(ϵ)-KENMOTSU MANIFOLDS ADMITTING A SEMI-SYMMETRIC METRIC CONNECTION

Venkatesha*
Vishnuvardhana S.V.
Department of Mathematics
Kuvempu University
Shankaraghatta - 577 451
Shimoga, Karnataka
INDIA
vensmath@gmail.com
svvishnuvardhana@gmail.com

Abstract. The object of the present paper is to study some properties of quasi-conformal and concircular curvature tensor on (ϵ)-Kenmotsu manifolds with respect to a semi-symmetric metric connection.
Keywords: Quasi-conformally flat, ϕ-concircularly flat, (ϵ)-Kenmotsu manifold, semi-symmetric metric connection, η-Einstein manifold.

1. Introduction

Takahashi [19] studied Sasakian manifold with associated pseudo-Riemannian metrics and are known as (ϵ)-Sasakian manifolds. Bejancu and Duggal [3] shows the existence of (ϵ)-almost contact metric structures and provide an example of (ϵ)-Sasakian manifolds. Further investigation on this manifold was taken up by Xufeng and Xiaoli [23] and Rakesh kumar et al [13].

The study of manifolds with indefinite metric has a great relevance from the standpoint of geometrization of physics and relativity. Recently De and Sarkar [7] introduced indefinite metrics on Kenmotsu manifold, and are called as (ϵ)-Kenmotsu manifolds. Here they studied conformally flat, Weyl semisymmetric, ϕ-recurrent (ϵ)-Kenmotsu manifolds. Further, Singh et al [18] and Haseeb et al [9] established the relation between Levi-Civita connection and semi-symmetric metric connection and obtained the relation between curvature tensors of Levi-Civita connection and semi-symmetric metric connection in an (ϵ)-Kenmotsu manifold.

After the introduction of an idea of semi-symmetric linear connection in a differentiable manifold [8], Hayden [10] defined a semi-symmetric metric connection on a Riemannian manifold and this was further studied by Yano [24], Barua et al [2], De and Biswas [5].

*. Corresponding author

Quasi-conformal curvature tensor and concircular curvature tensor are the important curvature tensors from the differential geometry point of view. A quasi-conformal transformation is the one which transforms infinitesimal circles into infinitesimal ellipses, where as a concircular transformation transforms every geodesic circle of an n-dimensional Riemannian manifold M into a geodesic circle.

The quasi-conformal curvature tensor $\bar{C}$ [25] and concircular curvature tensor $\bar{Z}$ [26] with respect to the semi-symmetric metric connection are respectively given by

$$\begin{aligned}\bar{C}(X,Y)Z = a\bar{R}(X,Y)Z + b[\bar{S}(Y,Z)X - \bar{S}(X,Z)Y \\ + g(Y,Z)\bar{Q}X - g(X,Z)\bar{Q}Y] \\ - \frac{\bar{r}}{n}\{\frac{a}{(n-1)} + 2b\}[g(Y,Z)X - g(X,Z)Y],\end{aligned} \tag{1.1}$$

$$\bar{Z}(X,Y)Z = \bar{R}(X,Y)Z - \frac{\bar{r}}{n(n-1)}[g(Y,Z)X - g(X,Z)Y]. \tag{1.2}$$

The paper is organized as follows: Section 2 contains the preliminaries of (ϵ)-Kenmotsu manifold and a semi-symmetric metric connection on an (ϵ)-Kenmotsu manifold. Section 3 and 4 are devoted to the study of quasi-conformally flat and quasi-conformally semisymmetric (ϵ)-Kenmotsu manifold admitting a semi-symmetric metric connection. In the next section we study ϕ-concircularly flat (ϵ)-Kenmotsu manifold admitting a semi-symmetric metric connection and shown that ϕ-concircularly flat (ϵ)-Kenmotsu manifold admitting a semi-symmetric metric connection is an η-Einstein manifold. Further in section 6, we prove (ϵ)-Kenmotsu manifold admitting a semi-symmetric metric connection satisfying $\bar{Z}(X,Y)\cdot\bar{S}(U,W)=0$ is an η-Einstein manifold.

2. Preliminaries

An almost contact structure on a n-dimensional differentiable manifold M is a triple (ϕ,ξ,η), where ϕ is a tensor field of type $(1,1)$, η is a 1-form and ξ is a vector field such that

$$\phi^2 = -I + \eta\circ\xi, \tag{2.1}$$

$$\eta(\xi)=1, \quad \phi\xi=0, \quad \eta\circ\phi=0. \tag{2.2}$$

A differential manifold with an almost contact structure is called an almost contact manifold. An almost contact metric manifold is an almost contact manifold endowed with a compatible metric g. An almost contact metric manifold M is said to be an (ϵ)-almost contact metric manifold if

$$g(\xi,\xi) = \pm 1 = \epsilon, \tag{2.3}$$

$$\eta(X) = \epsilon g(X,\xi), \quad rank(\phi) = n-1, \tag{2.4}$$

$$g(\phi X,\phi Y) = g(X,Y) - \epsilon\eta(X)\eta(Y), \quad \forall\ X,Y\in\Gamma(TM), \tag{2.5}$$

holds, where ξ is space-like or time-like but it is never a light like vector field. We say that (ϕ, ξ, η, g) is an (ϵ)-contact metric structure if we have

$$d\eta(X, Y) = g(X, \phi Y). \tag{2.6}$$

In this case, M is an (ϵ)-contact metric manifold. An (ϵ)-contact metric manifold is called an (ϵ)-Kenmotsu manifold [7] if

$$(\nabla_X \phi)Y = -g(X, \phi Y)\xi - \epsilon\eta(Y)\phi X, \tag{2.7}$$

holds, where ∇ is the Riemannian connection of g. An (ϵ)-almost contact metric manifold is a (ϵ)-Kenmotsu manifold if and only if

$$\nabla_X \xi = \epsilon(X - \eta(X)\xi). \tag{2.8}$$

The following conditions holds in an (ϵ)-Kenmotsu manifold [7]:

$$(\nabla_X \eta)(Y) = g(X, Y) - \epsilon\eta(X)\eta(Y), \tag{2.9}$$

$$\eta(R(X, Y)Z) = \epsilon\{g(X, Z)Y - g(Y, Z)X\}, \tag{2.10}$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \qquad R(\xi, X)Y = \eta(Y)X - \epsilon g(X, Y)\xi, \tag{2.11}$$

$$S(X, \xi) = -(n-1)\eta(X), \qquad Q\xi = -\epsilon(n-1)\xi, \tag{2.12}$$

$$S(\phi X, \phi Y) = S(X, Y) + \epsilon(n-1)\eta(X)\eta(Y). \tag{2.13}$$

A semi-symmetric metric connection $\bar{\nabla}$ on an n-dimensional (ϵ)-Kenmotsu manifold is given by [18],

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X, Y)\xi. \tag{2.14}$$

A relation between the curvature tensor $\bar{R}$, Ricci curvature $\bar{S}$ and the scalar curvature $\bar{r}$ of M with respect to semi-symmetric metric connection $\bar{\nabla}$ and R, S and r of M with respect to the Riemannian connection ∇ are given by

$$\begin{aligned} \bar{R}(X, Y)Z &= R(X, Y)Z + (2+\epsilon)[g(X, Z)Y - g(Y, Z)X] \\ &+ (1+\epsilon)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)]\xi \\ &+ (1+\epsilon)\eta(Z)[\eta(Y)X - \eta(X)Y], \end{aligned} \tag{2.15}$$

$$\begin{aligned} \bar{S}(Y, Z) &= S(Y, Z) + [(\epsilon+2)(\epsilon-n)+2]g(Y, Z) \\ &+ (1+\epsilon)(n-2\epsilon)\eta(Y)\eta(Z), \end{aligned} \tag{2.16}$$

$$\bar{r} = r + n[(\epsilon+2)(\epsilon-n)+2] + \epsilon(1+\epsilon)(n-2\epsilon). \tag{2.17}$$

3. Quasi-conformally flat (ϵ)-Kenmotsu manifold admitting a semi-symmetric metric connection

Definition 3.1. *An (ϵ)-Kenmotsu manifold admitting a semi-symmetric metric connection is said to be quasi-conformally flat if $\bar{C}(X, Y)Z = 0$.*

Suppose (ϵ)-Kenmotsu manifold admitting a semi-symmetric metric connection is quasi-conformally flat. Then from (1.1) we have

$$\begin{aligned} a\bar{R}(X,Y)Z &= b[\bar{S}(X,Z)Y - \bar{S}(Y,Z)X + g(X,Z)\bar{Q}Y - g(Y,Z)\bar{Q}X] \\ &+ \frac{\bar{r}}{n}\left\{\frac{a}{(n-1)} + 2b\right\}[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{3.1}$$

Taking an inner product of the above equation with ξ, we get

$$\begin{aligned} &ag(R(X,Y)Z,\xi) = a\epsilon\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\} + b[\epsilon S(X,Z)\eta(Y) \\ &- \epsilon S(Y,Z)\eta(X) + \{g(X,Z)Y - g(Y,Z)X\}\{4\epsilon - 3n\epsilon - 2n + 3\}] \\ &+ \epsilon\{\frac{\bar{r}}{n}\}\{\frac{a}{(n-1)} + 2b\}[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]. \end{aligned} \tag{3.2}$$

Setting $X = \xi$ in (3.2) and then using (2.2), (2.3), (2.10) and (2.12), we obtain

$$S(Y,Z) = Ag(Y,Z) + B\eta(Y)\eta(Z), \tag{3.3}$$

where

$$\begin{aligned} A &= \frac{2a}{b} - (4-3n) + (2n-3)\epsilon \\ &+ \left\{\frac{r + n[(\epsilon+2)(\epsilon-n)+2] + \epsilon(1+\epsilon)(n-2\epsilon)}{n}\right\}\left\{\frac{a}{(n-1)} + 2b\right\} \end{aligned}$$

and

$$\begin{aligned} B &= -\frac{2a}{b} + (4-3n)(1+\epsilon) \\ &+ \left\{\frac{\epsilon(r + n[(\epsilon+2)(\epsilon-n)+2] + \epsilon(1+\epsilon)(n-2\epsilon))}{n}\right\}\left\{\frac{a}{(n-1)} + 2b\right\}. \end{aligned}$$

Thus, we can state the following;

Theorem 3.2. *A quasi conformally flat n-dimensional (ϵ)-Kenmotsu manifold admitting a semi-symmetric metric connection is an η-Einstein manifold.*

4. Quasi-conformally semisymmetric (ϵ)-Kenmotsu manifold admitting a semi-symmetric metric connection

Theorem 4.1. *A quasi-conformally semisymmetric (ϵ)-Kenmotsu manifold admitting a semi-symmetric metric connection is an η-Einstein manifold.*

Proof. Suppose (ϵ)-Kenmotsu manifold admitting a semi-symmetric metric connection satisfies

$$\bar{R}(\xi,Y)\cdot\bar{C}(U,V)W = 0.$$

Which implies

$$\begin{aligned}&\bar{R}(\xi, Y)\bar{C}(U, V)W - \bar{C}(\bar{R}(\xi, Y)U, V)W \\ &- \bar{C}(U, \bar{R}(\xi, Y)V)W - \bar{C}(U, V)\bar{R}(\xi, Y)W = 0.\end{aligned} \tag{4.1}$$

By virtue of (2.15), (4.1) takes the form

$$\begin{aligned}&(1+\epsilon)[\eta(\bar{C}(U, V)W)Y - g(Y, \bar{C}(U, V)W)\xi - \eta(U)\bar{C}(Y, V)W \\ &+ g(Y, U)\bar{C}(\xi, V)W - \eta(V)\bar{C}(U, Y)W + g(Y, V)\bar{C}(U, \xi)W \\ &- \eta(W)\bar{C}(U, V)Y + g(Y, W)\bar{C}(U, V)\xi].\end{aligned} \tag{4.2}$$

Replacing Y by U in the above equation and then taking inner product with ξ, one can obtain

$$\begin{aligned}&(1+\epsilon)[g(U, \bar{C}(U, V)W) - g(U, U)\eta(\bar{C}(\xi, V)W) \\ &- \epsilon g(U, V)g(\bar{C}(\xi, W)\xi, U) - \epsilon\eta(W)g(\bar{C}(U, V)\xi, U)].\end{aligned} \tag{4.3}$$

Now putting $U = e_i$ in (4.3), where $\{e_i\}$, $i = 1, 2, \cdots, n$ is an orthonormal basis of the tangent space at each point of the manifold and sum up with respect to i and using (1.1), (2.11), (2.12), (2.15), (2.16) and (2.17), we get

$$S(V, W) = \alpha g(V, W) + \beta\eta(V)\eta(W), \tag{4.4}$$

where $\alpha = -[(\epsilon + 2)(\epsilon - n) + 2] + \frac{1}{(a-b)}[\{a + b(n-2)\}\frac{\bar{r}}{n} - (n-1)\{(a + (n-1)b)(1+\epsilon) - \frac{\bar{r}}{n(n-1)}(a + 2b(n-1))\}]$
and $\beta = -(\epsilon+1)(n-2\epsilon) + \frac{1}{(a-b)}[\{a + 2b(n-1)\}\frac{\bar{r}\epsilon}{n(n-1)} - \epsilon\{(a + (n-2)b)\}\{(n-1)(1+\epsilon) + \frac{\bar{r}\epsilon}{n}\}]$.

Hence the proof □

5. ϕ-concircularly flat (ϵ)-Kenmotsu manifold admitting a semi-symmetric metric connection

Definition 5.1. *An (ϵ)-Kenmotsu manifold admitting a semi-symmetric metric connection is said to be ϕ-concircularly flat if $\bar{Z}(\phi X, \phi Y)\phi Z = 0$.*

Assume that (ϵ)-Kenmotsu manifold admitting a semi-symmetric metric connection is ϕ-concircularly flat. Then from (1.2) we have

$$\begin{aligned}&g(\bar{R}(\phi X, \phi Y)\phi Z, \phi W) \\ &= \frac{\bar{r}}{n(n-1)}[g(\phi Y, \phi Z)g(\phi X, \phi W) - g(\phi X, \phi Z)g(\phi Y, \phi W)].\end{aligned} \tag{5.1}$$

Let $\{e_1, \cdots, e_{n-1}, \xi\}$ be a local orthonormal basis of vector fields in M. By using the fact that $\{\phi e_1, \cdots, \phi e_{n-1}, \xi\}$ is also a local orthonormal basis, if we put $X = W = e_i$ in (5.1) and sum up with respect to i, $1 \le i \le n-1$, we get

$$\sum_{i=1}^{n-1} g(\bar{R}(\phi e_i, \phi Y)\phi Z, \phi e_i)$$

$$= \frac{\bar{r}}{n(n-1)} \sum_{i=1}^{n-1} [g(\phi Y, \phi Z) g(\phi e_i, \phi e_i) - g(\phi e_i, \phi Z) g(\phi Y, \phi e_i)]. \tag{5.2}$$

It is easy to see that

$$\sum_{i=1}^{n-1} g(R(\phi e_i, Y)Z, \phi e_i) = S(Y, Z) + g(Y, Z), \tag{5.3}$$

$$\sum_{i=1}^{n-1} g(\phi e_i, Y) S(\phi e_i, Z) = S(Y, Z), \tag{5.4}$$

$$\sum_{i=1}^{n-1} g(\phi e_i, \phi e_i) = n-1, \tag{5.5}$$

$$\sum_{i=1}^{n-1} g(\phi e_i, Y) g(Z, \phi e_i) = g(Y, Z). \tag{5.6}$$

And by making use of (5.3)-(5.6), the equation (5.2) turns into

$$S(\phi Y, \phi Z) + g(\phi Y, \phi Z) = [2 + \epsilon + \frac{\bar{r}}{n(n-1)}](n-2) g(\phi Y, \phi Z). \tag{5.7}$$

Thus, by applying (2.5) and (2.13) into (5.7), we get

$$S(Y, Z)$$

$$= [-1 + \{2 + \epsilon + \frac{r + n[(\epsilon+2)(\epsilon-n)+2] + \epsilon(1+\epsilon)(n-2\epsilon)}{n(n-1)}\}(n-2)] g(Y, Z) \tag{5.8}$$

$$- [3 + \epsilon + \frac{r + n[(\epsilon+2)(\epsilon-n)+2] + \epsilon(1+\epsilon)(n-2\epsilon)}{n(n-1)}](n-2)\epsilon \eta(Y)\eta(Z).$$

Hence, we have the following;

Theorem 5.2. *A ϕ-concircularly flat (ϵ)-Kenmotsu manifold admitting a semi-symmetric metric connection is an η-Einstein manifold.*

6. (ϵ)-Kenmotsu manifold admitting a semi-symmetric metric connection satisfying $\bar{Z}(X, Y) \cdot \bar{S}(U, W) = 0$

Theorem 6.1. *An (ϵ)-Kenmotsu manifold admitting a semi-symmetric metric connection satisfying $\bar{Z}(X, Y) \cdot \bar{S}(U, W) = 0$ is an η-Einstein manifold.*

Proof. Let us assume that (ϵ)-Kenmotsu manifold admitting a semi-symmetric metric connection satisfies

$$\bar{Z}(X,Y)\cdot\bar{S}(U,W)=0. \tag{6.1}$$

Which implies that

$$\bar{S}(\bar{Z}(\xi,Y)U,W)+\bar{S}(U,\bar{Z}(\xi,Y)W)=0. \tag{6.2}$$

Using (1.2), (2.11) and (2.15) in (6.2), one can get

$$\begin{aligned}&-[(1+\epsilon)+\frac{\bar{r}}{n(n-1)}]g(Y,Z)\bar{S}(\xi,U)+[(1+\epsilon)+\frac{\bar{r}\epsilon}{n(n-1)}]\eta(Z)\bar{S}(Y,U)\\&-[(1+\epsilon)+\frac{\bar{r}}{n(n-1)}]g(Y,U)\bar{S}(Z,\xi)+[(1+\epsilon)+\frac{\bar{r}\epsilon}{n(n-1)}]\eta(U)\bar{S}(Z,Y)=0.\end{aligned} \tag{6.3}$$

Plugging $U=\xi$ in (6.3) and then taking into an account of (2.1), (2.3), (2.4), (2.12) and (2.16), we obtain

$$\begin{aligned}S(Y,Z)=&[-\{(2+\epsilon)(\epsilon-n)+2\}+(n-1)(1+\epsilon)]g(Y,Z)\\&-(1+\epsilon)(n-2\epsilon)\eta(Y)\eta(Z).\end{aligned}$$

□

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