \rightsquigarrow x^{n+1}/F = (x^n \to x^{n+1})/F = 1/F = 1_{H/F}$ and so $x^n \to x^{n+1} = 1$, hence H is a n-fold implicative hoop. □

4. n-fold positive implicative filter

In this section, we introduce the notion of n-fold positive implicative filter and n-fold positive implicative bounded hoop. Also, we derive some of their characterizations.

Definition 4.1. *Let F be a subset of H and $n \in N$. F is called a n-fold positive implicative filter of H if it satisfies:*

(1) $1 \in F$;

(2) $x \to ((y^n \to z) \to y) \in F$ *and* $x \in F$ *imply* $y \in F$, *for all* $x, y, z \in H$.

Example 4.2. In Example 3.2, $\{1\}$ is a 3-fold positive implicative filter. But $\{1\}$ is not a positive implicative filter, since $1 \to ((b \to 0) \to b) = 1 \in F$, while $b \notin F$.

Theorem 4.3. *Every n-fold positive implicative filter of H is a filter of H.*

Proof. Let F be a n-fold positive implicative filter of H, and $x, x \to y \in F$. We have that $x \to ((y^n \to 1) \to y) = x \to y \in F$ and so $x \to ((y^n \to 1) \to y) \in F$. Since $x \in F$ and F is a n-fold positive implicative filter of H, we get $y \in F$. Hence F is a filter. □

The following example shows that the converse of Theorem 4.3 is not true.

Example 4.4. Let $H = \{0, a, b, c, 1\}$, define $\odot$ and $\to$ as follows,

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	c	c	a
b	0	c	b	c	b
c	0	c	c	c	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	b	b	1
b	0	a	1	a	1
c	0	1	1	1	1
1	0	a	b	c	1

Then $(H, \odot, \to, 0, 1)$ is a hoop. It is clear that $F = \{b, 1\}$ is a filter of H, but it is not 2-fold positive implicative filter, since $1 \to ((a^2 \to 0) \to a) = 1 \in F$ and $a \notin F$.

Theorem 4.5. *Let F be a filter of H, the following conditions are equivalent for all $x, y, z \in H$:*

(1) *F is a n-fold positive implicative filter;*

(2) $(x^n \to y) \to x \in F$ *implies* $x \in F$, *for all* $x, y \in H$;

(3) $(x^n \to 0) \to x \in F$ *implies* $x \in F$, *for all* $x \in H$.

Proof. (1) ⇒ (2) Let F be a n-fold positive implicative filter of H and $(x^n \to y) \to x \in F$. Since $1 \to ((x^n \to y) \to x) = (x^n \to y) \to x$, we get $1 \to ((x^n \to y) \to x) \in F$. Now that $1 \in F$, thus $x \in F$.

(2) ⇒ (3) This is clear.

(3) ⇒ (1) Let $x \to ((y^n \to z) \to y) \in F$ and $x \in F$. Since F is a filter, we have $(y^n \to z) \to y) \in F$. Since $0 \leq z$, by Proposition 2.2 $y^n \to 0 \leq y^n \to z$ and so $(y^n \to z) \to y \leq (y^n \to 0) \to y$. Hence $(y^n \to 0) \to y \in F$ and so by assumption we get $y \in F$. Therefore F is a n-fold positive implicative filter. □

Theorem 4.6. *If F is a n-fold positive implicative filter of H, then it is a $(n+1)$-fold positive implicative filter.*

Proof. Let F be a n-fold positive implicative filter and $x \in H$, $(x^{n+1} \to 0) \to x \in F$. Since $x^{n+1} \leq x^n$, we have $x^n \to 0 \leq x^{n+1} \to 0$ and so $(x^{n+1} \to 0) \to x \leq (x^n \to 0) \to x$. Since F is a filter, we get $(x^n \to 0) \to x \in F$. Note that F is a n-fold positive implicative filterwe have $x \in F$ and so F is a $(n+1)$-fold positive implicative filter. □

The following example shows that the converse of Theorem 4.6 is not true.

Example 4.7. In Example 3.2, $\{1\}$ is a 3-fold positive implicative filter, while $\{1\}$ is not a 2-fold positive implicative filter, since $(b^2 \to 0) \to b = 1 \in F$ and $b \neq 1$.

Theorem 4.8. *Every n-fold positive implicative filter of H is a n-fold implicative filter of H.*

Proof. Let F be a n-fold positive implicative filter of H. By Theorem 4.3, F is a filter of H. Let $x, y \in H$ be such that $x^{n+1} \to y \in F$. Then by Proposition 2.2, we have

$$\begin{aligned}
&(x^{n+1} \to y)^n \to (x^n \to y) = ((x^{n+1} \to y)^{n-1} \odot (x^{n+1} \to y)) \to (x^n \to y) \\
&= (x^{n+1} \to y)^{n-1} \to ((x^{n+1} \to y) \to (x^n \to y)) \\
&= (x^{n+1} \to y)^{n-1} \to ((x^{n+1} \to y) \to ((x^{n-1} \to (x \to y))) \\
&= (x^{n+1} \to y)^{n-1} \to ((x^{n-1} \to ((x^{n+1} \to y) \to (x \to y))) \\
&= (x^{n+1} \to y)^{n-1} \to (x^{n-1} \to ((x \to (x^n \to y)) \to (x \to y))) \\
&\geq (x^{n+1} \to y)^{n-1} \to (x^{n-1} \to (x^n \to y) \to y) \\
&= (x^{n+1} \to y)^{n-1} \to ((x^n \to y) \to (x^{n-1} \to y)) \\
&= (x^n \to y)^2 \to ((x^{n+1} \to y)^{n-2} \to (x^{n-1} \to y)).
\end{aligned}$$

We show that $(x^{n+1} \to y)^n \to (x^n \to y) \geq (x^n \to y)^2 \to ((x^{n+1} \to y)^{n-2} \to (x^{n-2} \to y))$.

Note that

$$\begin{aligned}
&(x^n \to y) \odot (x^{n+1} \to y)^{n-1} \odot x^{n-1} \\
&= (x^n \to y) \odot (x^{n+1} \to y)^{n-2} \odot (x^{n+1} \to y) \odot x \odot x^{n-2} \\
&= (x^n \to y) \odot (x^{n+1} \to y)^{n-2} \odot x^{n-2} \odot x \odot (x^{n+1} \to y).
\end{aligned}$$

Since $x^{n+1} \to y \leq x^{n+1} \to y = x \to (x^n \to y)$, then $x \odot (x^{n+1} \to y) \leq x^n \to y$, we get $(x^n \to y) \odot (x^{n+1} \to y)^{n-1} \odot x^{n-1} \leq (x^n \to y)^2 \odot (x^{n+1} \to y)^{n-2} \odot x^{n-2}$.

Hence $((x^n \to y)^2 \odot (x^{n+1} \to y)^{n-2} \odot x^{n-2}) \to y \leq ((x^n \to y) \odot (x^{n+1} \to y)^{n-1} \odot x^{n-1}) \to y$ and so $(x^n \to y)^2 \to ((x^{n+1} \to y)^{n-2} \to (x^{n-2} \to y)) \leq (x^n \to y) \to ((x^{n+1} \to y)^{n-1} \to (x^{n-1} \to y))$.

Now we have $(x^{n+1} \to y)^n \to (x^n \to y) \geq (x^n \to y) \to ((x^{n+1} \to y)^{n-1} \to (x^{n-1} \to y))$ and $(x^{n+1} \to y)^n \to (x^n \to y) \geq (x^n \to y)^2 \to ((x^{n+1} \to y)^{n-2} \to (x^{n-2} \to y))$, and hence by repeating the process n times we get $(x^{n+1} \to$

$y)^n \to (x^n \to y) \geq (x^n \to y)^2 \to ((x^{n+1} \to y)^{n-2} \to (x^{n-2} \to y)) \geq \dots \geq (x^n \to y)^n \to ((x^{n+1} \to y)^0 \to (x^0 \to y)) = (x^n \to y)^n \to (1 \to (1 \to y)) = (x^n \to y)^n \to y$.

Hence $((x^n \to y)^n \to y) \to ((x^{n+1} \to y)^n \to (x^n \to y)) = 1$ and so $(x^{n+1} \to y)^n \to (((x^n \to y)^n \to y) \to (x^n \to y)) = 1$. Since F is a filter and $x^{n+1} \to y \in F$, we get $(x^{n+1} \to y)^n \in F$ and so $((x^n \to y)^n \to y) \to (x^n \to y)) \in F$. Note that F is a n-fold positive implicative filter, by Theorem 4.5 we have $x^n \to y \in F$. Hence by Theorem 3.7, F is a n-fold implicative filter. □

The following examples show that the converse of Theorem 4.8 is not true in general.

Example 4.9. In Example 4,4, $F = \{b, 1\}$ is a 2-fold implicative filter but it is not a 2-fold positive implicative filter, since $(a^2 \to 0) \to a = 1 \in F$ and $a \notin F$.

Example 4.10. Define for all $x, y \in H$, $x \odot y = \min\{x, y\}$,

$$x \to y = \begin{cases} 1, & (x \leq y) \\ y, & (x > y) \end{cases}.$$

It is clear that $H = ([0, 1], \odot, \to, 0, 1)$ is a hoop, $F = [\frac{1}{2}, 1]$ is a 2-fold implicative filter, but it is not 2-fold positive implicative filter, since $\frac{2}{3} \to ((\frac{1}{3})^2 \to \frac{1}{10}) \to \frac{1}{3} = 1 \in F$, $\frac{2}{3} \in F$, but $\frac{1}{3} \notin F$.

Definition 4.11. *A bounded hoop $H(H, \odot, \to, 0, 1)$ is called a n-fold positive implicative bounded hoop if it satisfies $(x^n \to 0) \to x = x$ for all $x \in H$.*

Example 4.12. In Example 3.2, H is a 3-fold positive implicative bounded hoop.

Theorem 4.13. *If H is a n-fold positive implicative bounded hoop, then it is a $(n+1)$-fold positive implicative bounded hoop.*

Proof. Since H is a n-fold positive implicative bounded hoop, $(x^n \to 0) \to x = x$ for all $x \in H$. Since $x^{n+1} \leq x^n$, then $x^n \to 0 \leq x^{n+1} \to 0$ and so $(x^{n+1} \to 0) \to x \leq (x^n \to 0) \to x = x$. Then by Proposition 2.2, $x \leq (x^{n+1} \to 0) \to x$. Hence $x = (x^{n+1} \to 0) \to x$, that is, H is a $(n+1)$-fold positive implicative bounded hoop. □

The following example shows that the converse of Theorem 4.13 is not true.

Example 4.14. In Example 3.2, H is a 3-fold positive implicative bounded hoop, but H is not a 2-fold positive implicative bounded hoop, since $(b^2 \to 0) \to b \neq b$.

Corollary 4.15. *In a n-fold positive implicative bounded hoop, the concepts of filter and n-fold positive implicative filter coincide.*

Proof. It follows from Theorem 4.3 and Theorem 4.5. □

Theorem 4.16. *The following conditions are equivalent:*

(1) *H is a n-fold positive implicative bounded hoop;*

(2) *Every filter of H is a n-fold positive implicative filter;*

(3) *$\{1\}$ is a n-fold positive implicative filter;*

(4) *If F is a filter of H, H/F is a n-fold positive implicative hoop.*

Proof. (1) $\Rightarrow$ (2) It is clear by the Corollary 4.15.

(2) $\Rightarrow$ (3) This is clear.

(3) $\Rightarrow$ (1) Let $\{1\}$ be a n-fold positive implicative filter. Consider $x \in H$ and let $t = ((x^n \to 0) \to x) \to x$. Then by Proposition 2.2 we have

$$\begin{aligned}(t^n \to 0) \to t &= (t^n \to 0) \to (((x^n \to 0) \to x) \to x)\\ &= ((x^n \to 0) \to x) \to ((t^n \to 0) \to x)\\ &\geq (t^n \to 0) \to (x^n \to 0)\\ &\geq x^n \to t^n.\end{aligned}$$

Since $x \leq (x^n \to 0) \to x = t$, then $x^n \leq t^n$, that is, $x^n \to t^n = 1$. Hence $(t^n \to 0) \to t = 1 \in \{1\}$, and since $\{1\}$ is a n-fold positive implicative filter, $t = ((x^n \to 0) \to x) \to x = 1$, that is, $(x^n \to 0) \to x \leq x$. On the other hand, by Proposition 2.2 we have $(x^n \to 0) \to x \geq x$. Hence we get $(x^n \to 0) \to x = x$ for all $x \in H$, that is, H is a n-fold positive implicative hoop.

(2) $\Rightarrow$ (4) Let F be a n-fold positive implicative filter of H. By (1) H is a n-fold positive implicative bounded hoop. Then for all $x \in H$, we have $(x^n \to 0) \to x = x$. Thus $(x^n/F \rightsquigarrow 0/F) \rightsquigarrow x/F = ((x^n \to 0) \to x)/F = x/F$, that is, H/F is a n-fold positive implicative bounded hoop.

(4) $\Rightarrow$ (2) Let F be a filter of H and $x \in H$, $(x^n \to 0) \to x \in F$. Since H/F is a n-fold positive implicative bounded hoop, then for all $x \in H$, $(x^n/F \rightsquigarrow 0/F) \rightsquigarrow x/F = x/F$. By Theorem 2.5 we have $((x^n \to 0) \to x) \to x \in F$. Since F is a filter, we get $x \in F$. Hence by Theorem 4.5, F is a n-fold positive implicative filter. □

Theorem 4.17. *Let F and G be filters of H such that $F \subseteq G$. If F is a n-fold positive implicative filter, then G is also a n-fold positive implicative filter.*

Proof. Let $x \in H$ be such that $(x^n \to 0) \to x \in G$. Since F is a n-fold positive implicative filter, by Theorem 4.15 H/F is a n-fold positive implicative bounded hoop. Then $((x^n \to 0) \to x)/F = (x^n/F \rightsquigarrow 0/F) \rightsquigarrow x/F = x/F$ and so $((x^n \to 0) \to x) \to x \in F \subseteq G$. Since G is a filter and $(x^n \to 0) \to x \in G$, we get $x \in G$. Hence by Theorem 4.5, G is a n-fold positive implicative filter. □

5. Conclusion and future research

In this paper, we introduce the concept of n-fold (positive) implicative filter and n-fold (positive)implicative (bounded) hoop. We prove that in any n-fold (positive)implicative (bounded) hoop, the notion of filter and n-fold (positive) implicative filter are coincide. Also, we prove that F is a n-fold (positive) implicative(bounded) hoop if and only if H/F is a n-fold (positive) implicative (bounded) hoop. Specially, we obtain that every n-fold positive implicative filter is a n-fold implicative filter. For our future research, we will construct n-fold implicative logic and n-fold positive implicative logic respect to many valued logic system correspondence to hoops and investigate under which conditions a n-fold implicative logic is a n-fold positive implicative logic.

Acknowledgements

The research is supported by the National Natural Science Foundation of China (11571281,11601302).

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Accepted: 22.06.2017